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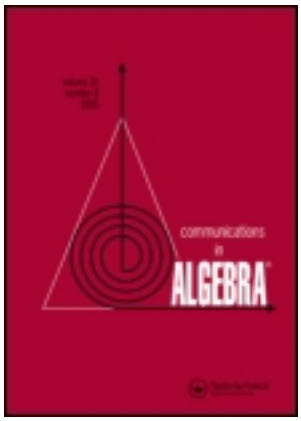

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When $\mathrm{Min}(A)^{-1}$ is Hausdorff

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WHEN Min(A)$^{-1}$ IS HAUSDORFF

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For a commutative ring with identity, say A, its collection of minimal prime ideals is denoted by* Min(A). *The hull-kernel topology on* Min(A) *is a well-studied structure. For example, it is known that the hull-kernel topology on* Min(A) *has a base of clopen subsets, and classifications of when* Min(A) *is compact abound. Recently, a program of studying the inverse topology on* Min(A) *has begun. This article adds to the growing literature. In particular, we characterize when* Min(A)$^{-1}$ *is Hausdorff. In the final section, we consider rings of continuous functions and supply examples.

Key Words: Hull-Kernel topology; Inverse topology; Minimal prime ideal; Rings of continuous functions.

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1. PRELIMINARIES

Throughout, A denotes a commutative ring with identity. Moreover, unless otherwise noted, we will assume that A is reduced, that is, it does not possess any nonzero nilpotent elements. We let Spec(A) denote the prime spectrum. Zorn's Lemma ensures that minimal prime ideals exist, and we designate the collection of all minimal prime ideals by Min(A). For $S \subseteq A$, let

$$\mathrm{U}(S) = \{P \in \mathrm{Min}(A) : S \nsubseteq P\}$$

and $\mathrm{V}(S) = \mathrm{Min}(A) \backslash \mathrm{U}(S)$. When S is a singleton, say $S = \{a\}$, we instead write $\mathrm{U}(a)$ and $\mathrm{V}(a)$. Notice that $\mathrm{U}(S) = \mathrm{U}(SA)$. Furthermore, it is known that the collection $\{\mathrm{U}(I) : I$ is an ideal of $A\}$ forms a topology on Min(A) known as the *hull-kernel topology* or the *Zariski topology*. The collection $\{\mathrm{U}(a) : a \in A\}$ is a base for the hull-kernel topology on Min(A). The hull-kernel topology is zero-dimensional and Hausdorff. (Recall that by a zero-dimensional topology we mean a topology with a base of clopen subsets.) The interested reader can consult [12] for more information on the hull-kernel topology on the space of minimal prime ideals.

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Lemma 1.1 (Corollary 2.2, [14]). *Let R be a reduced ring and $P \in Spec(R)$. $P \in \mathrm{Min}(R)$ if and only if for each $x \in P$ there exists an $r \in R \backslash P$ such that $xr = 0$.*

Lemma 1.2 (Corollary 2.3, [14]). *Let R be a reduced ring and let $P \in \mathrm{Min}(R)$. For a finitely generated ideal I of R, $I \subseteq P$ if and only if $\mathrm{Ann}_R(I) \nsubseteq P$.*

Lemma 1.1 and 1.2 are pivotal in proving the following properties of the operator $\mathrm{V}(\cdot)$. Recall that for a subset $S \subseteq A$, $\mathrm{Ann}(S) = \{x \in A : xs = 0 \text{ for all } s \in S\}$. An ideal I is called *dense* if $\mathrm{Ann}(I) = 0$.

Lemma 1.3. *Let A be a reduced ring, $a \in A$, and I, J finitely generated ideals of A. Then*:

(1) $\mathrm{V}(I) \cup \mathrm{V}(J) = \mathrm{V}(IJ)$;
(2) $\mathrm{V}(I) \cap \mathrm{V}(J) = \mathrm{V}(I + J)$;
(3) $\mathrm{V}(a) = \mathrm{Min}(A)$ *if and only if* $a = 0$;
(4) $\mathrm{V}(a) = \emptyset$ *if and only if a is not a zero-divisor*;
(5) $\mathrm{V}(I) = \emptyset$ *if and only if I is a dense ideal.*

It follows from Lemma 1.3 (i) and (ii) that the collection $\{\mathrm{V}(I) : I \text{ is a finitely generated ideal of } A\}$ is a base for a topology on $\mathrm{Min}(A)$. This topology is called the *inverse topology* and we denote by $\mathrm{Min}(A)^{-1}$ the space of minimal prime ideals of A equipped with the inverse topology. We now formally record some known facts regarding $\mathrm{Min}(A)^{-1}$.

Proposition 1.4 (Theorem 3.1, [16]). *Suppose A is a commutative reduced ring with identity. The inverse topology on $\mathrm{Min}(A)$ is compact and satisfies the T_1-separation axiom.*

Definition 1.5. From this point on by a inverse Hausdorff ring, we mean a ring A for which $\mathrm{Min}(A)^{-1}$ is a Hausdorff space.

Since for any finitely generated ideal I of A, $\mathrm{V}(I)$ is an open set in the hull-kernel topology, it follows that the hull-kernel topology is finer than the inverse topology. The next result characterizes when the two topologies coincide.

Proposition 1.6 (Proposition 3.2 [16]). *Suppose A is a reduced ring. The following are equivalent*:

(1) $\mathrm{Min}(A) = \mathrm{Min}(A)^{-1}$;
(2) $\mathrm{Min}(A)$ *is compact*;
(3) *For every $a \in A$ there is a finitely generated ideal $I \leq \mathrm{Ann}(a)$, such that* $\mathrm{Ann}(aA + I) = 0$.

Observe that if any of the equivalent conditions of Proposition 1.6 are satisfied, then $\mathrm{Min}(A)^{-1}$ is a compact zero-dimensional Hausdorff space, i.e., a *boolean space*. It is possible that $\mathrm{Min}(A)^{-1}$ is a boolean space without $\mathrm{Min}(A)$ being compact.

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The ring A is called *quasi-complemented* if whenever $a, b \in A$ and $ab = 0$, then there exist finitely generated ideals of A, say I, J, such that $a \in I$, $b \in J$, $IJ = 0$, and $I + J$ is a dense ideal of A. The notion of a quasi-complemented ring is a generalization of complemented rings (those rings satisfying for each $a \in A$ there is a $b \in A$ such that $ab = 0$ and $a + b$ is a non-zero-divisor of A). In fact, there is still yet another property that lies between complemented and quasi-complemented. The ring A is called *weakly complemented* if whenever $ab = 0$ there exists finitely generated ideals of A, say I, J, such that $a \in I, b \in J$, $IJ = 0$, and $I + J$ contains a regular element. For an example of a quasi-complemented, ring that is not weakly complemented see [16] (Quentel's Example). Quasi-complemented rings are reduced.

Theorem 1.7 (Theorem 3.4, [16]). *Suppose A is a reduced ring.* $\mathrm{Min}(A)^{-1}$ *is a boolean space if and only if A is a quasi-complemented ring. In particular, a quasi-complemented ring is inverse Hausdorff.*

As we close this section, we hope we have succeeded in giving the reader a taste of the style of theorems we are after. In particular, as the title of the article suggests we are interested in characterizing when $\mathrm{Min}(A)^{-1}$ is Hausdorff, i.e., when is A inverse Hausdorff. By Theorem 1.7, we gather that quasi-complemented rings are inverse Hausdorff.

2. INVERSE HAUSDORFF RINGS

In this section, we characterize when A satisfies the property that $\mathrm{Min}(A)^{-1}$ is a Hausdorff space. The class of *PF* rings will play an interesting role. For more information, see below and [20]. Some examples are in order.

Recall that a ring A is called a *weak Baer* ring if for each $a \in A$, $\mathrm{Ann}(a)$ is generated by an idempotent. These are also known as Rickart rings. It is known that a ring is a weak Baer ring if and only if it is a p.p. ring, that is, every principal ideal of A is a projective module. In [16] the authors generalized the notion of a weak Baer ring in the following manner. The ring A is called feebly Baer ring if whenever $a, b \in A$ and $ab = 0$, there is an idempotent $e^2 = e \in A$ such that $a \in eA$ and $b \in (1 - e)A$. A weak Baer ring is feebly Baer. Moreover, A is a feebly Baer ring if and only if it is an almost p.p. ring, that is, for each $a \in A$, $\mathrm{Ann}(a)$ is generated by idempotents. Every feebly Baer ring is weakly complemented, and thus inverse Hausdorff.

A is said to be a *PF-ring* if every principal ideal of A is a flat A-module. It immediately follows that a weak Baer ring is a PF-ring as projective modules are flat. Recall Theorem 2.13 of [16].

Theorem 2.1. *Suppose A is a commutative ring with identity. The following statements are equivalent*:

(i) *A is a feebly Baer ring*;
(ii) *A is an almost p.p. ring*;
(iii) *A is a weakly complemented PF-ring*;
(iv) *A is a quasi-complemented PF-ring.*

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Corollary 2.2. *A feebly Baer ring is a PF-ring.*

Our next result states a list of equivalent conditions for a ring to be a PF-ring. Recall that for a prime ideal $P \in \mathrm{Spec}(A)$, $\mathscr{O}(P)$ is the intersection of all prime ideals contained in P. Equivalently,

$$\mathscr{O}(P) = \{a \in P : \text{ there exists } x \in A \backslash P \text{ such that } xa = 0\}.$$

The proof of Theorem 2.3 can be found in [10], [20], [19], and [2].

Theorem 2.3. *Suppose A is a reduced ring. The following statements are equivalent*:

(1) *A is a PF-ring*;
(2) *For all $a, b \in A$, if $ab = 0$, then* $\mathrm{Ann}(a) + \mathrm{Ann}(b) = A$;
(3) *Every prime contains a unique minimal prime ideal*;
(4) *For each $P \in \mathrm{Spec}(A)$, $\mathscr{O}(P)$ is a prime ideal*;
(5) *For each $M \in \mathrm{Max}(A)$, $\mathscr{O}(M)$ is a prime ideal*;
(6) *For each $M \in \mathrm{Max}(A)$, A_M is an integral domain*;
(7) *For every $a, b \in A$,* $\mathrm{Ann}(a) + \mathrm{Ann}(b) = \mathrm{Ann}(ab)$.

We now turn our focus to characterizing when $\mathrm{Min}(A)^{-1}$ is Hausdorff.

Definition 2.4. Recall that a minimal prime ideal of A has the property that it does not contain any finitely generated dense ideal. Such primes shall play the starring role in our subsequent theorem. Let

$$\mathrm{Spec}_d(A) = \{P \in \mathrm{Spec}(A) : P \text{ does not contain any dense finitely generated ideals}\}$$

and

$$\mathrm{Spec}_r(A) = \{P \in \mathrm{Spec}(A) : P \text{ does not contain any non zero-divisors}\}.$$

Observe that $\mathrm{Min}(A) \subseteq \mathrm{Spec}_d(A) \subseteq \mathrm{Spec}_r(A)$. The set $\mathrm{Spec}_r(A)$ should be familiar to the reader in that this set is in bijective inclusion-preserving correspondence with the set of prime ideals of $q(A)$, the classical ring of quotients of A (a.k.a. the total ring of quotients). Zorn's Lemma may be applied to gather that both sets $\mathrm{Spec}_d(A)$ and $\mathrm{Spec}_r(A)$ contain maximal elements. Denote the set of maximal elements of $\mathrm{Spec}_d(A)$ (resp., $\mathrm{Spec}_r(A)$) by $\mathrm{Max}_d(A)$ (resp., $\mathrm{Max}_r(A)$).

The following lemma shall be useful in the proof of Theorem 2.6.

Lemma 2.5. *Suppose K is an ideal of A that does not contain a dense finitely generated ideal. Then there exists a $P \in \mathrm{Spec}_d(A)$ such that $K \subseteq P$. Furthermore, if K does not contain a nonzero-divisor, then there exists a $P \in \mathrm{Spec}_r(A)$ such that $K \subseteq P$.*

Proof. Notice that K is a proper ideal. Let S be the collection of all proper ideals containing K which do not contain dense finitely generated ideals. $S \neq \emptyset$. It is a straight forward argument to apply Zorn's Lemma and conclude that S has

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maximal elements. Let P be a maximal element of S. We demonstrate that P is a prime ideal of A. To that end, suppose that $cd \in P$. If $c \notin P$, then by maximality $cA + P$ contains a dense finitely generated ideal, say I. Similarly, if $d \notin P$, then $dA + P$ contains a dense finitely generated ideal, say J. Observe that $IJ \subseteq (cA + P)(dA + P)$. However, since $cd \in P$ it follows that $(cA + P)(dA + P) \subseteq P$. Thus, $IJ \subseteq P$. Since the product of dense finitely generated ideals is dense and finitely generated, this contradicts that $P \in S$. Therefore, either $c \in P$ or $d \in P$, whence P is a prime ideal. Moreover, $P \in \mathrm{Spec}_d(A)$.

We leave the proof of the second statement to the interested reader. □

We are now ready to state and prove the main theorem of this article. An ideal I of A is called a *radical ideal* (aka semiprime ideal) if whenever $x^n \in I$ for some $n \in \mathbb{N}$, then $x \in I$. By Zorn's Lemma an ideal is a radical ideal if and only if it is an intersection of prime ideals. For an ideal J, the *radical of* J is the ideal

$$\sqrt{J} = \{x \in A : x^n \in J\}.$$

By a *finitely generated radical ideal* we mean a radical ideal I for which there is a finite set, say $\{a_1, \ldots, a_n\}$, for which $I = \sqrt{a_1A + \cdots + a_nA}$. We make two final observations for a reduced ring. One, if I and J are ideals for which $IJ = 0$, then $\sqrt{I}\sqrt{J} = 0$. Two, I is a dense ideal of A if and only if $\sqrt{I}$ is a dense ideal of A.

Theorem 2.6. *Suppose A is a reduced ring. The following statements are equivalent*:

(1) $\mathrm{Min}(A)^{-1}$ *is a Hausdorff space*;
(2) $\mathrm{Min}(A)^{-1}$ *is a normal space*;
(3) *Whenever I, J are finitely generated ideals of A for which $IJ = 0$, there exists finitely generated ideals I', J' such that $IJ' = 0 = I'J$ while $I' + J'$ is a dense ideal*;
(4) *Whenever S, T are finitely generated radical ideals of A for which $ST = 0$, there exists finitely generated radical ideals S', T' such that $ST' = 0 = S'T$ while $S' + T'$ is a dense ideal*;
(5) *Whenever $a, b \in A$ and $ab = 0$, there exists finitely generated ideals of A, say I, J such that $a \in I \subseteq \mathrm{Ann}(b)$, $b \in J \subseteq \mathrm{Ann}(a)$ and $I + J$ is a dense finitely generated ideal*;
(6) *If $ab = 0$, then $\mathrm{Ann}(a) + \mathrm{Ann}(b)$ contains a dense finitely generated ideal*;
(7) *For every $M \in \mathrm{Max}_d(A)$, A_M is an integral domain*;
(8) *For every $P \in \mathrm{Spec}_d(A)$, A_P is an integral domain*;
(9) *For every $P \in \mathrm{Spec}_d(A)$, $\mathcal{O}(P)$ is a prime ideal*;
(10) *For every $M \in \mathrm{Max}_d(A)$, $\mathcal{O}(M)$ is a prime ideal.*

Proof. (1) ⇒ (2) If $\mathrm{Min}(A)^{-1}$ is Hausdorff, then since it is always compact and compact Hausdorff spaces are normal we gather that $\mathrm{Min}(A)^{-1}$ is normal.

(2) ⇒ (3) Suppose I and J are finitely generated ideals for which $IJ = 0$. Then $\mathrm{U}(I)$ and $\mathrm{U}(J)$ are disjoint closed subsets of $\mathrm{Min}(A)^{-1}$. By normality it follows that there are disjoint open subsets, say K_1, K_2 such that $\mathrm{U}(I) \subseteq K_1$ and $\mathrm{U}(J) \subseteq K_2$. Since $\mathrm{Min}(A)^{-1}$ is compact and T_1, under the assumption of normality it is also Hausdorff. Therefore, K_1 and K_2 can be chosen to be basic open subsets

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of $\mathrm{Min}(A)^{-1}$. Thus there are finitely generated ideals I', J' such that $K_1 = \mathrm{V}(J')$ and $K_2 = \mathrm{V}(I')$.

It follows that $\mathrm{U}(I) \cap \mathrm{U}(J') = \emptyset = \mathrm{U}(J) \cap \mathrm{U}(I')$, whence $IJ' = 0 = JI'$. Finally, since $\mathrm{V}(I' + J') = \mathrm{V}(I') \cap \mathrm{V}(J') = \emptyset$ it follows that $I' + J'$ is a finitely generated ideal which is not contained in any minimal prime ideal; Lemma 1.2 applies, and we conclude that $I' + J'$ is a dense ideal of A.

(3) $\Rightarrow$ (4) Suppose S and T are finitely generated radical ideals for which $ST = 0$. Let I and J be finitely generated ideals of A for which $\sqrt{I} = S$ and $\sqrt{J} = T$. Then $IJ = 0$ and by (3) there are finitely generated ideals, say I' and J', such that $IJ' = 0 = I'J$ while $I' + J'$ is a dense ideal. Set $S' = \sqrt{I'}$ and $T' = \sqrt{J'}$, finitely generated radical ideals. By the first observation prior to the theorem, we conclude that $ST' = 0 = S'T$. Since $I' + J'$ is a dense ideal and $I' + J' \leq S' + T'$, we conclude that $S' + T'$ is a dense ideal.

(4) $\Rightarrow$ (3) Suppose I and J are finitely generated ideals of A for which $IJ = 0$. Set $S = \sqrt{I}$ and $T = \sqrt{J}$, finitely generated radical ideals. Then $ST = 0$ and so by (4) there are finitely generated radical ideals, say S' and T' such that $ST' = 0 = S'T$ while $S' + T'$ is a dense ideal. Let I' and J' be finitely generated ideals of A for which $S' = \sqrt{I'}$ and $T' = \sqrt{J'}$. It is straightforward to check using containment arguments that $IJ' = 0 = I'J$. Finally, by the second observation prior to the theorem we conclude that $I' + J'$ is a dense ideal of A.

(3) $\Rightarrow$ (5) Patent.

(5) $\Rightarrow$ (1) Suppose $P, Q \in \mathrm{Min}(A)$ are distinct minimal prime ideals of A. Choose $a \in P \backslash Q$. By Lemma 1.1 there is some $x \in A \backslash P$ such that $ax = 0$. It follows that $x \in Q$. By (5) there are finitely generated ideals I, J such that $aI = xJ = 0$ and $I + J$ is a dense ideal. Then $Q \in \mathrm{V}(I)$, $P \in \mathrm{V}(J)$, and $\mathrm{V}(I) \cap \mathrm{V}(J) = \emptyset$. Consequently, $\mathrm{Min}(A)^{-1}$ is a Hausdorff space.

At this point, we have shown that the first five conditions are equivalent.

(3) $\Rightarrow$ (6) Suppose $a, b \in A$ for which $ab = 0$. Using (3), there exists finitely generated ideals I' and J' of A such that $aJ' = 0 = bI'$ while $I' + J'$ is a dense ideal. Let $I = I' + aA$ and $J = J' + bA$, then I and J are also finitely generated ideals of A and $I + J$ is a dense ideal. Thus, $a \in I \subseteq \mathrm{Ann}(b)$ and $b \in J \subseteq \mathrm{Ann}(a)$.

(6) $\Rightarrow$ (2) Let $a, b \in A$ with $\mathrm{U}(a) \cap \mathrm{U}(b) = \emptyset$, which implies that $ab = 0$. Using (6), there exists finitely generated ideals I and J of A such that $I \subseteq \mathrm{Ann}(b)$, $J \subseteq \mathrm{Ann}(a)$, and $I + J$ is a dense ideal. Since $I \subseteq \mathrm{Ann}(b)$, it follows that $\mathrm{U}(b) \subseteq \mathrm{V}(I)$. Similarly, $\mathrm{U}(a) \subseteq \mathrm{V}(J)$. Finally, $I + J$ a dense ideal implies that $\mathrm{V}(I) \cap \mathrm{V}(J) = \emptyset$. Therefore, any pair of disjoint subbasic closed sets are separated by disjoint open sets in the inverse topology. Consequently, $\mathrm{Min}(A)^{-1}$ is a normal space.

At this point, we have shown that the first six conditions are equivalent.

(6) $\Rightarrow$ (7) Let $a, b \in A$ and $m_1, m_2 \notin M$ such that $\frac{a}{m_1}\frac{b}{m_2} = \frac{0}{1}$. There exists some $s \notin M$ such that $(sa)b = s(ab) = 0$. By (6), $\mathrm{Ann}(sa) + \mathrm{Ann}(b)$ contains a dense finitely generated ideal. Since $M \in \mathrm{Max}_d(A)$, it follows that $\mathrm{Ann}(sa) + \mathrm{Ann}(b) \nsubseteq M$. So there exist $x_1 \in \mathrm{Ann}(sa)$ and $x_2 \in \mathrm{Ann}(b)$ such that $x_1 + x_2 \notin M$, which means either $x_1 \notin M$ or $y_1 \notin M$. If $x_1 \notin M$, then $x_1 s \notin M$ and $(x_1 s)a = 0$ implies that

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$\frac{a}{m_1} = \frac{0}{1}$. On the other hand, if $x_2 \notin M$, then $x_2 b = 0$ implies that $\frac{b}{m_2} = \frac{0}{1}$. Thus, A_M is an integral domain.

(7) $\Rightarrow$ (8) Since each $P \in \mathrm{Spec}_d(A)$ is contained in some M, $M \in \mathrm{Max}_d(A)$, and the localization A_P can be obtained as an appropriate localization of A_M it follows that each A_P is an integral domain.

(8) $\Rightarrow$ (9) Suppose $a, b \in A$ with $ab \in \mathcal{O}(P)$, then $ab \in P$ and there exist some $x \in A \setminus P$ such that $x(ab) = 0$. It follows that $\frac{a}{1}\frac{b}{1} = \frac{0}{1}$ in A_P. Since A_P is an integral domain, $\frac{a}{1} = \frac{0}{1}$ or $\frac{b}{1} = \frac{0}{1}$. Without any loss of generality, suppose that $\frac{a}{1} = \frac{0}{1}$; then there exists $s \notin P$ such that $sa = 0$. Therefore, $a \in \mathcal{O}(P)$, proving that $\mathcal{O}(P)$ is prime.

(9) $\Rightarrow$ (10) Straightforward.

(10) $\Rightarrow$ (6) Suppose that for each $M \in \mathrm{Max}_d(A)$, $\mathcal{O}(M)$ is a prime ideal, and let $ab = 0$. Suppose, by means of contradiction, that $\mathrm{Ann}(a) + \mathrm{Ann}(b)$ does not contain a dense finitely generated ideal. By Lemma 2.5 there is a $M \in \mathrm{Spec}_d(A)$ such that $\mathrm{Ann}(a) + \mathrm{Ann}(b) \subseteq M$. Without loss of generality, we can assume that $M \in \mathrm{Max}_d(A)$. By hypothesis, either $a \in \mathcal{O}(M)$ or $b \in \mathcal{O}(M)$. Without loss of generality we assume that $a \in \mathcal{O}(M)$. This means there is an $x \notin M$ such that $ax = 0$. But then $x \in \mathrm{Ann}(a) \subseteq \mathrm{Ann}(a) + \mathrm{Ann}(b) \subseteq M$, the desired contradiction. We are forced to conclude that $\mathrm{Ann}(a) + \mathrm{Ann}(b)$ does contain a dense finitely generated ideal. $\square$

Proposition 2.11 of [16] first investigated when $q(A)$ is a PF-ring. We add some new characterizations. Observe that in a reduced ring an ideal I contains a nonzero-divisor precisely when $\sqrt{I}$ contains a nonzero-divisor.

Theorem 2.7. *Suppose A is a commutative reduced ring. The following statements are equivalent*:

(1) *$q(A)$ is a PF-ring*;
(2) *If $ab = 0$, then there exist $x, y \in A$ such that $ay = 0 = bx$ while $x + y$ is a nonzero-divisor*;
(3) *If $ab = 0$, then $\mathrm{Ann}(a) + \mathrm{Ann}(b)$ contains a nonzero-divisor*;
(4) *Whenever I, J are finitely generated ideals of A for which $IJ = 0$, there exists finitely generated ideals I', J' such that $IJ' = 0 = I'J$, while $I' + J'$ contains a nonzero-divisor*;
(5) *Whenever I, J are finitely generated radical ideals of A for which $IJ = 0$, there exists finitely generated radical ideals I', J' such that $IJ' = 0 = I'J$, while $I' + J'$ contains a nonzero-divisor*;
(6) *Whenever $a, b \in A$ and $ab = 0$, there exists finitely generated ideals of A, say I, J such that $a \in I \subseteq \mathrm{Ann}(b)$, $b \in J \subseteq \mathrm{Ann}(a)$, and $I + J$ contains a nonzero-divisor*;
(7) *For every $M \in \mathrm{Max}_r(A)$, A_M is an integral domain*;
(8) *For every $P \in \mathrm{Spec}_r(A)$, A_P is an integral domain*;
(9) *For every $P \in \mathrm{Spec}_r(A)$, $\mathcal{O}(P)$ is a prime ideal*;
(10) *For every $M \in \mathrm{Max}_r(A)$, $\mathcal{O}(M)$ is a prime ideal.*

Corollary 2.8. *Suppose A is a reduced ring which satisfies Property A. Then $\mathrm{Min}(A)^{-1}$ is a Hausdorff space if and only if $q(A)$ is a PF-ring.*

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Example 2.9. Quentel's example (see Example 3.5 of [16]) supplies us with a commutative ring, say $\mathfrak{Q}$, which satisfies the equivalent conditions of Theorem 2.6 but not Theorem 2.7. Quentel's example is quasi-complemented (hence, inverse Hausdorff) but not weakly complemented. Moreover, $\mathfrak{Q}$ is a classical ring so it follows that $\mathfrak{Q}$ is not a PF-ring. In the next section, we will exhibit examples of inverse Hausdorff rings which are not quasi-complemented.

3. $C(X)$

We finish the article with a discussion of the ring $C(X)$ for a Tychonoff space X and find conditions when the ring $C(X)$ is inverse Hausdorff. First some definitions are in order.

Recall that for a topological space X we denote the collection of real-valued continuous functions on X by $C(X)$. In general, $C(X)$ is a subring of $\mathbb{R}^X$ under the coordinate-wise operations. It follows that $C(X)$ is a reduced ring. For the sake of studying the ring-theoretic properties of $C(X)$ it suffices to assume that X is a Tychonoff space, that is, completely regular and Hausdorff. In this case when X has more than one point, then $C(X)$ is a ring with zero-divisors. The standard reference for rings of continuous functions is the classic text [8]. Another way of viewing Tychonoff space is through the Stone–Čech compactification. In particular, Tychonoff spaces are precisely those spaces that can be densely embedded inside a compact Hausdorff space. As is customary, we use βX to denote the Stone–Čech compactification of X.

For $f \in C(X)$, the set $Z(f) = \{x \in X : f(x) = 0\}$ is called the *zero set of* f. A subset of X, say Z, is called a zeroset if there is some $f \in C(X)$ such that $Z(f) = Z$. The set $X \backslash Z(f)$ is called the *cozero set* of f and is denoted by $\mathrm{coz}(f)$. A *cozero set* of X is the cozero set for some function in $C(X)$. Recall that X is a Tychonoff space if and only if X is Hausdorff and the collection of cozero sets forms an open base for the topology on X.

In the articles [13] and [18] the authors determined which topological property classifies when $C(X)$ is complemented. The space X is said to be *cozero complemented* if for each cozero set C of X there is a disjoint cozero set D of X such that $C \cup D$ is a dense subset of X. X is cozero complemented if and only if $C(X)$ is a complemented ring. Observe that all perfectly normal spaces (every open set is a cozero set) are cozero complemented as given any cozero set O, the set $X \backslash cl_X O$ is a disjoint cozero to O and whose union with O is dense in X. It follows that every metric space is cozero complemented.

In a recent article, the authors of [16] show that for any Tychonoff space X, $C(X)$ is quasi-complemented if and only if $C(X)$ weakly complemented. Furthermore, $C(X)$ is weakly complemented if and only if X is weakly cozero complemented, that is, whenever C_1, C_2 are disjoint cozero sets then there are disjoint cozero sets D_1, D_2 such that $C_1 \subseteq D_1, C_2 \subseteq D_2$, and $D_1 \cup D_2$ is a dense subset of $C(X)$. The space $\beta\mathbb{N} \backslash \mathbb{N}$ is an example of a weakly cozero complemented space which is not cozero complemented. It follows that if X weakly cozero complemented, then $C(X)$ is inverse Hausdorff.

Theorem 5.7 of [16] states that $C(X)$ is a feebly Baer ring if and only if X is a strongly zero-dimensional F-space. Some definitions are in order. A space is said to be *strongly zero-dimensional* if βX is zero-dimensional. One of the equivalent

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characterizations for X to be an F-space is that every prime ideal of $C(X)$ contains a unique minimal prime ideal. Since $\beta\mathbb{N}\backslash\mathbb{N}$ is a strongly zero-dimensional F-space, $\beta\mathbb{N}\backslash\mathbb{N}$ is feebly Baer. Finally, if Z is the topological sum of $\beta\mathbb{N}\backslash\mathbb{N}$ and $\mathbb{R}$, then $C(Z)$ is a weakly complemented ring which is neither feebly Baer nor complemented.

Our last theorem gives topological conditions that characterize when $C(X)$ is an inverse Hausdorff ring. Since for any $f, g \in C(X)$, $Ann(f, g) = Ann(f^2 + g^2)$, therefore $C(X)$ always satisfies Property A.

Theorem 3.1. *Let X be a Tychonoff space. The following are equivalent*:

(a) $\mathrm{Min}(C(X))^{-1}$ *is a Hausdorff space*;
(b) $\mathrm{Min}(C(X))^{-1}$ *is a normal space*;
(c) *For each pair of disjoint cozero sets $C, D \subseteq X$ there exists a pair of cozero sets $C', D' \subseteq X$ such that $C \subseteq C'$, $D \subseteq D'$, $C \cap D' = \emptyset$, $D \cap C' = \emptyset$, and $C' \cup D'$ is a dense subset of X*;
(d) *For each pair of disjoint cozero sets $C, D \subseteq X$ there exist zero sets Z_1 and Z_2 such that $C \subseteq Z_1$, $D \subseteq Z_2$, and* $\mathrm{int}(Z_1) \cap \mathrm{int}(Z_2) = \emptyset$;
(e) *$q(X)$ is a PF-ring.*

Proof. The proof follows in two steps. First, that (a), (b), and (e) are equivalent follows from Theorems 2.6 and 2.7.

To see that (c) implies (d), let C and D be disjoint cozero sets. By (c) there are cozero sets, say $\mathrm{coz}(f)$, $\mathrm{coz}(g)$ such that $C \subseteq \mathrm{coz}(f)$, $D \subseteq \mathrm{coz}(g)$, $C \cap \mathrm{coz}(g) = \emptyset$, $D \cap \mathrm{coz}(f) = \emptyset$, and $\mathrm{coz}(f) \cup \mathrm{coz}(g)$ is a dense subset of X. Set $Z_1 = Z(g)$ and $Z_2 = Z(f)$. We leave it to the interested reader to check that Z_1 and Z_2 satisfy the condition of (d). The converse is similar.

Finally, that (e) and (c) are equivalent follows from the equivalence of (1) and (2) of Theorem 2.7 together with the following observations. For $f, g \in C(X)$

$$fg = 0 \text{ if and only if } \mathrm{coz}(f) \cap \mathrm{coz}(g) = \emptyset$$

and as mentioned before f is a nonzero-divisor if and only if $\mathrm{coz}(f)$ is a dense subset of X. □

Example 3.2. It is known that for an F-space X, βX is isomorphic to $\mathrm{Min}(C(X))^{-1}$ (see Proposition 7.3 of [21]). Furthermore, X is weakly cozero-complemented if and only if X is a strongly zero-dimensional space. Therefore, if X is an F-space which is not strongly zero-dimensional, then $C(X)$ is an inverse Hausdorff ring which is not quasi-complemented. For such an example consider $\mathbb{H} = \beta\mathbb{R}^+\backslash\mathbb{R}^+$, an F-space which is not zero-dimensional. Thus, $C(\mathbb{H})$ is an inverse Hausdorff ring which is not quasi-complemented.

REFERENCES

[1] Bhattacharjee, P. (2009). Minimal prime element space of an algebric frame. Dissertation, Bowling Green State University.
[2] Bhattacharjee, P. (2012). Two spaces of minimal primes. *J. Alg. and Its Appl.* 11(1).
[3] Bhattacharjee, P., Drees, K., McGovern, W. Extensions of commutative rings. *Top. Appl.* (to appear).

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[4] Bigard, A., Keimel, K., Wolfenstein, S. (1977). *Groupes et Anneaux Riticulés*, Lecture Notes in Mathematics. Vol. 608. Springer-Verlag.
[5] Darnel, M. (1995). *Theory of Lattice-Ordered Groups*. Monographs and Textbooks in Pure and Applied Mathematics. Vol. 187. New York: Marcel Dekker, Inc.
[6] De Marco, G., Orsatti, A. (1971). Commutative rings in which every prime ideal is contained in a unique maximal ideal. *Proc. Amer. Math. Soc.* 30(3):459–466.
[7] Dobbs, D. E., Picavet, G. (2003). Weak baer going-down rings. *Houston J. Math.* 29:559–581.
[8] Gillman, L., Jerison, M. (1976). *Rings of Continuous Functions*. Graduate Texts in Mathametics. Vol. 43. Berlin–Heidelberg–New York: Springer Verlag.
[9] Gilmer, R. (1975). Polynomial rings over a commutative von neumann regular ring. *Proceedings of the A.M.S.* 49(2):294–296.
[10] Glaz, S. (1989). *Commutative Coherent Rings*. Lecture Notes in Math. Vol. 1371. Berlin: Springer-Verlag.
[11] Hager, A. W., Martinez, J. (1993). Fraction-dense algebras and spaces. *Candian Journal of Mathematics* 45(5):977–996.
[12] Henriksen, M., Jerison, M. (1965). The space of minimal prime ideals of a commutative ring. *Trans. A.M.S.* 115:110–130.
[13] Henriksen, M., Woods, R. G. (2004). Cozero complemented spaces; when the space of minimal prime ideals of a C(X) is compact. *Top. Appl.* 141(1–3):147–170.
[14] Huckaba, J. (1988). *Commutative Rings with Zero Divisors*. Monographs and Textbooks in Pure and Applied Math. Vol. 117. New York: Marcel Dekker.
[15] Iberkleid, W., McGovern, W. (2009). A natural equivalence for the category of coherent frames. *Alg. Univers.* 62:247–258.
[16] Knox, M., Levy, R., McGovern, W. Wm., Shapiro, J. (2008). Generalizations of complemented rings with applications to rings of functions. *J. of Alg. and Appl.* 7(6):1–24.
[17] Lambek, J. (1976). *Lectures on Rings and Modules*. 2nd ed. New York: Chelsea Publishing Co.
[18] Levy, R., Shapiro, J. (2005). Rings of quotients of rings of functions. *Top. Appl.* 146/147:253–265.
[19] Marconi, U. (1984). On the Baer rings. *J. Pure Appl. Alg.* (2):159–161.
[20] Matlis, E. (1983). The minimal prime spectrum of a reduced ring. *Ill. J. Math.* 27(3):353–391.
[21] McGovern, W. Wm. (2006). Neat rings. *J. Pure Appl. Alg.* 205:243–265.

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