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p-Extensions

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Abstract. We consider extensions of unital commutative rings. We define an extension $R \hookrightarrow S$ to be a p-extension if every principally generated ideal of S is generated by an element of R. Examples are plentiful and localizations of regular multiplicative sets are p-extensions. We develop the theory of p-extensions.

1. Introduction

The study of extensions of rings has a long history. Localizations, integral extensions, and the "Going Up" Theorem are a few of the type of extensions that have been studied. Recently the authors of [**1**] studied extensions which preserve (finitely generated) principal annihilators; (quasi) rigid extensions. In this article we study a type of extension which resembles that of a localization of a set of non-zero-divisors. These extensions, called p-extensions, are examples of rigid extensions. One of the natural programs in localization theory is to determine which properties pass through; we do the same in the context of our extensions.

We are exclusively interested in an extension of rings $\Phi : R \hookrightarrow S$. We assume that all rings are commutative and with identity. Moreover, the extensions are unital. In this way we may view R as a unital subring of S.

We end this section with a discussion of commutative rings, setting forth the notation and definitions used throughout the article. Let R be a ring. The set of units of R is denoted by $\mathfrak{U}(R)$. The annihilator of a subset $T \subseteq R$ is denoted by $\mathrm{Ann}_R(T)$. When $T = \{a_1, \ldots, a_n\}$ then we instead write $\mathrm{Ann}_R(a_1, \ldots, a_n)$. The lattice of all ideals of R is denoted by $\mathcal{L}(R)$. The collection of all prime ideals of R is denoted by $\mathrm{Spec}(R)$. The subspaces consisting of all maximal (minimal prime) ideals, will be denoted by $\mathrm{Max}(R)$ ($\mathrm{Min}(R)$). The nilradical of R is denoted by $\mathfrak{N}(R)$ while the Jacobson radical is denoted by $\mathfrak{J}(R)$. A ring is called reduced if it has no nonzero nilpotent elements, i.e. $\mathfrak{N}(R) = 0$. An element of R is called regular if it is not a zero-divisor. The principal ideal of R generated by r will be denoted by rR. An idempotent of R is an element $e \in R$ for which $e^2 = e$.

R is called a *von Neumann regular ring* if for every $a \in R$ there is an $x \in R$ such that $a^2x = a$. This is known to be equivalent to the condition that every principal ideal of R is generated by an idempotent of R. Alternatively, R is von Neumann regular if and only if R is reduced and every prime ideal is maximal.

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R is called a *Bézout ring* if every finitely generated ideal is principally generated. A ring R is called a *chained ring* if its set of ideals is totally ordered by inclusion. A chained domain is called a *valuation domain*. A chained ring is a Bézout ring, and every von Neumann regular ring is also Bézout ring.

A ring R is called a *Baer ring* if for every $\emptyset \neq T \subseteq R$, there is an idempotent $e \in R$ such that $\mathrm{Ann}_R(T) = eR$. R is called a *weak Baer ring* if for every $a \in R$ there is an idempotent $e \in R$ such that $\mathrm{Ann}_R(a) = eR$. Every von Neumann regular ring is a weak Baer ring.

Our general references for topics in ring theory are [**6**], [**10**], [**11**], and [**12**].

2. p-extensions

We now turn to our main new concepts.

Definition 2.1. 1) We say the extension of rings $R \hookrightarrow S$ is a *p-extension* if it satisfies the following property:

for every $s \in S$ there is an $r \in R$ such that $sS = rS$.

This is equivalent to saying for each $s \in S$ there is an $r \in R$ and $t_1, t_2 \in S$ such that $r = st_1$ and $s = rt_2$. A third way to view a p-extension is order-theoretic. Denote the partially-ordered set of principal ideals of R by $\mathfrak{P}(R)$. An extension of rings $R \hookrightarrow S$ induces an order-preserving mapping of $p : \mathfrak{P}(R) \to \mathfrak{P}(S)$ defined by $p(rR) = rS$. In general, this map is well-defined and not necessarily either injective or surjective. The map p is a surjection precisely when $R \hookrightarrow S$ is a p-extension.

Observe that the map p is the restriction to $\mathfrak{P}(R)$ of the well-known extension map $e : \mathcal{L}(R) \to \mathcal{L}(S)$ defined by $e(I) = IS$ for each $I \in \mathcal{L}(R)$. If $R \hookrightarrow S$ is a p-extension, then e is a surjective map, but not conversely. That e be a surjective map is usually phrased by saying that every ideal of S is extended. In [**8**], the author examined many situations where the two notions coincide. For a more detailed discussion see Remark 2.2. For an example of an extension of rings in which every ideal of S is extended but the extension is not a p-extension see Example 5.6. We have more about extended ideals in Section 3.

2) We say the extension $R \hookrightarrow S$ is an *associate p-extension* if it satisfies the following property:

for every $s \in S$ there is an $r \in R$ and a unit $u \in \mathfrak{U}(S)$ such that $r = su$.

Condition 2) is saying that every element of S is associate to an element of R. Clearly, an associate p-extension is a p-extension. The converse holds whenever S is an integral domain.

3) We next consider a stronger condition than 2):

for every $s \in S$ there are $r, u \in R$ such that $u \in \mathfrak{U}(S)$ and $r = su$.

Condition 3) should look familiar. We call such an extension a *regular localization of R* (or, as it is also known, a *quotient ring* of R). We denote the classical ring of fractions of R by $q(R)$; $q(R)$ is also known as the total ring of quotients of R. Recall that an overring of R is a subring of $q(R)$ containing R. Notice that if U is a multiplicative system of regular elements, then $R \hookrightarrow R_U$ is a regular localization.

Note that an element $u \in R$ which is a unit of R_U is a regular element of R. Thus, if S is a regular localization of R, then every element $s \in S$ can be written in the form $\frac{r}{u}$ for some $r \in R$ and u a regular element of R, and so a regular localization of R is an overring. To say that there is a vast understanding on the

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subject of regular localizations (see [**7**], [**17**], [**20**], [**9**]) would be an understatement. Some historical remarks are in order.

REMARK 2.2. In [**14**] the author studied strongly homogeneous torsion-free abelian groups. The author had cause to study specific ring extensions of $\mathbb{Z}$ derived from such groups, and showed that such an extension ring R has the property that every element of R is an integral multiple of an element which is invertible in R. They called such a ring R a *strongly homogeneous ring*.

In [**19**] the author was interested in subrings of a finite dimensional division algebra over $\mathbb{Q}$ with the property that every element in said ring is rational multiple of an invertible element of said ring. The author, also, called these rings *strongly homogeneous*.

In [**8**] the author generalized the above notions to the context of an extension of domains; calling such an extension strongly homogeneous. What we have called a p-extension is the generalization of Goeters' strongly homogeneous extension to the context of arbitrary commutative rings. Goeters proved some fundamental properties of strongly homogeneous extensions. For example, given a p-extension of rings, say $R \hookrightarrow S$, every ideal of S is extended from R. In general, the converse is not true. Several results in the article aimed at describing situations when the two notions coincide as well as studying what the author called locally strongly homogeneous extensions.

(We would like to thank B. Olberding for pointing us in the direction of the above articles.)

Finally, in [**9**], the authors studied p-extensions of domains calling them *well-centered*. One of the things they were interested in was the interplay of flat overrings, generalized quotient rings, and well-centered overrings.

Given any overring $R \hookrightarrow S$ we set $U_S = \mathfrak{U}(S) \cap R$. As noted above U_S is a multiplicative system of regular elements of R and therefore we can construct R_{U_S}. It is straightforward to check that $R \hookrightarrow R_{U_S} \hookrightarrow S$. The map $R \hookrightarrow R_{U_S}$ is always a regular localization.

LEMMA 2.3. *Suppose S is an overring of R and consider $R \hookrightarrow R_{U_S} \hookrightarrow S$. The following statements are equivalent.*

(1) *The extension $R \hookrightarrow S$ is a regular localization.*
(2) *The extension $R_{U_S} \hookrightarrow S$ is a regular localization.*
(3) $S = R_{U_S}$.

PROOF. The proof of this is known but we include it for completeness sake.

(1) $\Rightarrow$ (2). This holds in general: if $s \in S$, then there are some $u, r \in R$ such that $u \in \mathfrak{U}(S)$ and $us = r$. Then $r, u \in R_{U_S}$.

(2) $\Rightarrow$ (3). Let $s = \frac{x}{y} \in S$ with $y \in R$ regular. By (2) there is some $\alpha = \frac{a}{b} \in R_{U_S}$ and $\mu = \frac{u}{v} \in U_{R_{U_S}}$ such that $\mu s = \alpha$. Observe that without loss of generality we can assume that $b, v \in U_S$. Notice that in S a simple multiplication yields $\frac{1}{u} = v^{-1}\mu^{-1} \in S$ and so $u \in U_S$. It follows that $\mu^{-1} \in R_{U_S}$ and therefore $s = \mu^{-1}\alpha \in R_{U_S}$. Consequently, $R_{U_S} = S$.

(3) $\Rightarrow$ (1). Patent.

□

Similar results follow for p-extensions and associate p-extensions.

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LEMMA 2.4. *Suppose S is an overring of R and consider $R \hookrightarrow R_{U_S} \hookrightarrow S$.*

(1) *$R \hookrightarrow S$ is a p-extension if and only if $R_{U_S} \hookrightarrow S$ is a p-extension.*
(2) *$R \hookrightarrow S$ is an associate p-extension if and only if $R_{U_S} \hookrightarrow S$ is an associate p-extension.*

PROOF. Let $R \hookrightarrow R_{U_S} \hookrightarrow S \hookrightarrow q(R)$.

(1) If $s \in S$, then there exists $r \in R$ such that $rS = sS$; so $r \in R_{U_S}$. Conversely, let $s \in S$, then there exists $\frac{x}{y} \in R_{U_S}$ such that $\frac{x}{y}S = sS$. So, there exist $t_1, t_2 \in S$ such that $\frac{x}{y} = st_1$ and $s = \frac{x}{y}t_2$. Therefore, $x = s(yt_1)$ and $s = x(t_2y^{-1})$, hence $xS = sS$ where $x \in R$.
(2) Clearly if $R \hookrightarrow S$ is an associate p-extension, then $R_{U_S} \hookrightarrow S$ is an associate p-extension. Suppose $R_{U_S} \hookrightarrow S$ is an associate p-extension. If $s \in S$, then there exists $\frac{x}{y} \in R_{U_S}$ and $u \in \mathfrak{U}(S)$ such that $su = \frac{x}{y}$. Therefore, $s(uy) = x \in R$. Notice that $y \in U_S$ means that $y \in \mathfrak{U}(S)$ and $u \in \mathfrak{U}(S)$, which implies $yu \in \mathfrak{U}(S)$.

□

The notions of localizations and of (associate) p-extensions are indeed transitive properties. The first is well-known. We prove the second.

LEMMA 2.5. *Suppose S is a regular localization of R while T is a regular localization of S. Then T is a regular localization of R.*

LEMMA 2.6. *Let R, S, and T be rings. If $R \hookrightarrow S$ and $S \hookrightarrow T$ are (associate) p-extensions, then $R \hookrightarrow T$ is also an (associate) p-extension.*

PROOF. We rove that the concept of associate p-extension is transitive. For the first case this follows from the fact that p-extension is equivalent to the map p defined in Definition 2.1 is surjective. A composition of surjective maps is surjective.

Now suppose $R \hookrightarrow S$ and $S \hookrightarrow T$ are associate p-extensions. Since $S \hookrightarrow T$ is an associate p-extension, there exist $s \in S$ and $u_1 \in \mathfrak{U}(T)$ such that $s = tu_1$. In addition, because $R \hookrightarrow S$ is an associate p-extension, there exist $r \in R$ and $u_2 \in \mathfrak{U}(S)$ such that $r = su_2$. Thus $r = su_2 = t(u_1u_2)$ where $u_1u_2 \in \mathfrak{U}(T)$, and we conclude that $R \hookrightarrow T$ is an associate p-extension □

DEFINITION 2.7. Recall that a ring R is said to satisfy the *QR-property* if every overring is a quotient ring (i.e. a regular localization). These rings have played an important role in the theory of Prüfer domains. In particular, a QR-domain is Prüfer (Theorem 1.1.1 [**4**]) and can be characterized within the class of Prüfer domains as follows.

THEOREM 2.8 ([**6**], Theorem 27.5). *Suppose R is a Prüfer domain. R satisfies the QR-property if and only if for each finitely generated ideal I of R, there is an $a \in R$ and $n \in \mathbb{N}$ such that $I^n \leq aR \leq I$.*

COROLLARY 2.9. A Bézout domain satisfies the QR-property.

A natural question is whether there is a nice characterization of those rings for which every overring is a p-extension; we call such rings *pR-rings*. Observe that for domains the QR-property is strong enough to force the domain to be integrally closed. This is not the case for pR-domains. In [**18**], the author produces an example of a non-integrally closed pR-domain. The example, not surprisingly, has

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few overrings. However, if D is integrally closed to start with, then we gather a new characterization of the QR-property.

THEOREM 2.10. *Suppose D is integrally closed in $q(D)$. D is a pR-domain if and only if D is a QR-domain. Furthermore, in this case D is a Prüfer domain*

PROOF. Clearly if D satisfies the QR-property, then every overring is a p-extension since every quotient ring is a p-extension. Conversely, suppose $D \hookrightarrow S$ is an overring of S. It is shown in [**6**] that if for each $s \in S, D \hookrightarrow D[s]$ is a quotient ring, then so is $D \hookrightarrow S$. Let $s \in S$ and by hypothesis, $D \hookrightarrow D[s]$ is a p-extension. We want to show it is a quotient ring. Notice that since D is integrally closed in $q(D)$, Corollary 3.8 of [**9**] yields that $D \hookrightarrow D[s]$ is a regular localization, and hence a p-extension. □

3. Fundamental Properties of p-extensions

A natural program is to determine which ring-theoretic properties transfer under passage of the above defined extensions. This program is standard for localizations. Our next definition will be useful.

DEFINITION 3.1. Like this manuscript, the article [**1**] also studied certain types of extensions. We recall the notion of a rigid and quasi rigid extension. The extension $R \hookrightarrow S$ is called a *rigid extension* if given $s \in S$ there exists an $r \in R$ such that $Ann_S(s) = Ann_S(r)$.

More generally, if for all $s \in S$ there exists a $r_1, \ldots, r_n \in R$ such that $Ann_S(s) = Ann_S(r_1, \ldots, r_n)$, then this extension is called a *quasi rigid extension*. We mirror this label by defining a *quasi p-extension* as an extension $R \hookrightarrow S$ with the property that for every $s \in S$ there are $r_1, \ldots, r_n \in R$ for which $sS = r_1S + \ldots + r_nS$. (Below, we relate quasi p-extensions to having extended ideals.)

It follows that every p-extension is a rigid extension, while every quasi p-extension is a quasi rigid extension.

For any ring R, the embedding of R into its polynomial ring, $R \hookrightarrow R[x]$ is never a quasi p-extension because there do not exist any $r_1, \ldots, r_n \in R$ such that $r_1R[x]+\ldots+r_nR[x] = xR[x]$. But according to Proposition 2.16 of [**1**], the extension of a reduced ring into its polynomial ring is always a quasi rigid extension.

Recall that R is said to satisfy the *annihilator condition* (or a.c.) if for every $a, b \in R$ there is a $c \in R$ such that $\mathrm{Ann}_R(a, b) = \mathrm{Ann}_R(c)$. Proposition 2.18 of [**1**] states that for a reduced ring R, the extension $R \hookrightarrow R[x]$ is a rigid extension precisely when R satisfies the a.c. Thus, if R is a reduced ring satisfying the a.c. then $R \hookrightarrow R[x]$ is a rigid extension which is not a quasi p-extension. We observe that Bézout rings and weak Baer rings satisfy the a.c.

LEMMA 3.2. *Given an extension $R \hookrightarrow S$ the following statements are equivalent.*

(1) *The extension is a quasi p-extension.*
(2) *The restriction of the extension map $e : \mathcal{L}(R) \to \mathcal{L}(S)$ to the subsets of finitely generated ideals is a surjective map.*
(3) *The ideals of S are extended.*

PROOF. First assume (1), that $R \hookrightarrow S$ is a quasi p-extension and let $s_1, ..., s_n \in S$. There is a collection $\{r_{i,j}\} \subseteq R$ such that $s_iS = r_{i,1}S + \ldots + r_{i,j_i}S$, for each

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$i = 1, \ldots, n$. Then

$$s_1S + \ldots + s_nS = \sum r_{i,j}S = (\sum r_{i,j}R)S.$$

Thus, every finitely generated ideal of S is an extension of a finitely generated ideal of R.

Next, suppose (2). Let J be an ideal of S. For each $x \in J$ there is a finite generated ideal of R, say I_x, such that $xS = I_xS$. Denote the ideal generated by the union of I_x for $x \in J$ by I. Then it is straightforward to check that $J = IS$, whence ideals of S are extended.

Finally, suppose that ideals of S are extended and let $s \in S$. By hypothesis there is an ideal I of R such that $IS = sS$. It follows that there is a finite collection $r_1, \cdots, r_n \in I$ and $s_1, \cdots, s_n \in S$ such that

$$r_1s_1 + \cdots + r_ns_n = s.$$

It is patent to check that $sS = r_1S + \cdots + r_nS$, whence the extension is a quasi p-extension.

□

Remark 3.3. It can be derived from section 2 of [**1**] that if $R \hookrightarrow S$ is a quasi rigid extension of reduced rings, then the inverse map $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ restricts to a homeomorphism $\mathrm{Min}(S) \to \mathrm{Min}(R)$. Therefore, this holds for quasi p-extensions. We can and do say more with regards to prime ideals.

Lemma 3.4. *Suppose $R \hookrightarrow S$ is a quasi p-extension and consider the contraction map from* $\mathrm{Spec}(S)$ *into* $\mathrm{Spec}(R)$*; that is, for $P \in \mathrm{Spec}(S)$ set $P' = P \cap R$. First, the contraction map is injective. Second, $P'S = P$ and so every prime ideal of S is the extension of some prime ideal of R.*

Proof. Suppose $P_1, P_2 \in \mathrm{Spec}(S)$ are distinct prime ideals of S. Without loss of generality there is an $s \in P_1 \smallsetminus P_2$. Choose $r_1, \ldots, r_n \in R$ such that $sS = r_1S + \ldots + r_nS$. Notice that each $r_1, \ldots, r_n \in P_1 \cap R = P_1'$. However, there must be some r_i satisfying $r_i \notin P_2$, and hence $r_i \notin P_2'$. So $r_i \in P_1' \smallsetminus P_2'$, whence contraction is injective.

Next, we prove that the ideal of S generated by $P' = P \cap R$ is P. Clearly, $P'S \subseteq P$. Conversely, let $s \in P$ and choose $r_1, \ldots, r_n \in R$ such that $sS = r_1S + \ldots + r_nS$. Then each $r_i \in P \cap R = P'$. Thus, it follows that $s \in P'S$. □

Theorem 3.5. *Suppose $R \hookrightarrow S$ is a quasi p-extension. The contraction mapping $\Psi : \mathrm{Spec}(S) \to \mathrm{Spec}(R)$ defined by $\Psi(P) = P \cap R$ is a homeomorphism between* $\mathrm{Spec}(S)$ *and its image (i.e. the set of contracted prime ideals of R with respect to S).*

Proof. Since we just proved that contraction mapping is injective and it clearly surjective onto its image we need to show that it is a homeomorphism between the topological spaces. Recall that for an arbitrary ring T a basic open subset of $\mathrm{Spec}(T)$ is of the form

$$U_T(t) = \{P \in \mathrm{Spec}(T) : t \notin P\}$$

for some $t \in T$. In general, contraction is continuous so we need to show that Ψ is open.

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Let $s \in S$ and choose $r_1, \ldots, r_n \in R$ for which $r_1S + \ldots r_nS = sS$.

$$\begin{aligned}\Psi(U_S(s)) &= \{P \cap R : P \in U_s(s)\} \\ &= \{P \cap R : s \notin P\} \\ &= \{P \cap R : r_i \notin P \text{ for some } i = 1, \ldots, n\} \\ &= (U_R(r_1) \cup \ldots \cup U_R(r_n)) \cap X\end{aligned}$$

which happens to be an open subset of X. it follows that Ψ is bi-continuous and hence a homeomorphism onto its image. □

As we have pointed out already every regular localization is a p-extension. The notion of a flat epimorphic extension is a generalization of that of a regular localization. In Proposition 4.8 of [**1**] it is shown that a flat epimorphic extension is a quasi rigid extension. In fact, a flat epimorphism is a special kind of a quasi p-extension as we now demonstrate.

Recall that an extension $R \hookrightarrow S$ is a *flat epimorphism* if for each $s \in S$ there are $r_1, \ldots, r_n \in R$ such that $r_is \in R$, for each $i = 1, \ldots, n$, and $r_1S + \ldots + r_nS = S$. This is equivalent to having S be a flat R-module.

Before we argue that every flat epimorphism is a quasi p-extension we note that $\mathbb{Z} \to \mathbb{R}$ is a p-extension which is not a flat epimorphism.

Proposition 3.6. Every flat epimorphism is a quasi p-extension.

Proof. Let $R \hookrightarrow S$ be a flat epimorphism. Let $s \in S$ and choose $r_1, \ldots, r_n \in R$ such that $r_is \in R$, for each $i = 1, \ldots, n$, and $r_1S + \ldots + r_nS = S$.

By the last part there are $s_1, \ldots, s_n \in S$ such that $r_1s_1 + \ldots + r_ns_n = 1$. Then

$$s = sr_1s_1 + \ldots + sr_ns_n$$

and so $s \in (r_1s)S + \ldots + (r_ns)S$. Clearly, each $r_is \in sS$ so that $sS = (r_1s)S + \ldots + (r_ns)S$. It follows that a flat epimorphism is a quasi p-extension. □

4. Passage via p-extensions

We now turn to considering the passage of ideal-theoretic properties under different types of extensions. We first consider passage via (quasi) rigid extensions as such properties will also pass under (quasi) p-extensions.

Proposition 4.1. Suppose $R \hookrightarrow S$ is a quasi rigid extension.

(1) If R satisfies the a.c., then the extension is a rigid extension. Moreover, S also satisfies the a.c.
(2) R is an integral domain if and only if S is an integral domain. Moreover, an integral domain is a quasi rigid extension of any of its subrings.
(3) R is reduced if and only if S is reduced.
(4) If R is a weak Baer ring, then S is a weak Baer ring. Moreover, R and S share the same idempotents.

Proof. (1) The second statement follows from the first and Proposition 2.4 of [**1**]. The first statement is Lemma 2.15 of [**1**].

(2) The sufficiency is clear. So, suppose R is an integral domain and suppose $s_1, s_2 \in S$ for which $s_1s_2 = 0$. There exist $r_1, \ldots r_n, t_1, \ldots, t_m \in R$ with $\mathrm{Ann}_S(s_1) = \mathrm{Ann}_S(r_1, r_2, ...r_n)$ and $\mathrm{Ann}_S(s_2) = \mathrm{Ann}_S(t_1, t_2, ...t_m)$. Notice then $r_it_j = 0$ for all $1 \leq i \leq n$, $1 \leq j \leq m$. Since R is an integral domain,

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if any of the $r_i \neq 0$, then $t_1 = \ldots = t_m = 0$ and so $s_2 = 0$. The other case is similar.

(3) Any subring of a reduced ring is reduced. So suppose that R is reduced. Let $s \in S$ for which $s^2 = 0$. By assumption there are $r_1, \ldots, r_n \in R$ such that $\mathrm{Ann}_S(s) = \mathrm{Ann}_S(r_1, \cdots, r_n)$. It follows from $s \cdot s = s^2 = 0$ that $sr_i = 0$. But then $r_i^2 = 0$ for each i and so $s = 0$. Therefore, S is reduced.

(4) Let $s \in S$ be nonzero and choose $r \in R$ for which $\mathrm{Ann}_S(s) = \mathrm{Ann}_S(r)$. Next, choose $e^2 = e \in R$ such that $\mathrm{Ann}_R(r) = \mathrm{Ann}_R(e)$. The argument used in Lemma 2.15 of [**1**] can be used here to show that $\mathrm{Ann}_S(s) = \mathrm{Ann}_S(e)$.

The second statement resembles condition (5) of Theorem 4.3 of [**1**]. We notice it holds in general. Let $e^2 = e \in S$. As we just showed, there is an idempotent $e' \in R$ such that $(1-e)S = \mathrm{Ann}_S(e) = \mathrm{Ann}_S(e') = (1-e')S$. It is straightforward to check that since both $1-e, 1-e'$ are idempotent and generate the same principal ideal, then $1-e = 1-e'$ and hence $e = e' \in R$.

□

Proposition 4.2. Suppose $R \hookrightarrow S$ is a quasi p-extension.

(1) If S is a field, then it is an associate p-extension of every subring.
(2) If R is a field, then so is S.
(3) If R is Bézout ring, then the extension is a p-extension and S is also a Bézout ring.
(4) If R is a von Neumann regular ring, then so is S.
(5) If R is a noetherian ring, then so is S.
(6) If R is principal ideal ring, then so is S.
(7) If R is a chained ring, then so is S.

Proof. (1) Let $s \in S$ be nonzero. Notice that if S is a field, then $1 \in R$ and $s^{-1} \in \mathfrak{U}(S)$ where $1 = ss^{-1}$.

(2) Let $s \in S$ be nonzero. By hypothesis, S is a quasi p-extension of R and so there exists nonzero $r_1, \ldots, r_n \in R$ with $sS = r_1S + \ldots + r_nS$. At least one of the r_i is nonzero and so by hypothesis a unit of R, and hence of S. Therefore, $sS = S$ whence s is a unit of S.

(3) By Lemma 3.2 if I is a finitely generated ideal of S, then there is a finitely generated ideal of R, say J, which extends to I. Since R is a Bézout ring, J is a principal ideal and thus I is a principal ideal as it is an extension of a principal ideal. Consequently, the extension is a p-extension and S is a Bézout ring.

(4) This is straightforward. Since a von Neumann regular ring is Bézout it follows that the extension is a p-extension and thus for each $s \in S$ there is an $r \in R$ such that $sS = rS$. Furthermore, there is an idempotent $e^2 = e \in R$ such that $eR = rR$ which then transfers to S and so $sS = rS = eS$. It follows that every principal ideal of S is generated by an idempotent.

(5) Let I be an ideal of S. To each $s \in S$ there is a finite subset, $F_s \subseteq R$ such that $sS = F_sS$. Let J be the ideal of R generated by the set $T = \bigcup_{s \in I} F_s$. Notice that I is the ideal of S generated by T. Since R is noetherian, there exists a finite set $r_1, \ldots, r_n \in R$ such that $J = r_1R + \cdots + r_nR$. It

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follows that the set $r_1, \ldots, r_n$ generates the ideal generated by T in S and therefore I is a finitely generated ideal of S.

(6) This follows from the fact that a principal ideal ring is precisely a noetherian Bézout ring together with an application of the two previous items.

(7) Let I and J be two ideals of S. We want to show that either $I \subseteq J$ or $J \subseteq I$. Assume, by means of contradiction, that there exist $s \in I \smallsetminus J$ and $t \in J \smallsetminus I$. Since $R \hookrightarrow S$ is a p-extension, there exist $r_1, r_2 \in R$ such that $r_1 S = sS \subseteq I$ and $r_2 S = tS \subseteq J$. However, R is a chained ring, so either $r_1 S \subseteq r_2 S$ or $r_2 S \subseteq r_1 S$. This implies either $s \in J$ or $t \in I$, which is a contradiction. Hence S is a chained ring.

□

Remark 4.3. 1) By (1) of Proposition 4.2, $\mathbb{Z} \hookrightarrow \mathbb{R}$ is an associate p-extension. Moreover, it is not an overring.

2) Proposition 3.1 of [**9**] proves that if A is noetherian domain and B is any overring which is a p-extension, then B is noetherian. Proposition 4.2 (5) generalizes this to rings with zero-divisors and any p-extension.

3) In [**18**] the author considers Prüfer-like conditions and their passage with regards to p-extensions.

4) We point out that if R is a Bézout domain, then $R \hookrightarrow R[x]$ is a rigid extension. However, $R[x]$ is not a Bézout domain unless R is a field. Therefore, (3) of Proposition 4.2 cannot be generalized to rigid extensions. The same type of example works to show that neither (4), (6) nor (7) of 4.2 can be generalized to rigid extensions.

5) If $R \hookrightarrow T$ and $S \hookrightarrow T$ are p-extensions, then it does not necessarily imply that $R \hookrightarrow S$ is a p-extension. For example, if R is a domain, then $R \hookrightarrow R(x)$ and $R[x] \hookrightarrow R(x)$ are both p-extensions since $R(x)$ is a field; whereas, $R \hookrightarrow R[x]$ is not a p-extension.

We end this section by showing that a quasi p-extension of a Euclidean domain is again a Euclidean domain. We begin with the definition of a Euclidean domain. There are several different (yet equivalent) definitions of Euclidean domains. We take ours from the book [**3**]. This seems to be the least restrictive definition.

Definition 4.4. Recall that a domain R is called a *Euclidean domain* if there is a function $N : R \to \mathbb{N}$ such that $N(0) = 0$ and for all $a, b \in R$ $(b \neq 0)$ there are $q, r \in R$ such that $a = bq + r$ and either $r = 0$ or $N(r) < N(b)$. The function N is called a norm on R.

Lemma 4.5. *Let R be a Euclidean domain with norm N. Then there is a norm on R, say $\bar{N}$, such that whenever $a, b \in R$ are associate, then $\bar{N}(a) = \bar{N}(b)$. In particular, for any unit $u \in \mathfrak{U}(R)$, $\bar{N}(u) = \bar{N}(1)$.*

Proof. Observe that being associate is an equivalence relation. We use $[r]$ to denote the set of associates of r. Define

$$\bar{N}(r) = \min\{N(x) : x \in [r]\}.$$

This defines a function $\bar{N} : R \to \mathbb{N}$ such that $\bar{N}(0) = 0$. Now, for each $r \in R$ choose $\bar{r} \in [r]$ such that $\bar{N}(r) = \bar{N}(\bar{r})$.

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We show that $\bar{N}$ is a norm on R making it into a Euclidean domain. Let $a, b \in R$ with $b \neq 0$. If b is a unit, then set $q = b^{-1}a$ and $r = 0$ and observe that $a = bq + r$ with $r = 0$ or $\bar{N}(r) < \bar{N}(b)$.

Next consider the case that b is not a unit. By hypothesis there are $q, r \in R$ such that $a = \bar{b}q + r$ and either $r = 0$ or $N(r) < N(\bar{b})$. Choose a unit $v \in \mathfrak{U}(R)$ such that $\bar{b} = vb$.

Then $a = \bar{b}q + r = b(qv) + r$. Notice that if $r = 0$ then $ru = 0$. On the other hand

$$\bar{N}(r) \leq N(r) < N(\bar{b}) = \bar{N}(\bar{b}) = \bar{N}(b).$$

Therefore, $\bar{N}$ is a norm on R.

□

THEOREM 4.6. *Suppose $R \hookrightarrow S$ is a quasi p-extension and R is a Euclidean domain. Then so is S.*

PROOF. First of all notice that S is a domain by Proposition 4.1 (2). And so quasi p-extension is equivalent to associate p-extension. Next, by Lemma 4.5 we can assume that N is a norm on R which agrees on associates.

Define $M : S \to \mathbb{N}$ as follows. To each $s \in S$ define

$$T(s) = \{r \in R : rS = sS\};$$

notice that $T(s) = [s] \cap R$. By hypothesis $T(s) \neq \emptyset$. Moreover there is an element of $T(s)$ of minimum norm, say x_s. Set $M(s) = N(x_s)$. Observe that if $r \in R$, then $M(r) \leq N(r)$.

Clearly $M(0) = 0$. Next, let $s, t \in S$. If $t \in \mathfrak{U}(S)$ then setting $q = t^{-1}s$ and $r = 0$ yields $s = tq + r$ where either $r = 0$ or $M(r) < M(t)$.

So we assume that t is not a unit. Choose $u, v \in \mathfrak{U}(S)$ such that $s = ux_s$ and $tv = x_t$. By hypothesis there are $q, r \in R$ such that $x_s = qx_t + r$ with either $r = 0$ or $N(r) < N(x_t)$. Then

$$s = ux_s = u(qx_t + r) = uqx_t + ur = t(uvq) + ur.$$

If $r = 0$ then also $ur = 0$. Otherwise, $M(ur) = N(x_r) \leq N(r) < N(x_t) = M(t)$. Consequently, S is a Euclidean domain. □

It is well-known that a localization of a UFD is again a UFD (for example see Exercise III.4.6. of [**11**]). At this point we have been unable to prove that the same holds for p-extensions.

PROPOSITION 4.7. Suppose $R \hookrightarrow S$ is an associate p-extension. If R is a Hermite ring, then so is R.

PROOF. Let $s_1, s_2 \in S$ and choose $r_1, r_2 \in R$ such that r_i is associate to s_i in S (for $i = 1, 2$). By hypothesis there are $d, u, v \in R$ such that $du = r_1, dv = r_2$, and $uR + vR = R$. In particular, $dR = r_1R + r_2R$. It follows that $s_1S + s_2S = r_1S + r_2S = dS$.

Next choose units $p, q \in S$ such that $s_1 = pr_1$ and $s_2 = qr_2$. Then $s_1 = pr_1 = d(pu)$ and $s_2 = qr_2 = d(qv)$ Now, since u, v are co-maximal in R they are co-maximal in S. Also since p, q are units it follows that pu and qv are also co-maximal in S. Thus S is a Hermite ring. □

We end this section with a few examples of some ring-theoretic properties which do not pass under p-extensions.

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EXAMPLE 4.8. Recall that a ring is called clean if every element is the sum of a unit and an idempotent. Every von Neumann regular ring is clean but not conversely. For more information on clean rings see [**16**].

In [**2**] and [**13**] the authors produce an example of a reduced clean ring whose classical ring of quotients is not clean. It follows that cleanliness does not pass through p-extensions.

EXAMPLE 4.9. This is a trivial example but it does illustrate that the boolean condition does not pass through p-extensions even though it does pass through regular localizations. That it passes through regular localizations is trivial since the only non-zero divisor is 1. However, consider the embedding of the field of two elements into the field of four elements: $\mathbb{F}_2 \hookrightarrow \mathbb{F}_4$. The former is a boolean ring however the latter is not.

5. Essential Extensions

PROPOSITION 5.1. Let $R \hookrightarrow S$ and suppose S is a von Neumann regular ring. The following are equivalent.

(1) S is an associate p-extension of R.
(2) S is a p-extension of R.
(3) S is a rigid extension of R.

PROOF. That (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) are clear.

(3) $\Rightarrow$ (2) Let $s \in S$, there exists some idempotent $e \in S$ such that $se = s$ and $sS = eS$. Using the hypothesis, there exists $r \in R$ such that $Ann_S(r) = Ann_S(e)$. So $r - re = (1-e)r = 0$, implies that $rS \subseteq eS = sS$. Next, there exists $a \in S$ such that $r^2a = r$. Therefore, $1 - ra \in Ann_S(r) = Ann_S(e)$. It follows that $s - r(sa) = s(1-ra) = se(1-ra) = 0$, leaving us with the reverse inclusion $sS \subseteq rS$.

(2) $\Rightarrow$ (1) Let $s \in S$, there exists $r \in R$ and $e \in S$ with $e^2 = e$ such that $sS = rS = eS$ and $se = s$, $re = r$. Also since S is a regular ring, we have $U(s) = U(r) = U(e)$ with respect to the hull-kernel topology on $Max(S) = Spec(S)$. Moreover, there exists $x, y \in S$ such that $r = sx = sxe$ and $s = ry = rye$. Notice that if $M \in Max(S)$ such that $xe \notin M$, then $e \notin M$ and so $s \notin M$; consequently, $U(xe) \subseteq U(s)$. For the other inclusion, if $M \in Max(S)$ and $xe \in M$, then $r \in M$ and so $s \in M$. We thus have, $U(xe) = U(s) = U(r) = U(e)$. Finally, we consider the element $xe + 1 - e \in S$. We first observe that $s(xe + 1 - e) = sxe + s - se = r$. If possible, suppose $xe + 1 - e \notin \mathfrak{U}(S)$. There exists some $M \in Max(S)$ such that $xe + 1 - e \in M$. Since $xe(1-e) = 0$ it follows that either $xe \in M$ or $1 - e \in M$. In either case both $xe, 1-e \in M$, which implies that $e, 1-e \in M$; a contradiction. Consequently, $xe + 1 - e \in \mathfrak{U}(S)$. $\square$

PROPOSITION 5.2. Let $R \hookrightarrow S$ and suppose S is a von Neumann regular ring. Then S is a quasi rigid extension of R if and only if S is a quasi p-extension. Moreover, if R satisfies the a.c., then S is a quasi rigid extension if and only if it is a rigid extension.

PROOF. Suppose S is a quasi rigid extension of R and let $s \in S$. Choose $r_1, ..., r_n \in R$ such that $\mathrm{Ann}_S(r_1, ..., r_n) = \mathrm{Ann}_S(s)$. Recall that in a von Neumann regular ring the action of taking a double annihilator of a finitely generated ideal

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is the identity. This produces the following equality:

$$r_1S + ... + r_nS = \mathrm{Ann}_S\,\mathrm{Ann}_S(r_1, ..., r_n) = \mathrm{Ann}_S\,\mathrm{Ann}_S(s) = sS.$$

Therefore, the extension if a quasi p-extension.

As for the last statement if R satisfies the a.c., then a quasi rigid extension is equivalent to a rigid extension (see Lemma 2.15 of [**1**]). □

Remark 5.3. Another interesting topic discussed in [**1**] is the embedding of a ring R into its maximal ring of quotients; denoted by $Q(R)$. If R is reduced, then so is $Q(R)$, and moreover, $Q(R)$ is a von Neumann regular ring.

For a reduced ring R, $Q(R)$ can also be described as the largest essential extension of R. Recall that an extension of R-modules, say $M \leq N$, is said to be *essential* if for any R-submodule $N' \leq N$, the equation $N' \cap M = 0$ implies $N' = 0$. Every R-module has a maximal essential extension. In the case of R its maximal essential extension is $Q(R)$.

The epimorphic hull of the reduced ring R, denoted $E(R)$, is the intersection of the set of von Neumann regular rings lying intermediate between R and $Q(R)$. It is known that $E(R)$ is a von Neumann regular ring. $E(R)$ can also be described as the subring of $Q(R)$ generated by R and the quasi-inverses of elements of R.

The following are corollaries to Theorem 4.3 and Theorem 4.6 of [**1**].

Corollary 5.4. For a reduced ring R, the following statements are equivalent.

(1) $Q(R)$ is an associate p-extension of R.
(2) $Q(R)$ is a p-extension of R.
(3) $Q(R)$ is a rigid extension of R.
(4) $\mathrm{Min}(R)$ is a compact extremally disconnected space and R satisfies the a.c.
(5) $q(R)$ is a Baer ring.
(6) $q(R)$ and $Q(R)$ have the same idempotents.

Corollary 5.5. For a reduced ring R, the following statements are equivalent.

(1) $E(R)$ is a regular localization of R.
(2) $E(R)$ is an associate p-extension of R.
(3) $E(R)$ is a p-extension of R.
(4) $E(R)$ is a rigid extension of R.
(5) $Min(R)$ is compact and R satisfies the a.c.
(6) $q(R)$ is von Neumann regular.
(7) $q(R) = E(R)$.
(8) $q(R)$ is a weak Baer ring.

Example 5.6. Suppose R is a reduced ring for which $\mathrm{Min}(R)$ is compact and R does not satisfy the a.c., e.g. Quentel's Example (see Example 6 of Chapter 27 [**10**]). Then since $\mathrm{Min}(R)$ is compact it is known that $E(R)$ is a flat epimorphism (see Theorem 4.11 [**1**]) and hence $R \hookrightarrow E(R)$ is a quasi p-extension. Furthermore, by the previous corollary it is not a p-extension. This supplies us with an example of an extension $R \hookrightarrow S$ in which all ideals of S are extended (i.e. quasi p) but not a p-extension.

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Unresolved Questions.

(1) Is there a nice internal classification for when $q(R) = Q(R)$, i.e. $Q(R)$ is a regular localization of R.
(2) Is it the case that if S is a p-extension of R and R is a UFD, then so is S? What about GCD-rings?
(3) If $R \hookrightarrow S$ is a (quasi) p-extension is it the case that the extension of a prime ideal of R is a prime ideal of S?
(4) Suppose $R \hookrightarrow S$ is a p-extension of rings. Is it the case that if R is a Hermite ring (or elementary divisor ring) then so is S?

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