Complex numbers and the exponential function

This homework is not connected with the more recent classwork, but does use some theory of divisibility and residue classes. Let $i$ denote the complex number $\sqrt{-1}$.

Just about the only things you need to know about complex numbers are

A. For any real number $\alpha$, $e^{i\alpha} = \cos \alpha + i \sin \alpha$ (Demoivre’s theorem),

B. For any two complex numbers $z$ and $w$ we have $e^{z+w} = e^z e^w$.

For each real number $\alpha$ define $e(\alpha) = e^{2\pi i \alpha}$.

1. Prove that $e(\alpha)$ is periodic with period 1.

2. Prove that $e(\alpha) = 1$ if and only if $\alpha \in \mathbb{Z}$.

3. Let $m \in \mathbb{N}, a, b \in \mathbb{Z}$. Prove that $a \equiv b \pmod{m}$ if and only if $e(a/m) = e(b/m)$.

4. Prove that
\[
\sum_{k=0}^{m-1} e(ak/m) = \begin{cases}
  m & \text{when } m|a, \\
  0 & \text{when } m \nmid a.
\end{cases}
\]

5. Let
\[
c_m(a) = \sum_{\substack{k=1 \leq \text{gcd}(k, m) = 1}} e(ak/m),
\]
i.e. a sum in which the index $k$ is restricted to a set of reduced residues modulo $m$. This is Ramanujan’s sum. Prove that if $p \nmid a$, then $c_p(a) = -1$ and that if $p|a$, then $c_p(a) = p - 1$.

6. Show that if $(m_1, m_2) = 1$, then $c_{m_1m_2}(a) = c_{m_1}(a)c_{m_2}(a)$. Hint: Use the fact that $k_1m_2 + k_2m_1$ ranges over a reduced set of residues modulo $m_1m_2$ as $k_1$ and $k_2$ range over a reduced set of residues modulo $m_1$ and $m_2$ respectively.