pg-EXTENSIONS AND p-EXTENSIONS WITH APPLICATIONS TO $C(X)$

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Abstract. Essential extensions and p-extensions have been studied for commutative rings with identity in various articles, such as [2], [3], and [13]. The present article applies these concepts to certain subrings of $C(X)$. Moreover, the article introduces a new ring extension, called a pg-extension, and determines its relation to both essential extension and p-extension. It turns out that the pg-extension $R \hookrightarrow S$ induces a well-defined contraction map between principal ideals $\mathcal{P}(S)$ and $\mathcal{P}(R)$.

1. Introduction

In [2] and [3] we looked at a type of ring extension called a p-extension. We say $R \hookrightarrow S$ is a p-extension if for every $s \in S$ there is an $r \in R$ such that $sS = rS$. We are particularly interested in applying this concept to rings of continuous functions, which is discussed in section 4. Furthermore, in section 2, we introduce some new ring extensions called pg-extension and pgs-extension and investigate them in contrast to p-extension. We assume that all rings are commutative and with identity. We are exclusively interested in an extension of rings $R \hookrightarrow S$. Moreover, the extensions are unital. In this way we may view R as a unital subring of S.

For a ring R, the set of units of R is denoted by $\mathfrak{U}(R)$. A ring is called reduced if it has no nonzero nilpotent elements. An element of R is called regular if it is not a zero-divisor. The principal ideal of R generated by r will be denoted by rR. An idempotent of R is an element $e \in R$ for which $e^2 = e$.

R is called a *von Neumann regular ring* if for every $a \in R$ there is an $x \in R$ such that $a^2x = a$. This is known to be equivalent to the condition that every principal ideal of R is generated by an idempotent of R. Alternatively, R is von Neumann regular if and only if R is reduced and every prime ideal is maximal. The annihilator of a subset $T \subseteq R$ is denoted by $Ann_R(T)$. Our general references for topics in ring theory are [6], [8], [9], and [10].

In this paper, all spaces X are assumed to be Tychonoff. Let $C(X)$ denote the ring of continuous functions from a space X into $\mathbb{R}$, $C^*(X)$ is the subring of $C(X)$ consisting of all bounded functions, $C(X, \mathbb{Q})$ the ring of all continuous rational-valued functions on X, $C(X, \mathbb{Z})$ the ring of all continuous integer-valued functions on X, $C_c(X)$ the ring of continuous functions with countable image, and $A(X) = \{f \in C(X) : f^{-1}(O)$ is a countable union of clopen sets for every open subset O of $\mathbb{R}\}$. After our investigation of pg-extensions and pgs-extensions in section 2 and section 3 for general ring extensions, we will focus our attention

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in section 4 to how these extensions, as well as p-extensions, work for extensions involving the subrings of $C(X)$ defined above.

If Y is a dense subspace of a Tychonoff space X, then the function $\Psi : C(X) \to C(Y)$ is an extension of rings induced by restriction. Since Y is dense in X, Ψ is injective. In [2], the authors investigated some situations in which $\Psi : C(X) \to C(Y)$ is a p-extension.

A subspace Y of X is called *C-embedded* if for every $f \in C(Y)$ there is some $g \in C(X)$ such that for all $y \in Y$, $f(y) = g(y)$. This is equivalent to saying $\Psi : C(X) \hookrightarrow C(Y)$ is a surjection. So a dense subspace of X is C-embedded if and only if Ψ is an isomorphism. Given $f \in C(X)$, we will use the notation $f|_Y$ instead of $\Psi(f)$.

Similarly, we say Y is *C^*-embedded* in X if for every $f \in C^*(Y)$ there is some $g \in C^*(X)$ such that for all $y \in Y$, $f(y) = g(y)$, and this is equivalent to saying that the map $\Psi^* : C^*(X) \hookrightarrow C^*(Y)$ induced by restriction is a surjection. It is easy to see that every C-embedded subspace is C^*-embedded, but the converse fails.

For $f \in C(X)$, the *zeroset* of f is the set $Z(f) = \{x \in X : f(x) = 0\}$. Its set-theoretic complement is denoted by $\mathrm{coz}(f)$ and is called a *cozeroset*. A subspace of X is called a *zeroset* (resp. *cozeroset*) if it is of the form $Z(f)$ (resp. $\mathrm{coz}(f)$) for some $f \in C(X)$. Obviously zerosets are closed. Since we are assuming that X is a Tychonoff space, it follows that the collection of cozerosets of X form a base for the topology of open sets of X. For more topological information or information on rings of continuous functions, we urge the reader to consult [5].

Recall that a *π-base* for X is a collection of open sets $\mathcal{B}$ such that for any open subset O of X there is a $B \in \mathcal{B}$ such that $B \subseteq O$. If there is a π-base consisting of clopen sets, then X is said to have a clopen π-base. The reader can consult [4] for more on these topological definitions.

The Stone-Čech compactification of X is denoted by βX. It is known that X is a C^*-embedded subspace of βX, so the induced embedding of $C(\beta X)$ into $C(X)$ is an isomorphism. If X has a base of clopen sets, then X is said to be *zero-dimensional*, and if βX is zero-dimensional, then X is called *strongly zero-dimensional*. So X is strongly zero-dimensional if and only if disjoint zerosets of X can be separated by a clopen set. The following characterization of strongly zero-dimensional spaces will be useful.

Lemma 1.1. [12] *Suppose X is a Tychonoff space. X is a strongly zero-dimensional space if and only if every cozero set is a countable union of clopen subsets of X.*

2. p-EXTENSIONS AND pgs-EXTENSIONS

In [2] the extension of rings $R \hookrightarrow S$ is defined as a p-extension if for every $s \in S$ there is an $r \in R$ such that $sS = rS$. This is equivalent to saying that for each $s \in S$ there is an $r \in R$ and $t_1, t_2 \in S$ such that $r = st_1$ and $s = rt_2$. Moreover, the condition is saying that the principal ideals of S are generated by elements of R. We will see in section 4 that extensions such as $C(X, \mathbb{Q}) \hookrightarrow C_c(X)$ are always p-extensions, while $C(X, \mathbb{Q}) \hookrightarrow C(X)$ being a p-extension is equivalent to X being strongly zero-dimensional.

We also define the extension $R \hookrightarrow S$ as an associate p-extension if for every $s \in S$ there is an $r \in R$ and a unit $u \in S$ such that $r = su$. In other words, every element of S is associate to an element of R. Clearly, an associate p-extension is a p-extension. The converse holds whenever S is an integral domain. It is still unknown

if there exists a p-extension which is not an associate p-extension. Observe that if Y is a C^*-embedded subspace of X, then $C(X) \hookrightarrow C(Y)$ will be an associate p-extension. If Y is a cozeroset of X, then $C(X) \hookrightarrow C(Y)$ will be a p-extension (see [3]).

We say the extension $R \hookrightarrow S$ is a regular localization or R is localized in S if for every $s \in S$ there are $r, u \in R$ such that u is a unit of S and $r = su$. It is known that if Y is a C^*-embedded subspace of X, then $C(Y)$ is a regular localization of $C(X)$. Furthermore, if $C(Y)$ is a regular localization of $C(X)$, then $C(Y)$ is an associate p-extension of $C(X)$.

Recall from [1] that an extension $R \hookrightarrow S$ is called a rigid extension if given $s \in S$ there exists an $r \in R$ such that $Ann_S(s) = Ann_S(r)$. It is straightforward to check that a p-extension is a rigid extension. In the case of von Neumann regular rings, the conditions are equivalent as we see in the next proposition, which is Proposition 5.1 from [2].

Proposition 2.1. *Let $R \hookrightarrow S$ and suppose S is a von Neumann regular ring. The following are equivalent.*

(1) *S is an associate p-extension of R.*
(2) *S is a p-extension of R.*
(3) *S is a rigid extension of R.*

Notice that for any ring R, the extension $R \hookrightarrow R[x]$ does not satisfy the properties mentioned in Proposition 2.1. This leads to the definition of the following ring extensions. An extension $R \hookrightarrow S$ is called

(1) an **Essential extension** if for all $s \in S$, sS nonzero implies that $sS \cap R$ is non zero.
(2) a *pgs*-**extension** if for all $s \in S$, sS nonzero implies that $sS \cap R$ is non zero and principally generated.
(3) a *pg*-**extension** if for all $s \in S$, sS nonzero implies that $sS \cap R$ is principally generated.

It follows immediately that an extension is a *pgs*-extension if and only if it is both an essential extension and a *pg*-extension. Observe that p-extensions are essential extensions, but not conversely. For example, consider $\mathbb{Q} \hookrightarrow \mathbb{R}$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, it follows that $C(\mathbb{Q}) \subset C(\mathbb{R})$ is an essential extension (Proposition 4.1). However this extension is not a p-extension, as proved in Proposition 4.9 of [3]. In addition, essential extensions need not be *pg*-extensions. In section 4 we will show that for a zero-dimensional space which is not a P-space, $C(X, \mathbb{Z}) \hookrightarrow C(X, \mathbb{Q})$ is an essential extension which is not a *pg*-extension.

The following characterization of essential extensions can be found in 1.1 of [13].

Proposition 2.2. *The following are equivalent for $R \hookrightarrow S$.*

(1) *$R \hookrightarrow S$ is an essential extension.*
(2) *Every non zero ideal of S intersects R in a non zero ideal.*
(3) *For each $s \in S$ with $s \neq 0$, there exists $t \in S$ such that $st \neq 0$ and $st \in R$.*

We continue by considering when some common ring extensions, such as $R \hookrightarrow R[x]$, are essential extensions, p-extensions, or *pg*-extensions.

Lemma 2.3. *If R is an integral domain, then $R \hookrightarrow R[x]$ is a pg-extension.*

Proof. Let $f = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n \in R[x]$ be nonzero with $a_0, a_1, \cdots, a_n \in R$. We consider the following cases:

Case 1. $f = a_0$ is a constant polynomial. It follows that $f \in R$ and therefore $fR \subseteq fR[x] \cap R$. On the other hand, suppose $g = b_0 + b_1x + \cdots + b_mx^m \in R[x]$ with $fg \in R$. So, $a_0b_0 + a_0b_1x + \cdots + a_0b_mx^m \in R$, which is possible if $a_0b_i = 0$ for all $1 \leq i \leq m$. Since R is a domain we have $b_1 = b_2 = \cdots = b_m = 0$. Consequently $fg \in fR$, that is, $fR[x] \cap R \subseteq fR$.

Case 2. f is not a constant polynomial, that is, f has a positive degree. Since R is an integral domain, the degree of the product of any two polynomials is the sum of the degrees of the factor polynomials. It follows that the ideal $fR[x]$ does not contain any nonzero constant; hence $fR[x] \cap R = 0 = 0R$.

From the two cases it follows that $R \hookrightarrow R[x]$ is a *pg*-extension. □

Example 2.4.

(1) $R \hookrightarrow R[x]$ is never an essential extension or a *pgs*-extension since $xR[x] \cap R = 0$. Using Lemma 2.3 it follows that $\mathbb{Z} \hookrightarrow \mathbb{Z}[x]$ is an example of a *pg*-extension which is not an essential extension.

(2) We claim that $\mathbb{Z} \hookrightarrow \mathbb{Z}[\sqrt{2}]$ is a *pgs*-extension that is not a *p*-extension. Since $\mathbb{Z}$ is a principal ideal domain, the extension is clearly a *pg*-extension. Let $a + b\sqrt{2} \in \mathbb{Z}[\sqrt{2}]$ for some $a, b \in \mathbb{Z}$. Since $a^2 - 2b^2 = (a + b\sqrt{2})(a - b\sqrt{2}) \in \mathbb{Z}$, it follows that this is an essential extension. Finally, notice that $\sqrt{2} \in \mathbb{Z}[\sqrt{2}]$ does not have an associate in $\mathbb{Z}$, since an associate of $\sqrt{2}$ has a norm of 2. Therefore, the extension is not an associate *p*-extension. Since $\mathbb{Z}$ is an integral domain, the extension is not a *p*-extension.

Lemma 2.5. *Suppose that R, S, and T are three rings with $R \hookrightarrow S \hookrightarrow T$. If $R \hookrightarrow S$ and $S \hookrightarrow T$ are both pg-extensions, then $R \hookrightarrow T$ is also a pg-extension.*

Proof. Let $t \in T$ be a nonzero element. Since $S \hookrightarrow T$ is a *pg*-extension, there exists some $s \in S$ such that $tT \cap S = sS$. If $s = 0$, then $tT \cap R \subseteq tT \cap S = 0$ and hence $tT \cap R = 0R$. Finally if $s \neq 0$, then there exists $r \in R$ such that $sS \cap R = rR$, since $R \hookrightarrow S$ is a *pg*-extension. Notice that $rR = sS \cap R = tT \cap S \cap R = tT \cap R$. Therefore, $R \hookrightarrow T$ is a *pg*-extension. □

Corollary 2.6. *Suppose that R, S, and T are three rings with $R \hookrightarrow S \hookrightarrow T$. If $R \hookrightarrow S$ and $S \hookrightarrow T$ are both pgs-extensions, then $R \hookrightarrow T$ is also a pgs-extension.*

Proof. It is known from [13] that if $R \hookrightarrow S$ and $S \hookrightarrow T$ are essential extensions, then $R \hookrightarrow T$ is an essential extension. It follows that if $R \hookrightarrow S$ and $S \hookrightarrow T$ are *pgs*-extensions, then by the previous Lemma $R \hookrightarrow T$ is also a *pgs*-extension. □

Remark 2.7.

(1) It can be easily verified that if R is a field, then $R \hookrightarrow S$ is an essential extension if and only if S is a field. (Assuming that the rings are commutative with identity.) However, this result does not hold if essential extension is replaced by *pg*-extension. For example, $\mathbb{R} \hookrightarrow \mathbb{R}[x]$ is a *pg*-extension but not an essential extension.

(2) If S is a field, then $R \hookrightarrow S$ is a *pgs*-extension for any R. Let $R \hookrightarrow S$ and $s \in S$ is nonzero. Since S is a field, $sS = S$, and so $sS \cap R = R = 1R$.

(3) If R is a principal ideal ring or a field, then $R \hookrightarrow S$ is a *pg*-extension. The result follows from the fact that for any nonzero $s \in S$, $sS \cap R$ is an ideal of R.

Proposition 2.8. *Let $R \hookrightarrow S$ be an essential extension. If R is an integral domain, then S is an integral domain.*

The proof of the preceding proposition is easily verified using Proposition 2.2. The proposition is not true if we replace 'essential extension' with 'pg-extension' as we see in the next example.

Example 2.9. Let $S = \mathbb{Z} \oplus \mathbb{Z}$ be the ring with pointwise addition and multiplication. Let $R = \{(a, a) : a \in \mathbb{Z}\}$ be a subring of S. Since $(1, 1) \in R$, we have an extension $R \hookrightarrow S$. Notice that R is an integral domain, whereas S is not. We observe the following properties of the two rings R and S.

Let $(a, b) \in S$ with $(1, 0)(a, b) \in R$. Then, $a = 0$, concluding that $(1, 0)S \cap R = (0, 0)$. Therefore, $R \hookrightarrow S$ is not an essential extension. Furthermore, $R \cong \mathbb{Z}$, which is a principal ideal domain. So, $R \hookrightarrow S$ is a pg-extension that is not an essential extension.

Proposition 2.10. *Suppose R, S, and T are rings with $R \hookrightarrow S \hookrightarrow T$. If $R \hookrightarrow T$ is a pg-extension and $R \hookrightarrow S$ is a p-extension, then $S \hookrightarrow T$ is a pg-extension.*

Proof. Let $t \in T$ be a nonzero element. By hypothesis there exists $r \in R$ such that $tT \cap R = rR$. Notice that $rR = tT \cap R \subseteq tT \cap S$, and therefore, $rS \subseteq tT \cap S$. To show the other inclusion let $y = tt_1 \in S$, for some $t_1 \in T$. Since $R \hookrightarrow S$ is a p-extension, there exists $q \in R$ such that $qS = yS$. It follows that there exists $s_1, s_2 \in S$ such that $q = ys_1$ and $y = qs_2$. Observe that $q = ys_1 = t(t_1s_1) \in tT \cap R = rR$. So $q = rr_1$, for some $r_1 \in R$. Finally $y = qs_2 = r(r_1s_2) \in rS$, proving that $tT \cap S \subseteq rS$. □

From [2] we have noticed some interesting extension-related results when we restrict our rings to von Neumann regular rings. In this article we now try to do the same and look at von Neumann regular rings in light of these new extensions.

Proposition 2.11. *Suppose R, S, and T are rings with $R \hookrightarrow T$ is a pg-extension. If S is a von Neumann regular ring with $R \hookrightarrow S \hookrightarrow T$, then $R \hookrightarrow S$ is also a pg-extension.*

Proof. Let s be a nonzero element of S. Since S is a von Neumann regular ring, there exists $y \in S$ such that $s = s^2y$. Therefore, sy is a nonzero idempotent element of S. By hypothesis, there exists $r \in R$ such that $syT \cap R = rR$. So $r = syt$, for some $t \in T$. Notice that

$$r = syt = (sy)^2t = s[y(syt)] \in sS \cap R.$$

Consequently, $rR \subseteq sS \cap R$. On the other hand, $s = (sy)s \in syS$ which implies that $sS \subseteq syS \subseteq syT$. Hence, $sS \cap R \subseteq syT \cap R = rR$. This proves that $sS \cap R = rR$, that is, $R \hookrightarrow S$ is a pg-extension. □

We do not have an example where S is not a von Neumann regular ring where the preceding proposition fails. So, we leave this as an open question.

From Proposition 4.2 of [2] we know that if $R \hookrightarrow S$ is a p-extension and R is a von Neumann regular ring, then S is also von Neumann regular. The question is: when will similar results be true if we replace p-extension with pg-extensions or essential extensions?

Proposition 2.12. *Let R be a von Neumann regular ring with $R \hookrightarrow S$ an essential extension. The following are equivalent.*

(1) *S is an integral domain.*
(2) *R is an integral domain.*
(3) *R is a field.*
(4) *S is a field.*

Proof. (1) $\Rightarrow$ (2) Since R is isomorphic to a subring of S, and S is an integral domain, it follows that R is an integral domain. It is easy to show that a von Neumann regular integral domain is a field, so (2) $\Rightarrow$ (3). The implication (3) $\Rightarrow$ (4) is clear since $R \hookrightarrow S$ is an essential extension. Clearly (4) $\Rightarrow$ (1). □

Recall that a ring R is *reduced* if $rad(R) = 0$, where $rad(R)$ is the radical ideal of R. In other words, R is reduced if whenever $r \in R$ satisfies that $r^n = 0$ for some positive integer n, then $r = 0$. It is known that a reduced ring is von Neumann regular if and only if every prime ideal is a maximal ideal. Also, an essential extension of a reduced ring is reduced (see Lemma 1.2 of [13]).

Proposition 2.13. *Let $R \hookrightarrow S$ be a pg-extension, S a reduced ring, and R a von Neumann regular ring that is not a field. If every prime ideal of S is principal (for example, if S is a principal ideal ring), then S is also a von Neumann regular ring.*

Proof. We only need to show that every prime ideal of S is a maximal ideal. Let $P = sS$ be a prime ideal of S, for some nonzero $s \in S$, and so $P \cap R$ is a prime ideal of R. There exists some $r \in R$ such that $P \cap R = sS \cap R = rR$. Since R is a von Neumann regular ring, $P \cap R$ is a maximal ideal of R. Also since R is not a field it follows that $r \neq 0$. Next we will show that sS is a maximal ideal of S. Let $s_0 \in S \setminus sS$ be arbitrary and consider the ideal $(sS + s_0S) \cap R$. Since $r \in sS$, $r \in sS + s_0S \cap R$, and so $rR \subseteq (sS + s_0S) \cap R$. It follows that $(sS + s_0S) \cap R = R$, because rR is a maximal ideal of R. So $1 \in sS + s_0S$, concluding that $sS + s_0S = S$. Hence, $P = sS$ is a maximal ideal of S. □

Definition 2.14. For a commutative ring R with identity, we denote by $\mathcal{I}(R)$ the collection of all ideals of R, by $Spec(R)$ the collection of all prime ideals of R, and by $\mathcal{P}(R)$ the collection of all principal ideals of R. Suppose $R \hookrightarrow S$ and I is an ideal of S. We say $I \cap R$ is the *contraction* of I in R, following the same concept of 'contraction' in the theories of frames and lattice-ordered groups. Notice that the contraction map $I \mapsto I \cap R$ is well-defined between $\mathcal{I}(S) \to \mathcal{I}(R)$ and between $Spec(S) \to Spec(R)$. Also, if $R \hookrightarrow S$ is a *pg*-extension, then the contraction map restricted of $\mathcal{P}(S)$, denoted $\mathcal{C} : \mathcal{P}(S) \to \mathcal{P}(R)$, is a well-defined map.

Proposition 2.15. *Suppose $R \hookrightarrow S$ is an essential extension, and suppose the contraction map from $\mathcal{I}(S)$ to $\mathcal{I}(R)$ defined by $I \mapsto I \cap R$ is an injective map. If R is a von Neumann regular ring, then S is a von Neumann regular ring.*

Proof. Since R is von Neumann regular and hence reduced, S is also reduced under essential extension. Let P be a prime ideal of S. It follows that $P \cap R$ is a prime ideal of R and hence is maximal in R. Let I be an ideal of S with $P \subset I$. Therefore, $I \cap R$ is an ideal of R with $P \cap R \subseteq I \cap R$. Since $P \cap R$ is a maximal ideal of R, it follows that either $I \cap R = R$ or $I \cap R = P \cap R$. If $I \cap R = P \cap R$, then by the injectivity of the contraction map it follows that $I = P$, which is not the case; thence, $I \cap R = R$. Consequently $1 \in I$, proving that $I = S$. Hence, P is a maximal ideal of S. Since every prime ideal of S is maximal, S is a von Neumann regular ring. □

Theorem 2.16. *Let $R \hookrightarrow S$ be a pg-extension. The contraction map $\mathcal{C} : \mathcal{P}(S) \to \mathcal{P}(R)$ is an injective map if and only if $R \hookrightarrow S$ is a p-extension.*

Proof. Assume $\mathcal{C} : \mathcal{P}(S) \to \mathcal{P}(R)$ is an injective map, and let $s \in S$ be nonzero. Using the definition of a pg-extension, there exists $r \in R$ such that $sS \cap R = rR$. It is easy to check that $sS \cap R = rR = rS \cap R$. In other words, $\mathcal{C}(sS) = \mathcal{C}(rS)$. Since $\mathcal{C}$ is an injective map, we have $sS = rS$, concluding that $R \hookrightarrow S$ is a p-extension.

Conversely, suppose $R \hookrightarrow S$ is a p-extension. Let $s, t \in S$ with $\mathcal{C}(sS) = \mathcal{C}(tS)$, that is, $sS \cap R = tS \cap R$. Using the fact that $R \hookrightarrow S$ is a p-extension, there exist $r, q \in R$ such that $sS = rS$ and $tS = qS$. Consequently, $rS \cap R = qS \cap R$. It is straightforward to check that $rS = qS$. Hence, $sS = tS$. □

Corollary 2.17. *Let $R \hookrightarrow S$ be a pg-extension and let R be a von Neumann regular ring. If $\mathcal{C} : \mathcal{P}(S) \to \mathcal{P}(R)$ is an injective map, then S is also von Neumann regular.*

Proof. The result follows from the fact that if $R \hookrightarrow S$ is a p-extension and R is von Neumann regular, then S is also von Neumann regular. □

3. Absorbing Sets

We know that a p-extension is an essential extension, and Theorem 2.16 gave us a condition when a pgs-extension is a p-extension. We now ask the question: when is a p-extension a pgs-extension? To investigate further, notice that in general for $R \hookrightarrow S$, if $r \in R$ is nonzero, then $rS \cap R \neq rR$. For example, consider $\mathbb{Z} \hookrightarrow \mathbb{R}$ and let $2 \in \mathbb{Z}$. Then, $2\mathbb{R} \cap \mathbb{Z} = \mathbb{R} \cap \mathbb{Z} = \mathbb{Z} \neq 2\mathbb{Z}$. Trivially, if $r = 0$, then it is always true that $rS \cap R = rR$.

Definition 3.1. Let R be a ring with $I \subseteq R$. If $R \setminus I \neq \emptyset$ and for every $x \in I$ and nonzero $r \in R \setminus I$, $xr \in I$, then we call I an *absorbing subset* of R.

Example 3.2.

(1) Consider $\mathbb{Q} \hookrightarrow \mathbb{R}$. The set of irrational numbers, $\mathbb{R} \setminus \mathbb{Q}$, is an absorbing subset of $\mathbb{R}$.
(2) Consider $\mathbb{Z} \hookrightarrow \mathbb{R}$. Although $\mathbb{Z}$ is a domain, $\mathbb{R} \setminus \mathbb{Z}$ is not an absorbing subset of $\mathbb{R}$. Notice further that if $n \in \mathbb{Z}$ is nonzero and non-unit, then $n\mathbb{R} \cap \mathbb{Z} = \mathbb{R} \cap \mathbb{Z} = \mathbb{Z} \neq n\mathbb{Z}$.
(3) Consider $\mathbb{Z} \hookrightarrow \mathbb{Z}[x]$. Notice that $\mathbb{Z}$ is a domain and $\mathbb{Z}[x] \setminus \mathbb{Z}$ is an absorbing subset of $\mathbb{Z}[x]$.
(4) Let α be an algebraic number and consider the extension $\mathbb{Z} \hookrightarrow \mathbb{Z}[\alpha]$, where $\mathbb{Z}[\alpha] \setminus \mathbb{Z} \neq \emptyset$. Let $s = m + n\alpha \in \mathbb{Z}[\alpha] \setminus \mathbb{Z}$, where $m, n \in \mathbb{Z}$, then $n \neq 0$. For any nonzero $r \in \mathbb{Z}$ we have that $rs = rm + rn\alpha \in \mathbb{Z}[\alpha] \setminus \mathbb{Z}$, since $rn \neq 0$. Hence $\mathbb{Z}[\alpha] \setminus \mathbb{Z}$ is an absorbing subset of $\mathbb{Z}[\alpha]$. Similarly, $\mathbb{Z}[\alpha] \setminus \mathbb{Z}$ is an absorbing subset of $\mathbb{Z}[\alpha]$ for any transcendental number α.

The above examples lead us to some general results about absorbing subsets of commutative rings.

Proposition 3.3. *A ring R is an integral domain if and only if $R[x] \setminus R$ is an absorbing subset of R.*

Proof. Suppose R is a domain and let $f(x) = a_0 + a_1x + \cdots + a_nx^n \in R[x] \setminus R$. Consequently, $a_i \neq 0$ for some $1 \leq i \leq n$. For any nonzero $r \in R$ it follows that $ra_i \neq 0$. Hence, $rf(x) \in R[x] \setminus R$.

Conversely, suppose R is not a domain. Let $r \in R$ be a nonzero zero divisor, that is, there exists nonzero $q \in R$ such that $rq = 0$. It follows that $0 = x(rq) = (xr)q \notin R[x] \setminus R$. Consequently, $R[x] \setminus R$ is not an absorbing subset of $R[x]$. □

Theorem 3.4. *For $R \hookrightarrow S$, the following are equivalent:*

(1) *$S \setminus R$ is an absorbing subset of S.*
(2) *R is a domain and for every $r \in R$, $rS \cap R = rR$.*

Proof. Assume $S \setminus R$ is an absorbing subset of S. If $r = 0$, then $rS \cap R = rR$. Suppose that $r \in R$ is nonzero. Since $r \in rS$, it follows that $rR \subseteq rS \cap R$. For the other inclusion, let $x \in rS \cap R$. There exists $s \in S$ with $x = rs \in R$. If $s \in S \setminus R$, then $rs \in S \setminus R$ by definition, which contradicts the fact that $x \in R$. Therefore, $x = rs$ with $s \in R$, which means that $x \in rR$.

To show that R is a domain, suppose by contradiction that $x, y \in R$ is nonzero with $xy = 0$. Let $s \in S \setminus R$. By definition, $sx \in S \setminus R$ is nonzero. Furthermore, $(sx)y \in S \setminus R$ since y is also a nonzero element of R. This means that $0 = s(xy) \in S \setminus R$, which is a contradiction. Hence, one of x or y must be 0.

To prove the converse, let $s \in S \setminus R$ and $r \in R$ be nonzero. Since $rS \cap R = rR$, we have that $rs \in rS$. If $rs \in R$, then $rs \in rR$ and so $rs = rq$, for some $q \in R$. Since R is a domain, $s = q \in R$, a contradiction. Therefore, $rs \in S \setminus R$ concluding that $S \setminus R$ is an absorbing subset of S. □

Corollary 3.5. *Let $R \hookrightarrow S$. If R is a field, then $S \setminus R$ is an absorbing subset of S.*

Corollary 3.6. *Let $R \hookrightarrow S$ and suppose $S \setminus R$ is an absorbing subset of S. Then:*

(1) *If S is a field, then R is a field.*
(2) *If $R \hookrightarrow S$ is a pgs-extension, then $\mathcal{C} : \mathcal{P}(S) \to \mathcal{P}(R)$ is a surjection.*
(3) *If $R \hookrightarrow S$ is a p-extension, then it is a pgs-extension. Furthermore, $\mathcal{C} : \mathcal{P}(S) \to \mathcal{P}(R)$ is a bijection.*

Proof. (1) Let $r \in R$ be nonzero. From Theorem 3.4 it follows that R is a domain and $rS \cap R = rR$. Since S is a field, $rS = S$, which means that $rR = S \cap R = R$. So $1 \in rR$, concluding that r is a unit.

(2) It follows immediately that if $rR \in \mathcal{P}(R)$, then $rS \in \mathcal{P}(S)$ such that $\mathcal{C}(rS) = rS \cap R = rR$.

(3) We already know that a p-extension is an essential extension. Let $s \in S$ be nonzero. Using the definition of p-extension, there exists some nonzero $r \in R$ such that $sS = rS$. From Theorem 3.4 we know that $rS \cap R = rR$. Therefore $sS \cap R = rR$, concluding that $R \hookrightarrow S$ is a pgs-extension. □

The dual concept of a contraction map $\mathcal{C} : \mathcal{P}(S) \to \mathcal{P}(R)$ is an order-preserving extension map $p : \mathcal{P}(R) \to \mathcal{P}(S)$ defined by $p(rR) = rS$, for $r \in R$. This is a well-defined map but need not be injective or surjective. Notice that the map p is surjective precisely when $R \hookrightarrow S$ is a p-extension. The question we ask at this point is, when is the map p an injective map?

Proposition 3.7. *Suppose $R \hookrightarrow S$ is a pg-extension. The following are equivalent:*

(1) *$rS \cap R = rR$, for each $r \in R$.*
(2) *$p : \mathcal{P}(R) \to \mathcal{P}(S)$ is an injective map.*

Proof. Let $r, q \in R$ such that $rS = qS$; then $rS \cap R = qS \cap R$. Consequently, $rR = qR$.

For the converse, given nonzero $r \in R$ there exists some $q \in R$ such that $rS \cap R = qR$, by pg-extension. Notice that $r \in qR \subseteq qS$, which means that $rS \subseteq qS$. Again, $q \in rS$ implies that $qS \subseteq rS$. So, $rS = qS$. Since p is an injective map, it follows that $rR = qR = rS \cap R$. □

Corollary 3.8. *Let $R \hookrightarrow S$ with $S \setminus R$ an absorbing subset of S, then $p : \mathcal{P}(R) \to \mathcal{P}(S)$ is an injective map.*

We summarize the relationship between the p-extension, the pgs-extension, the contraction map $\mathcal{C}$, and the map p in the following theorem:

Theorem 3.9. *Let $R \hookrightarrow S$ with $S \setminus R$ an absorbing subset of S. The following are equivalent:*

(1) *$R \hookrightarrow S$ is a p-extension.*
(2) *$R \hookrightarrow S$ is a p-extension and a pgs-extension.*
(3) *$p : \mathcal{P}(R) \to \mathcal{P}(S)$ and $\mathcal{C} : \mathcal{P}(S) \to \mathcal{P}(R)$ are bijective maps.*
(4) *$p : \mathcal{P}(R) \to \mathcal{P}(S)$ is a bijective map.*
(5) *$p : \mathcal{P}(R) \to \mathcal{P}(S)$ is a surjective map.*

Notice that since $S \setminus R$ is an absorbing subset of S only if R is a domain (from Proposition 3.4), the preceding theorem applies in this specific situation where R is an integral domain. Furthermore, we will observe in Section 4, Example 4.8, an example of $R \hookrightarrow S$ where R is not a domain and the extension is a p-extension that is not a pgs-extension.

Remark 3.10. We make a few observations regarding the contraction map $\mathcal{C}$ and the extension map p. Suppose $R \hookrightarrow S$ is a pgs-extension with $S \setminus R$ an absorbing subset of S, then $\mathcal{C} : \mathcal{P}(S) \to \mathcal{P}(R)$ is well defined. Furthermore, $\mathcal{C} \circ p : \mathcal{P}(R) \to \mathcal{P}(R)$ gives us, for $r \in R$,

$$(\mathcal{C} \circ p)(rR) = \mathcal{C}(rS) = rS \cap R = rR.$$

If $R \hookrightarrow S$ is also a p-extension, then given $s \in S$ there exists $r \in R$ such that $sS = rS$. So, $p \circ \mathcal{C} : \mathcal{P}(S) \to \mathcal{P}(S)$ gives us,

$$(p \circ \mathcal{C})(sS) = (p \circ \mathcal{C})(rS) = p(rS \cap R) = p(rR) = rS = sS.$$

In conclusion, if $R \hookrightarrow S$ is a p-extension and a pgs-extension with $S \setminus R$ an absorbing subset of S, then the contraction map $\mathcal{C}$ and the extension map p are inverses of each other. In other words, $\mathcal{P}(S)$ and $\mathcal{P}(R)$ are isomorphic.

We end the section investigating the relationship between various extensions for von Neumann regular rings. Recall that an $R \hookrightarrow S$ is an associate p-extension if given $s \in S$ there exist $r \in R$ and unit $u \in S$ such that $r = su$.

Lemma 3.11. *Let $R \hookrightarrow S$ be an essential extension. If $S \setminus R$ is an absorbing subset of S, then S is a domain.*

Proof. Since $S \setminus R$ is an absorbing subset of S, it follows that R is a domain. By Proposition 2.8, S is a domain □

Lemma 3.12. *Suppose $R \hookrightarrow S$ is an essential extension, and S is a von Neumann regular ring. If $S \setminus R$ is an absorbing subset of S, then $R \hookrightarrow S$ is a pgs-extension.*

Proof. From Lemma 3.11 S is a domain, so S does not contain any nontrivial idempotent elements. Since S is a von Neumann regular ring, it follows that S is a field. Using Corollary 3.6 R is also a field. Thus, the extension is a *pgs*-extension. □

Finally, we have the following theorem for von Veumann regular rings, in contrast to Proposition 2.1.

Theorem 3.13. *Suppose $R \hookrightarrow S$ with S a von Neumann regular ring. If $S \setminus R$ is an absorbing subset of S, then the following are equivalent.*

(1) *$R \hookrightarrow S$ is an essential extension.*
(2) *$R \hookrightarrow S$ is a pgs-extension.*
(3) *$R \hookrightarrow S$ is a p-extension.*
(4) *$R \hookrightarrow S$ is a rigid extension.*
(5) *$R \hookrightarrow S$ is an associate p-extension.*

Proof. (1) $\Rightarrow$ (2) From Lemma 3.12.

(2) $\Rightarrow$ (3) Since $R \hookrightarrow S$ is an essential extension, S is a domain using Lemma 3.11. Therefore, S is a field, which implies that $R \hookrightarrow S$ is a p-extension.

(3) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5) From Proposition 2.1.

(5) $\Rightarrow$ (1) Let $s \in S$ be nonzero. There exists $r \in R$ and unit $u \in S$ such that $r = su$. So, $r \neq 0$. Hence, $sS \cap R \neq 0$. □

Observe that the ring extension $\mathbb{Z} \hookrightarrow \mathbb{Z}[\sqrt{2}]$ is a *pgs*-extension that is not a p-extension. Furthermore, $\mathbb{Z}[\sqrt{2}] \setminus \mathbb{Z}$ is an absorbing subset of $\mathbb{Z}[\sqrt{2}]$, but $\mathbb{Z}[\sqrt{2}]$ is not a von Neumann regular ring.

The question we ask is if it is possible to drop the condition "$S \setminus R$ an absorbing subset of S" in the preceding theorem and have a similar result for von Neumann regular rings. Let us denote the set of all idempotents of a ring R by $\mathcal{B}(R)$, then $\mathcal{B}(R)$ is a Boolean ring.

Theorem 3.14. *Suppose R and S are von Neumann regular rings with $R \hookrightarrow S$. The following are equivalent.*

(1) *$R \hookrightarrow S$ is an associate p-extension.*
(2) *$R \hookrightarrow S$ is a p-extension.*
(3) *$R \hookrightarrow S$ is a rigid extension.*
(4) *$p : \mathcal{P}(R) \to \mathcal{P}(S)$ is a surjective map.*
(5) *$\mathcal{B}(R) = \mathcal{B}(S)$.*

Proof. (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) From Proposition 2.1.

(2) $\Leftrightarrow$ (4) Clear.

(4) $\Leftrightarrow$ (5) Suppose $\mathcal{B}(R) = \mathcal{B}(S)$ and let $s \in S$ be nonzero. Since S is a von Neumann regular ring, there exists $e \in \mathcal{B}(S) = \mathcal{B}(R)$ such that $sS = eS$. So, $e \in R$ and $p(eR) = eS = sS$.

On the other hand, let us assume (4) and choose $e \in \mathcal{B}(S)$. There exists some $r \in R$ such that $rS = eS$. Since R is a von Neumann regular ring, we can assume $r \in \mathcal{B}(R)$. It follows that $e = er = r$; that is, $e \in \mathcal{B}(R)$. □

Lemma 3.15. *Suppose R and S are von Neumann regular rings. If $R \hookrightarrow S$ is an associate p-extension, then $R \hookrightarrow S$ is a pgs-extension.*

Proof. Let $s \in S$ be nonzero, there exists $r \in R$ and unit $u \in S$ such that $r = su$. So, $r \neq 0$ and $rS = sS$. Since R is a von Neumann regular ring, there exists an idempotent $e \in R$ with $r = re$ and $eR = rR$. It follows that $sS = rS = eS$; consequently, $eR \subseteq sS \cap R$. On the other hand if $x \in eS \cap R$, then there exists $t \in S$ with $x = et = e(et) = ex \in eR$. Therefore, $sS \cap R = eS \cap R = eR$. □

Lemma 3.16. *Suppose R and S are von Neumann regular rings with $\mathcal{B}(R) = \mathcal{B}(S)$. If $R \hookrightarrow S$ is a pgs-extension, then $R \hookrightarrow S$ is a p-extension.*

Proof. Let $s \in S$ be nonzero, there exists $r \in R$ nonzero such that $sS \cap R = rR$. Since S is a von Neumann regular ring, $sS = eS$ for some $e \in \mathcal{B}(S)$. So, $e \in R$ with $eS \cap R = rR$. Consequently, $eS = rS$. □

Theorem 3.17. *Suppose R and S are von Neumann regular rings with $R \hookrightarrow S$. If $\mathcal{B}(R) = \mathcal{B}(S)$, then the following are equivalent.*

(1) *$R \hookrightarrow S$ is an associate p-extension.*
(2) *$R \hookrightarrow S$ is a p-extension.*
(3) *$R \hookrightarrow S$ is a rigid extension.*
(4) *$R \hookrightarrow S$ is a pgs-extension.*

Proof. The equivalence of (1), (2), and (3) follows from Proposition 2.1. (1) implies (4) by Lemma 3.15, and (4) implies (2) by Lemma 3.16. □

4. Applications to $C(X)$

Now that we have considered pg-extensions and pgs-extensions for rings $R \hookrightarrow S$, we are ready to investigate these extensions (along with p-extensions and essential extensions) concerning certain subrings of $C(X)$, specifically

$$C(X, \mathbb{Z}) \hookrightarrow C(X, \mathbb{Q}) \hookrightarrow C_c(X) \hookrightarrow A(X) \hookrightarrow C(X).$$

We would like to note that $C(X)$ will not be a domain as long as X has at least two points, so many of the results regarding absorbing subsets in section 3 will not apply here.

We start with a few results concerning $C(X) \hookrightarrow C(Y)$ when Y is a subspace of X. The following proposition is Theorem 2.1 in [14].

Proposition 4.1. *If Y is a dense subspace of X, then $C(X) \hookrightarrow C(Y)$ is an essential extension.*

Proposition 4.2. *If Y is C^*-embedded in X, then $C(X) \hookrightarrow C(Y)$ is a pg-extension. If Y is also dense in X, then $C(X) \hookrightarrow C(Y)$ is a pgs-extension.*

Proof. Let $f \in C(Y)$ be nonzero, then there exists $f^* \in C^*(Y)$ such that $fC(Y) = f^*C(Y)$. There also exists $g \in C(X)$ such that $g|_Y = f^*$. Hence $fC(Y) \cap C(X) = f^*C(y) \cap C(X) = gC(X)$. □

We have been unable to find an example of a pgs-extension that is not a C^*-extension.

Example 4.3. Let D be an uncountable discrete space. αD denotes the one-point compactification of D, and λD denotes the space in which all but one point, d, are isolated and a neighborhood of d is any set containing d whose complement is countable.

(1) $C(\lambda D) \hookrightarrow C(D)$ is not a p-extension. To see this, define $f = \chi_{\{d\}} \in C(D)$. Assume, by way of contradiction, that $C(\lambda D) \hookrightarrow C(D)$ is a p-extension, then there exists $g \in C(\lambda D)$ with $fC(D) = gC(D)$. It follows that $Z(f) = Z(g) = D \smallsetminus \{d\}$, however $g \in C(\lambda D)$ and $g(D \smallsetminus \{d\})$ impy $g(d) = 0$.
(2) $C(\lambda D) \hookrightarrow C(D)$ is not an essential extension. With f as defined in (1), let $ff' \in fC(D) \cap C(\lambda D)$ for some $f' \in C(D)$. We see that $Z(f) \subseteq Z(ff')$, however $ff' \in C(\lambda D)$ with $Z(ff') \supseteq D \smallsetminus \{d\}$ forces $ff' = \mathbf{0}$. Hence $C(\lambda D) \hookrightarrow C(D)$ is not a pgs-extension.
(3) $C(\lambda D) \hookrightarrow C(D)$ is not a pg-extension. Select $h \in C(D)$ such that $d \notin coz(h)$ and $coz(h)$ is uncountable. Note that $\chi_{coz(h)} \notin C(\lambda D)$. Assume, by means of contradiction, that there exists $k \in C(\lambda D)$ with $hC(D) \cap C(\lambda D) = kC(\lambda D)$. Then $Z(h) \subseteq Z(k)$. Observe that $h\chi_{\{}a\} \in C(\lambda D)$ for every $a \in coz(h)$, and hence $Z(k) \subseteq D \smallsetminus \{a\}$ for every $a \in coz(h)$. Thus $Z(k) = Z(h)$, which is a contradiction since $coz(h)$ is uncountable and $d \in Z(k)$.
(4) It is easy to see that if $C(\alpha D) \hookrightarrow C(D)$ is essential or a p-extension, then so must be $C(\lambda D) \hookrightarrow C(D)$. Hence $C(\alpha D) \hookrightarrow C(D)$ is not an essential extension, pgs-extension, or a p-extension.

Observe that if X has a clopen π-base, then $C(X, \mathbb{Q})$, $C_c(X)$, and $A(X)$ are essential extensions of $C(X, \mathbb{Z})$. To see this, let f be a nonzero function in $A(X)$, then there is a clopen set $K \subseteq coz(f)$. It follows that $\chi_K \in fA(X) \cap C(X, \mathbb{Z})$, hence $fA(X) \cap C(X, \mathbb{Z})$ is non zero. A similar argument applies if $f \in C_c(X)$ or $f \in C(X, \mathbb{Q})$.

Proposition 4.4. *The following are equivalent:*
(1) *X has a clopen π-base.*
(2) *$C(X, \mathbb{Z}) \hookrightarrow C(X)$ is an essential extension.*
(3) *$C(X, \mathbb{Q}) \hookrightarrow C(X)$ is an essential extension.*
(4) *$C_c(X) \hookrightarrow C(X)$ is an essential extension.*
(5) *$A(X) \hookrightarrow C(X)$ is an essential extension.*

Proof. The equivalence of (1)-(3) can be found in Theorem 2.9 of [11]. To show (1) implies (4), if X has a clopen π-base, then by Proposition 4.4 we know $C(X, \mathbb{Q}) \hookrightarrow C(X)$ is an essential extension and therefore $C_c(X) \hookrightarrow C(X)$ is an essential extension. Clearly (4) implies (5). To show (5) implies (1), assume $A(X) \hookrightarrow C(X)$ is an essential extension, and let O be a cozeroset of X, say $O = coz(f)$ for some $f \in C(X)$. By assumption, we know there exists a nonzero $g \in fC(X) \cap A(X)$. Since $g \in A(X)$, there exist clopen sets K_n such that $coz(g) = \bigcup_{n=1}^{\infty} K_n$. It follows that $O = coz(f) \supseteq coz(g) = \bigcup_{n=1}^{\infty} K_n$, and thus X has a clopen π-base. □

The next lemma will be useful in proving Theorems 4.6 and 4.7, in which we will characterize when certain subrings of $C(X)$ are p-extensions and pg-extensions of $C(X, \mathbb{Z})$.

Lemma 4.5. *Let X be a space with a clopen π-base, and let S represent any subring of $C(X)$ containing $C(X, \mathbb{Z})$. For any $f \in S$, $fS \cap C(X, \mathbb{Z}) = hC(X, \mathbb{Z})$ for some $h \in C(X, \mathbb{Z})$ if and only if $Z(f)$ is a clopen subset of X.*

Proof. Let $f \in S$. Clearly if $Z(f)$ is clopen, then $fS \cap C(X, \mathbb{Z}) = \chi_{coz(f)} C(X, \mathbb{Z})$. So suppose $fS \cap C(X, \mathbb{Z}) = hC(X, \mathbb{Z})$ for some $h \in C(X, \mathbb{Z})$. It follows that $Z(f) \subseteq Z(h)$. Assume, by way of contradiction, that $Z(f) \neq Z(h)$. Since $h \in$

$C(X,\mathbb{Z})$, $Z(f)$ is an open subset of X. So there exists a clopen set K such that $K \subseteq Z(h) \smallsetminus Z(f)$. Note that $\chi_K \in C(X,\mathbb{Z})$ and $\frac{1}{f}\chi_K \in S$. Hence $\chi_K = f(\frac{1}{f}\chi_K) \in fS \cap C(X,\mathbb{Z}) = hC(X,\mathbb{Z})$. However, χ_K is not a multiple of h because $Z(h)$ is not a subset of $Z(\chi_K) = X \smallsetminus K$. Therefore $Z(f) = Z(h)$ is clopen. □

Recall that X is called a P-space if $C(X)$ is a von Neumann regular ring. It is equivalent to say $Z(f)$ is open for all $f \in C(X)$, or to say every cozeroset of X is C-embeddded.

Theorem 4.6. *For any space X with a clopen π-base, the following are equivalent:*

(1) *X is a P-space.*
(2) *$C(X,\mathbb{Z}) \hookrightarrow C(X)$ is a p-extension.*
(3) *$C(X,\mathbb{Z}) \hookrightarrow C(X)$ is a pg-extension.*

Proof. If X is a P-space and $f \in C(X)$, then clearly $fC(X) = \chi_{coz(f)}C(X)$ and thus $C(X,\mathbb{Z}) \hookrightarrow C(X)$ is a p-extension. The equivalence of (2) and (3) follow from Lemma 4.5. Suppose $C(X,\mathbb{Z}) \hookrightarrow C(X)$ is a p-extension, and let $f \in C(X)$. Then there exists $g \in C(X,Z)$ with $fC(X) = gC(X)$, which implies $Z(f) = Z(g)$ is clopen. □

Theorem 4.7. *For any space X with a clopen π-base, the following are equivalent:*

(1) *$C(X,\mathbb{Z}) \hookrightarrow C(X,\mathbb{Q})$ is a p-extension.*
(2) *$C(X,\mathbb{Z}) \hookrightarrow C(X,\mathbb{Q})$ is a pg-extension.*
(3) *$C(X,\mathbb{Z}) \hookrightarrow C_c(X)$ is a p-extension.*
(4) *$C(X,\mathbb{Z}) \hookrightarrow C_c(X)$ is a pg-extension.*
(5) *$C(X,\mathbb{Z}) \hookrightarrow A(X)$ is a p-extension.*
(6) *$C(X,\mathbb{Z}) \hookrightarrow A(X)$ is a pg-extension.*
(7) *For any sequence $\{K_n\}$ of clopen subsets of X, $\bigcup_{n=1}^{\infty} K_n$ is clopen.*

Proof. Lemma 4.5 tells us that (1) and (2) are equivalent, (3) equivalent with (4), and (5) equivalent with (6). Assume $C(X,\mathbb{Z}) \hookrightarrow C(X,\mathbb{Q})$ is a p-extension, and we will show that $C(X,\mathbb{Z}) \hookrightarrow C_c(X)$ is a p-extension. Let $f \in C_c(X)$, then it is straightforward to prove there exists $g \in C(X,\mathbb{Q})$ such that $coz(f) = coz(g)$. By (1) there exists $h \in C(X,\mathbb{Z})$ with $coz(f) = coz(g)$, where $coz(h)$ is a clopen set. Hence $fC_c(X) = \chi_{coz(f)}C_C(X)$. The proof of (3) implies (5) is similar to that of (1) implies (3).

Next assume (5) holds, and let $\{K_n\}$ be a sequence of clopen subsets of X. Define $f \in A(X)$ by

$$g(x) = \begin{cases} \frac{1}{n}, & \text{if } x \in K_n \\ 0 \text{ otherwise .} \end{cases}$$

Then by (5) there exists $h \in C(X,\mathbb{Z})$ such that $hA(X) = gA(X)$, in which case $coz(f) = coz(g) = \bigcup_{n=1}^{\infty} K_n$ is clopen.

Finally, assume (7) holds, and let $g \in C_c(X)$. Then by (7) we know that $coz(g)$ is clopen. Hence $gC_c(X) = \chi_{coz(g)}C_c(X)$.

□

Using the previous theorem, we can give an example of an essential extension which is not a pg-extension. Note that $C(\mathbb{Q},\mathbb{Z}) \hookrightarrow C(\mathbb{Q},\mathbb{Q})$ is essential (see the paragraph preceding Proposition 4.4) and $\mathbb{Q}$ has a clopen π-base, but $C(\mathbb{Q},\mathbb{Z}) \hookrightarrow C(\mathbb{Q},\mathbb{Q})$ is not a pg-extension (and hence not a pgs-extension) since a countable

union of clopen subsets of $\mathbb{Q}$ need not be clopen. The following is an example of a p-extension which is not a pgs-extension.

Example 4.8. $\mathbb{R}^* = \mathbb{R} \setminus 0$ is a cozeroset of $\mathbb{R}$, so $C(\mathbb{R}) \subset C(\mathbb{R}^*)$ is a p-extension and hence an essential extension. However, this is not a pgs-extension. Consider $\chi = \chi_{(0,\infty)} \in C(\mathbb{R}^*)$, then $\chi C(\mathbb{R}^*) \cap C(\mathbb{R}) = \{g \in C(\mathbb{R}) : (-\infty, 0] \subseteq Z(g)\}$, which we claim is not a principal ideal of $C(\mathbb{R})$. To see this, suppose to the contrary that $\chi C(\mathbb{R}^*) \cap C(\mathbb{R})$ is generated by $h \in C(\mathbb{R})$. If $(-\infty, 0] \subset Z(h)$, then there exists $c \in \mathbb{R}$ with $c > 0$ and $h(c) \neq 0$. Select any $k \in C(\mathbb{R})$ such that $k(c) = 0$, then k cannot be a multiple of h because $Z(h)$ is not a subset of $Z(k)$. So we must have $Z(h) = (\infty, 0]$. Now define a function $f \in C(\mathbb{R})$ with the following properties for any $n \in \mathbb{N}$: $f(x) = h(x)$ if $x \in [\frac{1}{2n}, \frac{1}{2n-1}]$ or if $x \leq 0$, and $f(x) = -h(x)$ if $x \in [\frac{1}{2n+1}, \frac{1}{2n}]$. If $f = hh_1$ for some $h_1 \in C(\mathbb{R})$, then $h_1(x) = 1$ if $x \in [\frac{1}{2n}, \frac{1}{2n-1}]$, and $h_1(x) = -1$ if $x \in [\frac{1}{2n+1}, \frac{1}{2n}]$. The function h_1 is not continuous at 0, a contradiction.

Proposition 4.9. *For any space X, $C(X, \mathbb{Q}) \hookrightarrow C_c(X)$ is a p-extension.*

Proof. Let $f \in C_c(X)$ be a nonunit. If $f(X)$ is finite, then clearly $fC(X, \mathbb{Q}) = \chi_{coz(f)} C(X, \mathbb{Q})$ and we are done. So assume $f(X)$ is countably infinite. Without loss of generality, we may assume $f \in C_c(X)^+$. Choose a sequence x_n of real numbers such that $x_n \notin f(X)$ and $\frac{1}{n+1} < x_n < \frac{1}{n}$ for each $n \in \mathbb{N}$. Define

$$g(x) = \begin{cases} 1 & \text{if } x \in f^{-1}(x_1, \infty) \\ \frac{1}{n+2} & \text{if } x \in f^{-1}(x_{n+1}, x_n),\ n \in \mathbb{N} \\ 0 & \text{if } x \in Z(f). \end{cases}$$

Clearly $g \in C(X, \mathbb{Q})$ with $Z(f) = Z(g)$. Define $h_1 : X \to \mathbb{R}$ by

$$h_1(x) = \begin{cases} \frac{f(x)}{g(x)} & \text{if } x \in coz(f) \\ 1 & \text{if } x \in Z(f) \end{cases}$$

Then for $x \in f^{-1}(x_{n+1}, x_n)$ we have $h_1(x) = \frac{f(x)}{g(x)} \leq \frac{x_n}{\frac{1}{n+2}} < \frac{\frac{1}{n}}{\frac{1}{n+2}}$ which converges to 1. Hence $h_1 \in C_c(X)$. Let $h_2 = \frac{1}{h_1} \in C_c(X)$. It is clear that $f = gh_1$ and $g = fh_2$. □

Proposition 4.10. *For any space X, $C(X, \mathbb{Q}) \hookrightarrow C_c(X)$ is a pg-extension.*

Proof. Let $f \in C_c(X)^+$, and pick a sequence $\{r_n\} \subset X \setminus f(X)$ with $0 < r_{n+1} < r_n$ and $\frac{1}{n+1} < r_n < \frac{1}{n}$ for each $n \in \mathbb{N}$. Define clopen sets $K_1 = f^{-1}((r_1, \infty))$ and $f^{-1}((r_n, r_{n-1}))$ for $n \geq 2$, and we see that $coz(f) = \bigcup_{n=1}^{\infty} K_n$. Next define $h \in C(X, \mathbb{Q})$ by

$$h(x) = \begin{cases} \frac{1}{n+1}, & \text{if } x \in K_n \\ 0, & \text{otherwise .} \end{cases}$$

We will show that $hC(X, \mathbb{Q}) = fC_c(X) \cap C(X, \mathbb{Q})$.

To show that h is a multiple of f in $C_c(X)$, define $f' : X \to \mathbb{R}$ by

$$f'(x) = \begin{cases} (n+1)f(x)k(x), & \text{if } x \in K_n \\ 1, & \text{otherwise .} \end{cases}$$

For $x \in K_n = f^{-1}((r_n, r_{n-1}))$ we have

$$\frac{n-1}{n+1} < \frac{1}{(n+1)f(x)} < 1,$$

and so $f' \in C_c(X)$. Since $ff' = h$, $hC(X,\mathbb{Q}) \subseteq fC_c(X) \cap C(X,\mathbb{Q})$.

Suppose $fk \in C_c(X) \cap C(X,\mathbb{Q})$, and define $h' : X \to \mathbb{Q}$ by

$$h'(x) = \begin{cases} (n+1)f(x)k(x), & \text{if } x \in k^{-1}([0,s)) \\ k(x), & \text{otherwise .} \end{cases}$$

For $x \in K_n$ we have

$$k(x) < (n+1)r_n k(x) < (n+1)f(x)k(x) < (n+1)r_n k(x) < \frac{n+1}{n-1}k(x).$$

Thus $h' \in C(X,\mathbb{Q})$ with $hh' = fk$ as needed. □

Before we can prove that $C_c(X) \hookrightarrow A(X)$ is a p-extension for zero-dimensional spaces X, we need the next two results. The Lemma is (i) from Chapter 4 of [12]. The proof of Proposition 4.12 is modeled after the proof of Theorem (j) in Chapter 4 of [12]

Lemma 4.11. [12] *Let A and B be disjoint closed subsets of the zero-dimensional Lindelöf space X. Then there exists a clopen set C such that $A \subseteq C$ and $B \cap C = \emptyset$.*

Proposition 4.12. *Let X be a zero-dimensional space. Let $f \in A(X)$ and let $r, s \in \mathbb{R}$ with $r < s$. There exists a clopen set K of X such that $f^{-1}((-\infty, r]) \subseteq C$ and $C \cap f^{-1}([s,\infty)) = \emptyset$.*

Proof. Let $Z_1 = f^{-1}((-\infty, r])$ and $Z_2 = f^{-1}([s,\infty)$. Since $f \in A(X)$, there exist clopen sets A_n and B_n for each $n \in \mathbb{N}$ such that $Z_1 = \bigcap_{n\in\mathbb{N}} A_n$ and $Z_2 = \bigcap_{n\in\mathbb{N}} B_n$. Let $L = \beta_0 X \smallsetminus \bigcap\{cl_{\beta_0 X} A_n \cap cl_{\beta_0 X} B_n : n \in \mathbb{N}\}$. Then $X \smallsetminus L = Z_1 \cap Z_2 = \emptyset$, hence $X \subseteq L \subseteq \beta_0 X$. Note that L is an F_σ-subset of the compact space $\beta_0 X$, thus L is Lindelöf. Observe that $L \cap [\bigcap\{cl_{\beta_0 X} A_n : n \in \mathbb{N}\}]$ and $L \cap [\bigcap\{cl_{\beta_0 X} B_n : n \in \mathbb{N}\}]$ are each the intersection of countably many clopen sets of L, so they are disjoint zerosets of L. By Lemma 4.11 there exists a clopen set C with $L \cap [\bigcap\{cl_{\beta_0 X} A_n : n \in \mathbb{N}\}] \subseteq C$ and $L \cap [\bigcap\{cl_{\beta_0 X} B_n : n \in \mathbb{N}\}] \cap C = \emptyset$. Let $K = C \cap X$, then K is a clopen subset of X, $Z_1 \subseteq K$, and $K \cap Z_2 = \emptyset$. □

Theorem 4.13. *For any zero-dimensional space X,*

1. *$C_c(X) \hookrightarrow A(X)$ is a p-extension.*
2. *$C(X,\mathbb{Q}) \hookrightarrow A(X)$ is a p-extension.*
3. *$C(X,\mathbb{Q}) \hookrightarrow A(X)$ is a pgs-extension.*
4. *$C_c(X) \hookrightarrow A(X)$ is a pgs-extension.*

Proof. First, to prove $C_c(X) \hookrightarrow A(X)$ is a p-extension, let $f \in A(X)$ be a nonunit. If $f(X)$ is finite, then clearly $fC_c(X) = \chi_{coz(f)} C_c(X)$ and we are done. So assume $f(X)$ is countably infinite. Without loss of generality, we may assume $f \in A(X)^+$. By Proposition 4.12, for each $n \in \mathbb{N}$ there exists a clopen set K_n with $f^{-1}([0, \frac{1}{n+1})) \subseteq K_n$ and $K_n \cap f^{-1}([\frac{1}{n}, \infty)) = \emptyset$. Observe that $K_{n+1} \subseteq K_n$ for each $n \in \mathbb{N}$. Let $C_n = K_n \smallsetminus K_{n+1}$, then $\{C_n\}$ is a collection of disjoint clopen subsets of X, and for any $x \in C_n$ we have $\frac{1}{n+2} \leq f(x) \leq \frac{1}{n}$. Define $g \in C(X,\mathbb{Q})$ as follows:

$$g(x) = \begin{cases} \frac{1}{n+2} & \text{if } x \in K_n \\ 0 & \text{otherwise.} \end{cases}$$

Define h_1 and h_2 as in Proposition 4.9, then $h_1, h_2 \in A(X)$ with $f = gh_1$ and $g = fh_2$.

Since $C(X,\mathbb{Q}) \hookrightarrow C_c(X)$ is a p-extension by Proposition 4.9, and $C_c(X) \hookrightarrow A(X)$ by part (1), transitivity gives us that $C(X,\mathbb{Q}) \hookrightarrow A(X)$ is a p-extension.

Next we will prove that $C(X,\mathbb{Q}) \hookrightarrow A(X)$ is a pgs-extension. It is easy to see that $C(X,\mathbb{Q}) \hookrightarrow A(X)$ is an essential extension. Since $A(X)$ is an essential extension of $C(X,Z)$, if $f \in C(X)$ with $fC(X)$ nonzero, then $fC(X) \cap C(X,\mathbb{Z})$ nonzero implies $fC(X) \cap C(X,\mathbb{Q})$ is nonzero. We will now show that $C(X,\mathbb{Q}) \hookrightarrow A(X)$ is a pg-extension. Let $f \in A(X)$ with $fA(X) \neq 0A(X)$. Then by Proposition 4.12, there exists a sequence of pairwise disjoint clopen sets $\{K_n\}$ such that $coz(f) = \bigcup_{n=1}^{\infty} K_n$ and $\frac{1}{n+2} < f(x) < \frac{1}{n}$ for each $x \in K_n, n \in \mathbb{N}$. Define $g \in C(X,\mathbb{Q})$ as above, then we claim that $g \in fA(X)$. To see this, define $h : X \to \mathbb{R}$ by

$$h(x) = \begin{cases} \frac{1}{nf(x)} & \text{if } x \in K_n \\ 1 & \text{otherwise.} \end{cases}$$

Since $1 < \frac{1}{nf(x)} < \frac{n+2}{n}$ for each $x \in K_n, n$
$in\mathbb{N}$, we have that $h \in A(X)$. Since $g = fh$, we have $gC(X,\mathbb{Q}) \subseteq fA(X) \cap C(X\mathbb{Q})$.

To show that $fA(X) \cap C(X\mathbb{Q}) \subseteq gC(X,\mathbb{Q})$, let $fj \in C(X,\mathbb{Q})$ for some $j \in A(X)$. There exists a sequence of pairwise disjoint clopen sets $\{C_n\}$ such that $coz(fj) = \bigcup_{n=1}^{\infty} C_n$ and $\frac{1}{n+2} < f(x)j(x) < \frac{1}{n}$ for each $x \in C_n, n \in \mathbb{N}$. Define $k : X \to \mathbb{Q}$ by

$$k(x) = \begin{cases} nf(x)j(x) & \text{if } x \in K_n \cap C_m \\ 1 & \text{otherwise.} \end{cases}$$

Then $k \in C(X,\mathbb{Q})$ with $fj = gk$ as needed.

Finally we will prove that $C_c(X) \hookrightarrow A(X)$ is a pgs-extension. Note that $C(X,\mathbb{Q}) \hookrightarrow A(X)$ is a pgs-extension by part (2) of this Proposition, and $C(X,\mathbb{Q}) \hookrightarrow C_c(X)$ is a p-extension by Proposition 4.10. Hence by Proposition 2.10 $C_c(X) \hookrightarrow A(X)$ is a pgs-extension. □

Theorem 4.14. *Let X be zero-dimensional. The following are equivalent:*

(1) *X is strongly zero-dimensional.*
(2) *Every cozeroset is a countable union of clopen subsets of X.*
(3) *$A(X) = C(X)$.*
(4) *$A(X) \hookrightarrow C(X)$ is a p-extension.*
(5) *$A(X) \hookrightarrow C(X)$ is a pgs-extension.*
(6) *$C_c(X) \hookrightarrow C(X)$ is a p-extension.*
(7) *$C_c(X) \hookrightarrow C(X)$ is a pgs-extension.*
(8) *$C(X,\mathbb{Q}) \hookrightarrow C(X)$ is a p-extension.*
(9) *$C(X,\mathbb{Q}) \hookrightarrow C(X)$ is a pgs-extension.*

Proof. The equivalence of (1) through (3) can be found in [12] and [7]. Clearly if $A(X) = C(X)$, then $A(X) \hookrightarrow C(X)$ is a p-extension and a pgs-extension, so (1) implies (4) and (5).

Assume $A(X) \hookrightarrow C(X)$ is a p-extension, and we will show that X is strongly zero-dimensional. Let $f \in C(X)$, then there exists $g \in A(X)$ such that $fA(X) = gA(X)$. But then $coz(f) = coz(g)$ is a countable union of clopen sets, hence X is strongly zero-dimensional. A similar argument proves if $C_c(X) \hookrightarrow C(X)$ or $C(X,\mathbb{Q}) \hookrightarrow C(X)$ is a p-extension, then X is strongly zero-dimensional.

Suppose $A(X) \hookrightarrow C(X)$ is a pgs-extension, and we will show that X is strongly zero-dimensional. Let $f \in C(X)$, then there exists $g \in A(X)$ with $fC(X) \cap A(X) =$

$gA(X)$. It follows that $Z(f) \subseteq Z(g)$. We will show that $Z(f) = Z(g)$. Suppose, by way of contradiction, that there exists $x \in Z(g) \smallsetminus Z(f)$. Since X is zero-dimensional, there is a clopen set K with the property that $x \in K \subseteq coz(f)$. Now define $f' \in C(X)$ by

$$f'(x) = \begin{cases} \frac{1}{f(x)}, & \text{if } x \in K \\ 0, & \text{otherwise .} \end{cases}$$

Then $\chi_K = ff' \in fC(X) \cap A(X) = gA(X)$ implies $Z(g) \subseteq Z(\chi_K) = X \smallsetminus K$. However $x \in Z(g) \cap K$, which is a contradiction. Therefore $Z(f) = Z(g)$ and so $coz(f)$ is a countable union of clopen sets, as needed. A similar argument also works assuming $C_c(X) \hookrightarrow C(X)$ or $C(X, \mathbb{Q}) \hookrightarrow C(X)$ is a pgs-extension.

Assume (3) holds. From Theorem 4.13 we know that $C_c(X) \hookrightarrow A(X)$ is a p-extension and a pgs-extension. We also know that $C(X, \mathbb{Q}) \hookrightarrow A(X)$ is a p-extension and a pgs-extension. Hence (6), (7), (8) and (9) all follow from (3).

□

Open Questions:

(1) Is there an example of commutative rings $R \hookrightarrow S \hookrightarrow T$ with S not a von Neumann regular ring, where Proposition 2.11 fails?
(2) In Theorem 3.9 can we replace "$S \setminus R$ an absorbing subset of S" with "R is a domain"?
(3) In Theorem 3.13 can we omit "$S \setminus R$ an absorbing subset of S"?
(4) Is there an example of $R \hookrightarrow S$ with R a domain, that is a p-extension but not a pgs-extension?
(5) Is there an example of $R \hookrightarrow S$ with R and S von Neumann regular rings, that is a pgs-extension but not a p-extension?
(6) Is there an example of a pgs-extension that is not a C^*-extension?
(7) Is there a space X with clopen π-base that is not a P-space?

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