

## TWO SPACES OF MINIMAL PRIMES

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This paper studies algebraic frames  $L$  and the set  $\text{Min}(L)$  of minimal prime elements of  $L$ . We will endow the set  $\text{Min}(L)$  with two well-known topologies, known as the Hull-kernel (or Zariski) topology and the inverse topology, and discuss several properties of these two spaces. It will be shown that  $\text{Min}(L)$  endowed with the Hull-kernel topology is a zero-dimensional, Hausdorff space; whereas,  $\text{Min}(L)$  endowed with the inverse topology is a  $T_1$ , compact space. The main goal will be to find conditions on  $L$  for the spaces  $\text{Min}(L)$  and  $\text{Min}(L)^{-1}$  to have various topological properties; for example, compact, locally compact, Hausdorff, zero-dimensional, and extremally disconnected. We will also discuss when the two topological spaces are Boolean and Stone spaces.

*Keywords:* Algebraic frame; minimal prime; Hull-kernel topology; inverse topology.

### 1. Introduction

The study of frame theory started in the 1930s; although the term “frame” was first used in the late 1970s, prior to that they were known as complete Brouwerian lattices (refer to [13]). A popular example of frames is a topology (considering the open sets, without referring the points of the space), and hence frames are also called point-free topologies. On the other hand, a different approach in terms of lattice-ordered group (or  $\ell$ -group) gave examples of frames; it was found that the collection of all convex  $\ell$ -subgroups of an  $\ell$ -group  $G$  is a frame and it became a very common example of frames other than topologies. In 1974 Speed’s paper [16] defined a topology on the space of minimal prime ideals of a distributive lattice, which was the inverse topology in the theory of frames. In this paper, Speed proved some basic facts on this topology in the context of distributive lattices. In the current paper, we will expand on this theory. The general approach in terms of frame theory will give us more information about the inverse topology. We will discuss some of Speed’s results in the context of frame theory, and prove some new and interesting results on the spaces of minimal primes.

A *frame*  $L$  is a complete lattice which satisfies the following distributive law called the *frame law*: for each  $a \in L$  and  $\{b_i\}_{i \in I} \subseteq L$  for some index set  $I$ ,

$$a \wedge \left( \bigvee_{i \in I} b_i \right) = \bigvee_{i \in I} (a \wedge b_i).$$

Notice that the frame law implies that a frame is a distributive lattice.

We wish to recall some general definitions from frame theory. Note, throughout this paper  $L$  will denote a frame.

- (1) A subset  $M$  of  $L$  is a *subframe* if  $M$  is closed under finite infima and arbitrary suprema. Closure under the empty infimum and supremum implies, respectively, that  $M$  inherits top and bottom elements.

Note that the arbitrary infima in  $M$  need not agree with those in  $L$ . In general, if  $S \subseteq M$ , then

$$\bigwedge S = \bigvee \{x \in M : x \leq s, \text{ for all } s \in S\}.$$

- (2) The elements  $x, y \in L$  are *disjoint* if  $x \wedge y = 0$ .
- (3) An element  $c \in L$  is *compact* if  $c \leq \bigvee_{i \in I} b_i$  implies that  $c \leq \bigvee_{i \in I_0} b_i$  for some finite subset  $I_0$  of  $I$ . We denote the collection of all compact elements of the frame  $L$  by  $\mathfrak{K}(L)$ .
- (4)  $L$  is *algebraic* if every element in  $L$  is the supremum of compact elements.
- (5)  $L$  is said to satisfy the *finite intersection property* (FIP) if  $x, y \in \mathfrak{K}(L)$  implies that  $x \wedge y \in \mathfrak{K}(L)$ .
- (6) For each  $x \in L$  define

$$x^\perp = \bigvee \{y \in L : y \wedge x = 0\}.$$

The element  $x^\perp$  is called the *polar* (a.k.a *pseudocomplement*) of  $x$ , and due to the frame law  $x^\perp$  is the largest element of  $L$  which is disjoint from  $x$ . Elements of the form  $x^\perp$  are known as the *polars* of  $L$ . An element  $x \in L$  is said to be *complemented* if  $x \vee x^\perp = 1$ .

- (7) An element  $x \in L$  is *dense* if  $x^\perp = 0$ . We call a compact dense element of  $L$  a *unit*.
- (8)  $L$  is called a *normal frame* if for all  $x, y \in L$  such that  $x \vee y = 1$  there exist disjoint  $a, b \in L$  such that  $a \vee x = 1 = b \vee y$ .

Some basic properties of polars in a frame are stated below. These properties will be used often in this paper.

Let  $x, y \in L$ ,

- (i)  $(x \vee y)^\perp = x^\perp \wedge y^\perp$ .
- (ii)  $(x \wedge y)^\perp \geq x^\perp \vee y^\perp$ .
- (iii)  $x \leq x^{\perp\perp}$ .
- (iv)  $x \vee y$  is dense if and only if  $x^\perp \wedge y^\perp = 0$ .

Thus it follows that for any  $x$  in  $L$ ,

$$(x \vee x^\perp)^\perp = x^\perp \wedge x^{\perp\perp} = 0,$$

demonstrating the fact that  $x \vee x^\perp$  is a dense element.

Finally, we state some topological definitions which will also be used in the paper. Let  $(X, \tau)$  be a topological space.

- (a)  $(X, \tau)$  is a *zero-dimensional* space if it has a basis consisting of clopen (that is, closed and open) subsets.
- (b)  $(X, \tau)$  is *totally disconnected* if the only connected subsets of  $X$  are the singletons sets.
- (c)  $(X, \tau)$  is an *extremally disconnected* space if the closure in  $X$  of every open set is clopen.
- (d) A totally disconnected, compact, Hausdorff topological space is a *Boolean* space.
- (e) An extremally disconnected, compact, Hausdorff topological space is a *Stone* space.

## 2. Minimal Prime Elements

Let  $L$  be an algebraic frame which satisfies the FIP. From this point on such a frame will be called an  $M$ -frame. It follows then that an  $M$ -frame is, up to isomorphism, the ideal lattice of a distributive lattice with 0. It is in this sense that the study of  $M$ -frames can be taken from two equivalent directions.

The first is to consider the sublattice  $\mathfrak{K}(L)$  and then to investigate the properties of the ideal lattice of  $\mathfrak{K}(L)$  which is frame isomorphic to  $L$ . The other direction is to study the elements of  $L$  without mentioning that every element of  $L$  corresponds to an ideal of  $\mathfrak{K}(L)$ . Historically, the former approach has been the prevailing mode of thought. We have found that there is an ease to studying the elements of  $L$  without having to mention ideals. For example, in studying the frame of radical ideals of a commutative ring with identity, it is easier to consider ring-theoretic ideals instead of a lattice-theoretic ideal of finitely generated ring ideals. It is this approach that we will undertake throughout the rest of the paper. In particular, we will study prime elements of an  $M$ -frame, rather than the prime ideals of a distributive lattice with 0.

An element  $p < 1$  in  $L$  is called *prime* if for any  $x, y \in L$ ,  $x \wedge y \leq p$  implies that  $x \leq p$  or  $y \leq p$ . Since  $L$  is a distributive lattice, this definition is equivalent to the condition that  $x \wedge y = p$  implies that  $x = p$  or  $y = p$ . A prime  $p$  is *minimal* if there does not exist any other prime element  $q \neq p$  with the property  $q < p$ . The *prime spectrum* of  $L$  is the collection of all prime elements of  $L$  and is denoted  $\text{Spec}(L)$ . We denote by  $\text{Min}(L)$  the set of all minimal prime elements of  $L$ .

We notice here that for an algebraic frame  $L$  the set  $\text{Spec}(L)$  is nonempty. Given a compact element in  $L$ , say  $c$ , the usual Zorn's Lemma argument (or an appeal to the Boolean prime ideal theorem) assures us of the existence of an element  $m$

maximal with respect to  $c \not\leq m$ . Such an element is prime. Zorn's lemma can also be used to guarantee that minimal primes exist.

It turns out that to check the primality of an element in an  $M$ -frame we just need to check the prime condition with respect to compact elements.

**Proposition 2.1.** *Let  $L$  be an  $M$ -frame and  $p \in L$ . Then  $p \in \text{Spec}(L)$  if and only if for any compact elements  $c, d \in L$ ,  $c \wedge d \leq p$  implies that  $c \leq p$  or  $d \leq p$ .*

The *Lemma on Ultrafilters* is a fundamental result that connects the minimal primes and ultrafilters of  $M$ -frames. This result has been used numerous times in the literature of frames. As for references we suggest the reader consult [1], [6], or [14]. Before stating the result we remind the reader of the notions of filters and ultrafilters.

Let  $L$  be a frame and let  $\mathfrak{F}$  a nonempty subset of  $L$ .  $\mathfrak{F}$  is a *filter* if:

- (1)  $0 \notin \mathfrak{F}$ ,
- (2) if  $x, y \in \mathfrak{F}$ , then  $x \wedge y \in \mathfrak{F}$ , and
- (3) if  $x \in \mathfrak{F}$  and  $y \in L$  with  $x \leq y$ , then  $y \in \mathfrak{F}$ .

A maximal filter is called an *ultrafilter*. Zorn's lemma forces every filter to be contained in an ultrafilter.

**Lemma 2.2 (Lemma on Ultrafilters).** *Let  $L$  be an  $M$ -frame. The prime element  $p \in \text{Spec}(L)$  is a minimal prime element if and only if*

$$F_p = \{c \in \mathfrak{K}(L) : c \not\leq p\}$$

*is an ultrafilter on  $\mathfrak{K}(L)$ . In this case,  $p = \bigvee \{c^\perp : c \in F_p\}$ .*

For the proof of the lemma we refer to [13]. A partial converse to Lemma 2.2 and an immediate corollary follows in the next two results.

**Lemma 2.3.** *Suppose  $L$  is an  $M$ -frame and  $\mathfrak{U}$  is an ultrafilter on  $\mathfrak{K}(L)$ . The element  $p = \bigvee \{c^\perp : c \in \mathfrak{U}\}$  is a minimal prime element of  $L$ .*

**Lemma 2.4.** *Let  $L$  be an  $M$ -frame and  $p \in \text{Min}(L)$ . For any  $x \in \mathfrak{K}(L)$  either  $x \not\leq p$  or  $x^\perp \not\leq p$ .*

### 3. The Hull-Kernel Topology on $\text{Min}(L)$

In [16] the author topologized the collection of minimal prime ideals of a distributive lattice via the Hull-kernel topology. The author then demonstrated that this topology is zero-dimensional and Hausdorff. In this section we translate many of the results from [16] as well as add some results to the theory of minimal primes.

The *Hull-kernel topology* on  $\text{Min}(L)$  is the topology generated by the collection  $\mathcal{B} = \{U(a) \mid a \in \mathfrak{K}(L)\}$  where  $U(a) = \{p \in \text{Min}(L) \mid a \not\leq p\}$ . Notice that by primality  $U(a) \cap U(b) = U(a \wedge b)$ , and so since  $L$  satisfies the FIP, the collection  $\mathcal{B}$  is in fact a base for the Hull-kernel topology on  $\text{Min}(L)$ . We set  $V(a)$  as the set-theoretic

complement of  $U(a)$ . We recall some interesting properties of the operators  $U(\cdot)$  and  $V(\cdot)$ .

**Lemma 3.1.** *Each of the following holds:*

- (1) For any  $x, y \in L$ ,  $U(x) \cap U(y) = U(x \wedge y)$  and  $U(x) \cup U(y) = U(x \vee y)$ .
- (2) For any  $x, y \in L$ ,  $V(x) \cup V(y) = V(x \wedge y)$  and  $V(x) \cap V(y) = V(x \vee y)$ .
- (3)  $\bigcup_{\alpha} U(x_{\alpha}) = U(\bigvee_{\alpha} x_{\alpha})$ ,  $x_{\alpha} \in L$ .
- (4) Similarly,  $\bigcap_{\alpha} V(x_{\alpha}) = V(\bigvee_{\alpha} x_{\alpha})$ ,  $x_{\alpha} \in L$ .
- (5)  $U(x) = \emptyset \Leftrightarrow x = 0 \Leftrightarrow V(x) = \text{Min}(L)$ .
- (6) The compact element  $x \in L$  is a unit of  $L$  if and only if  $V(x) = \emptyset$ .
- (7) For any  $x \in \mathfrak{K}(L)$ ,  $U(x) = V(x^{\perp})$  and  $U(x^{\perp}) = V(x)$ .

**Proposition 3.2.** *The open (closed) subsets of  $\text{Min}(L)$  endowed with the Hull-kernel topology are precisely of the form  $U(x)$  ( $V(x)$ ) for some  $x \in L$ . Therefore, by (7) of Lemma 3.1, each basic open set is clopen.*

Our next result is Proposition 2.3 and Corollary 2.6 of [16].

**Theorem 3.3.** *The Hull-kernel topology on  $\text{Min}(L)$  is zero-dimensional Hausdorff.*

**Proposition 3.4.** *For any two distinct minimal prime elements  $p, q \in L$  there exist compact elements  $c, d \in L$  with  $c \wedge d = 0$  such that*

- (i)  $c \leq p$ ,  $c \not\leq q$  and
- (ii)  $d \leq q$ ,  $d \not\leq p$ .

**Proof.** Suppose  $p, q \in \text{Min}(L)$  with  $p \neq q$ . Since  $\text{Min}(L)$  is Hausdorff there exists  $c, d \in \mathfrak{K}(L)$  such that  $p \in U(d)$ ,  $q \in U(c)$  and  $U(c) \cap U(d) = \emptyset$ . So,  $d \not\leq p$ ,  $c \not\leq q$  and  $c \wedge d = 0$ , the last conclusion being a consequence of Lemma 3.1. Finally,  $p \notin U(c)$  implies that  $p \in V(c)$ , and so  $c \leq p$ . Similarly,  $d \leq q$ .  $\square$

One of the main results of [16] is a characterization of when  $\text{Min}(L)$  is compact. Speed also gives a characterization of when  $\text{Min}(L)$  is a Stone space, that is, an extremally disconnected compact Hausdorff space. (Recall a space is *extremally disconnected* if the closure of every open set is clopen.) We shall address these results in the next section. For now we characterize when  $\text{Min}(L)$  is a discrete space. We start with a useful lemma.

**Lemma 3.5.** *For any  $x \in L$ ,  $\text{cl}(U(x)) = V(x^{\perp})$  with respect to the Hull-kernel topology.*

**Proof.** It is apparent that primality forces  $U(x) \subseteq V(x^{\perp})$ ; so  $\text{cl}(U(x)) \subseteq V(x^{\perp})$ .

To show the other inclusion, let  $p \in V(x^{\perp})$ . So,  $x^{\perp} \leq p$ . Let  $y \in \mathfrak{K}(L)$  satisfy  $p \in U(y)$ . Suppose, by way of contradiction, that  $U(y) \cap U(x) = \emptyset$ ; then,  $U(y \wedge x) = \emptyset$  which implies that  $x \wedge y = 0$  by Lemma 3.1. It thus follows that  $y \leq x^{\perp} \leq p$ .

Consequently,  $p \in V(y)$ , which is a contradiction. Therefore, every neighborhood of  $p$  intersects  $U(x)$ , thence  $p \in \text{cl}(U(x))$ .  $\square$

Using the preceding lemma we can immediately conclude when  $\text{Min}(L)$  is an extremally disconnected space.

**Proposition 3.6.**  *$\text{Min}(L)$ , endowed with the Hull-kernel topology, is an extremally disconnected space if and only if  $V(x^\perp)$  is open for all  $x \in L$ .*

The next result is a generalization of [6, Lemma 2.6]. We include a proof for completeness sake.

**Proposition 3.7.** *Let  $L$  be an  $M$ -frame and let  $p \in \text{Min}(L)$ . The following statements are equivalent.*

- (i)  $p$  is an isolated point of  $\text{Min}(L)$ .
- (ii) For some  $c \in \mathfrak{K}(L)$ ,  $p = c^\perp$ .
- (iii)  $p$  is a polar.

**Proof.** (i)  $\Rightarrow$  (ii) Suppose  $p$  is an isolated point of  $\text{Min}(L)$ , i.e.  $\{p\} = U(c)$  for some  $c \in \mathfrak{K}(L)$ . Since  $c \not\leq p$  and  $p$  is a prime element it follows that  $c^\perp \leq p$ . We claim that  $p \leq c^\perp$ . Notice that  $\{p\} = U(c)$  implies that the only ultrafilter of  $L$  containing  $c$  is  $F_p$ . Let  $d \in \mathfrak{K}(L)$  satisfy  $d \leq p$ . Notice that if  $d \wedge c \neq 0$ , then there would be an ultrafilter containing the compact element  $d \wedge c$ . Any such ultrafilter would necessarily contain  $c$ , and therefore would be equal to  $F_p$ . Thus  $d \in F_p$ , giving a contradiction. The only recourse we have is that  $d \wedge c = 0$ , and so  $d \leq c^\perp$ . Since  $L$  is an algebraic frame it follows that  $p \leq c^\perp$ , thence  $p = c^\perp$ .

(ii)  $\Rightarrow$  (iii) It is clear.

(iii)  $\Rightarrow$  (i) Suppose  $p$  is a polar. Then  $p = p^{\perp\perp}$ . Since  $p < 1$  it follows that  $p^\perp \neq 0$ . Therefore, we can choose a compact element  $c$  satisfying  $0 < c \leq p^\perp$ . First of all observe that this implies  $p \leq c^\perp$ . On the other hand, we also know that  $c \not\leq p$ , and so  $c \in F_p$ . This means that  $c^\perp \leq p$ . It follows that  $p = c^\perp$ . We claim  $\{p\} = U(c)$ . We have already seen why  $\{p\} \subseteq U(c)$ . As to the reverse containment, observe that if  $q \in U(c)$ , then  $p = c^\perp \leq q$ . So, since they are both minimal primes we conclude that  $p = q$ . Consequently  $U(c) = \{p\}$ , and hence  $p$  is an isolated point of  $\text{Min}(L)$ .  $\square$

**Corollary 3.8.**  *$\text{Min}(L)$  endowed with the Hull-kernel topology is a discrete space if and only if every minimal prime element of  $L$  is a polar.*

#### 4. $\text{Min}(L)^{-1}$ , the Inverse Topology on $\text{Min}(L)$

Since  $V(c_1) \cap V(c_2) = V(c_1 \vee c_2)$  for any  $c_1, c_2 \in \mathfrak{K}(L)$ , the collection of sets  $\{V(x) \mid x \in \mathfrak{K}(L)\}$  forms a base for a topology on  $\text{Min}(L)$  which we shall call the *inverse topology* on  $\text{Min}(L)$ , and denote by  $\text{Min}(L)^{-1}$ . The first observation that should be made is that  $\text{Min}(L)$  is finer than  $\text{Min}(L)^{-1}$ .

In [16] the author called the inverse topology the *dual spectral* topology but the author only considered the case when  $\text{Min}(L) = \text{Min}(L)^{-1}$ . In [15] the author investigated the inverse topology on the collection of minimal prime subgroups of an arbitrary abelian lattice-ordered group. This was later generalized to arbitrary  $\ell$ -groups in [11]. In [12] the authors investigated the inverse topology on the collection of minimal prime ideals of a commutative ring with identity. It is our goal in this section to generalize each of these cases by investigating  $\text{Min}(L)^{-1}$ , for general  $M$ -frames  $L$ .

In contrast to the Hull-kernel topology, we will see that  $\text{Min}(L)^{-1}$  is generally not a Hausdorff space nor is it zero-dimensional. However,  $\text{Min}(L)^{-1}$  satisfies an important topological property: compactness.

**Lemma 4.1.**  $\text{Min}(L)^{-1}$  is a compact,  $T_1$ -space for any  $M$ -frame  $L$ .

**Proof.** We first show that  $\text{Min}(L)^{-1}$  is a  $T_1$ -space. Let  $p, q \in \text{Min}(L)^{-1}$  with  $p \neq q$ . Since  $L$  is algebraic,  $p = \bigvee_{\alpha \in I} c_\alpha$  and  $q = \bigvee_{\beta \in J} d_\beta$ , for some index sets  $I$  and  $J$  where,  $\{c_\alpha \mid \alpha \in I\}, \{d_\beta \mid \beta \in J\} \subseteq \mathfrak{K}(L)$ . It follows that there exist at least one  $c_{\alpha_0}$  and one  $d_{\beta_0}$  from the two respective subcollections such that  $c_{\alpha_0} \not\leq q$  and  $d_{\beta_0} \not\leq p$ . Hence,  $q \in U(c_{\alpha_0}) \cap V(d_{\beta_0})$  and  $p \in U(d_{\beta_0}) \cap V(c_{\alpha_0})$ . So,  $p \in V(c_{\alpha_0}) \setminus V(d_{\beta_0})$  and  $q \in V(d_{\beta_0}) \setminus V(c_{\alpha_0})$ . Hence,  $\text{Min}(L)^{-1}$  is  $T_1$ .

To show compactness, suppose  $\{V(c)\}_{c \in K}$  is a basic open cover of  $\text{Min}(L)^{-1}$  for some  $K \subseteq \mathfrak{K}(L)$ . Consider the set

$$S = \{c_1 \wedge c_2 \wedge \cdots \wedge c_n : c_i \in K, n \in \mathbb{N}, i = 1, \dots, n\}.$$

We first notice that  $S$  is closed under finite meets. To complete the proof it suffices to show that  $0 \in S$  in which case

$$\text{Min}(L)^{-1} = V(0) = V(c_1 \wedge c_2 \wedge \cdots \wedge c_n) = V(c_1) \cup V(c_2) \cup \cdots \cup V(c_n).$$

Suppose, by way of contradiction, that  $0 \notin S$ . Define

$$\overline{S} = \{x \in \mathfrak{K}(L) : s \leq x \text{ for some } s \in S\},$$

and observe that  $\overline{S}$  is a filter of compact elements of  $L$ .

Applying Zorn's lemma,  $\overline{S}$  can be extended to an ultrafilter, say  $\mathfrak{F}$ , of compact elements of  $L$ . Therefore we have  $\overline{S} \subseteq \mathfrak{F}$ . Now, let us consider the element  $p = \bigvee \{c^\perp : c \in \mathfrak{F}\}$  in  $L$ . Using Lemma 2.3 we conclude that  $p \in \text{Min}(L)$ . Given our basic open cover, there exists some  $c \in K$  such that  $p \in V(c)$ ; that is,  $c \leq p$ . However,  $c \in S$ , which in turn says that  $c \in \mathfrak{F}$ . This implies that  $c^\perp \leq p$ . Consequently,  $c \not\leq p$  by Lemma 2.4, the desired contradiction.  $\square$

We now recall Speed's theorem in all its glory (see [16, Proposition 5.1]).

**Theorem 4.2.** Suppose  $L$  is an  $M$ -frame. The following statements are equivalent.

- (i) The Hull-kernel topology on  $\text{Min}(L)$  is compact.

- (ii)  $\text{Min}(L) = \text{Min}(L)^{-1}$ .
- (iii) For each  $x \in \mathfrak{K}(L)$  there exist  $y \in \mathfrak{K}(L)$  such that  $x \wedge y = 0$  and  $x \vee y$  is a unit.

An immediate observation is that if  $L$  satisfies the property of Theorem 4.2, then  $\text{Min}(L)^{-1}$  is a compact, Hausdorff, and zero-dimensional space. We ask several questions at this point: Can we characterize each of these topological properties for  $\text{Min}(L)^{-1}$ ? When is  $\text{Min}(L)^{-1}$  a Boolean space (compact, Hausdorff, totally disconnected)? We proceed to answer these questions first.

We shall have several occasions to use the following lemma.

**Lemma 4.3.** *A subset  $K$  of  $\text{Min}(L)^{-1}$  is clopen if and only if  $K = V(y) = U(z)$  for some  $y, z \in \mathfrak{K}(L)$ . Moreover,  $y \vee z$  is a unit while  $y \wedge z = 0$ .*

**Proof.** Suppose  $K$  is a clopen subset of  $\text{Min}(L)^{-1}$ . Since  $K$  is an open subset,  $K = \bigcup \{V(c) \mid c \in \mathfrak{K}(L)\}$ . Also  $K$  is compact since it is a closed subset of the compact space  $\text{Min}(L)^{-1}$ . Thus,  $K = V(c_1) \cup V(c_2) \cup \dots \cup V(c_n)$  for some  $n$ . Letting  $y = c_1 \wedge \dots \wedge c_n$  we observe that  $y \in \mathfrak{K}(L)$  and  $K = V(y)$ . Once again since  $\text{Min}(L) \setminus K$  is a clopen subset, by a similar argument we gather that  $\text{Min}(L) \setminus K = V(z)$  for some compact  $z \in L$ . So,  $K = \text{Min}(L) \setminus V(z) = U(z)$ .

On the other hand, if  $K \subseteq \text{Min}(L)^{-1}$  with  $K = V(y) = U(z)$  for some compact  $y, z \in L$ , then  $K$  is a clopen subset of  $\text{Min}(L)^{-1}$ .

Next, since  $V(y \wedge z) = V(y) \cup V(z) = \text{Min}(L)$  it follows that  $y \wedge z = 0$ . Finally,  $V(y \vee z) = V(y) \cap V(z) = \emptyset$  so that  $y \vee z$  is a unit.  $\square$

**Definition 4.4.** Let  $x \in \mathfrak{K}(L)$ . If there exists a  $y \in \mathfrak{K}(L)$  for which  $x \wedge y = 0$  and  $x \vee y$  is a unit, then we call  $x$  a *component* of  $L$ . In ring theory and the theory of lattice-ordered groups these types of elements are actually referred to as complemented elements. Since this has a different meaning in the context of lattice theory we have chosen to use a different word.

Observe then that for any  $x \in \mathfrak{K}(L)$ ,  $U(x)$  is clopen in  $\text{Min}(L)^{-1}$  if and only if  $x$  is a component.

In contrast to Proposition 3.7, we state our next result.

**Theorem 4.5.** *Let  $L$  be an  $M$ -frame and  $p \in \text{Min}(L)^{-1}$ .  $p$  is isolated with respect to the inverse topology if and only if  $p = c^\perp$ , for a component  $c \in \mathfrak{K}(L)$ .*

**Proof.** Suppose that  $p$  is isolated. Since  $\text{Min}(L)^{-1}$  is a  $T_1$ -topological space, it follows that  $\{p\}$  is a clopen subset. Using Lemma 4.3, there exist a component  $c \in \mathfrak{K}(L)$  such that  $\{p\} = U(c)$ . Following a similar argument from Proposition 3.7 we can conclude that  $p = c^\perp$ .

On the other hand, suppose that  $p = c^\perp$ , for a component  $c \in \mathfrak{K}(L)$ . It follows that,

$$\{p\} = V(p) = V(c^\perp) = U(c),$$

which is a clopen subset since  $c$  is a component.  $\square$



Recall that a topological space  $X$  is *totally disconnected* if the only nonempty connected subsets of  $X$  are the singletons.

**Theorem 4.6.** *Suppose  $L$  is an  $M$ -frame. The following statements are equivalent.*

- (i)  $\text{Min}(L)^{-1}$  is zero-dimensional (has a base of clopen subsets).
- (ii) For every  $p \in \text{Min}(L)^{-1}$  and  $x \in \mathfrak{K}(L)$  with  $x \leq p$ , there exist  $y, z \in \mathfrak{K}(L)$  with  $x \leq y$  such that  $y \leq p$ , and furthermore,  $y \wedge z = 0$  and  $y \vee z$  is dense.
- (iii) For each pair of distinct minimal primes  $p$  and  $q$  there exists a component  $x \in \mathfrak{K}(L)$  such that  $x \leq p$  but  $x \not\leq q$ .
- (iv)  $\text{Min}(L)^{-1}$  is totally disconnected.
- (v) For all  $a, b \in \mathfrak{K}(L)$  with  $a \wedge b = 0$ , there exist  $x, y \in \mathfrak{K}(L)$ , with  $x \wedge y = 0$  and  $x \vee y$  is a unit such that  $a \leq x$  and  $b \leq y$ .

**Proof.** The structure of the proof is as follows. (i)  $\Rightarrow$  (ii)  $\Rightarrow$  (iii)  $\Rightarrow$  (iv)  $\Rightarrow$  (i) and (i)  $\Rightarrow$  (v)  $\Rightarrow$  (iii).

(i)  $\Rightarrow$  (ii) Let  $p \in \text{Min}(L)^{-1}$ , and let  $x \in \mathfrak{K}(L)$  with  $x \leq p$ . Therefore,  $p \in V(x)$ . Since  $\text{Min}(L)^{-1}$  is zero-dimensional, there exists a clopen subset  $K$  such that  $p \in K \subseteq V(x)$ . By Lemma 4.3 we have that  $K = V(y_1) = U(z)$ , for some  $y_1, z \in \mathfrak{K}(L)$ . Let  $y = y_1 \vee x$ . We first notice that since  $y_1, x \in \mathfrak{K}(L)$ , we have  $y \in \mathfrak{K}(L)$ , and also  $x \leq y$ , by definition of  $y$ . Now,

$$p \in V(x) \cap V(y_1) = V(x \vee y_1) = V(y),$$

and so  $y \leq p$ . It remains to show that  $y \wedge z = 0$  and  $y \vee z$  is dense in  $L$ . This follows from Lemma 4.3 since  $V(y) = V(y_1 \vee x) = V(y_1) \cap V(x) = V(y_1) = U(z)$ .

(ii)  $\Rightarrow$  (iii) Let  $p, q \in \text{Min}(L)^{-1}$  with  $p \neq q$ . Since  $\text{Min}(L)^{-1}$  is  $T_1$ , there exists a compact element  $x_1$  in  $L$  such that  $p \in V(x_1)$  and  $q \notin V(x_1)$ . So,  $x_1 \leq p$  and  $x_1 \not\leq q$ . By (ii) there exist  $y, z \in \mathfrak{K}(L)$  with  $x_1 \leq y$  such that  $y \leq p$ , and furthermore,  $y \wedge z = 0$  and  $y \vee z$  is dense. Letting  $x = x_1 \vee y$  we have  $x \in \mathfrak{K}(L)$ . We also observe that  $p \in V(y) \cap V(x_1) = V(y \vee x_1) = V(x)$ , and so  $x \leq p$ . However,  $x \not\leq q$  because  $x_1 \not\leq q$  and  $x_1 \leq x$ . Finally, we make the following two observations. Since  $x_1 \wedge z = 0$ , we have

$$x \wedge z = (x_1 \vee y) \wedge z = (x_1 \wedge z) \vee (y \wedge z) = 0,$$

and

$$(x \vee z)^\perp = (x_1 \vee y \vee z)^\perp = x_1^\perp \wedge (y^\perp \wedge z^\perp) = x_1^\perp \wedge 0 = 0.$$

Hence,  $x \vee z$  is dense.

(iii)  $\Rightarrow$  (iv) Let  $p, q \in \text{Min}(L)^{-1}$  with  $p \neq q$ . We verify that  $\{p, q\}$  is disconnected. From (iii), there exist  $x, y \in \mathfrak{K}(L)$  with  $x \leq p$ ,  $x \not\leq q$  such that  $x \wedge y = 0$  and  $x \vee y$  is dense. Using Lemma 4.3 we conclude that  $V(x) = U(y)$  and  $U(x) = V(y)$ . Since  $x \leq p$  and  $x \not\leq q$ , it follows that  $p \in V(x)$  and  $q \in U(x) = V(y)$ . Therefore, we have

two disjoint open subsets  $V(x) \cap \{p, q\}$  and  $V(y) \cap \{p, q\}$  which separates  $\{p, q\}$ . Consequently,  $\text{Min}(L)^{-1}$  is a totally disconnected space.

(iv)  $\Rightarrow$  (i) Choose a basic open subset  $V(x)$  for some compact element  $x$ , and let  $p \in V(x)$ . We want to find a clopen subset  $K$  such that  $p \in K \subseteq V(x)$ . Since  $\text{Min}(L)^{-1}$  is totally disconnected it follows that for all  $q \in U(x)$ ,  $q \neq p$ , there exists a clopen subset  $K_q$  such that  $q \in K_q$  but  $p \notin K_q$ . Therefore,

$$\text{Min}(L)^{-1} = \left( \bigcup_{q \in U(x)} K_q \right) \cup V(x),$$

where  $\{K_q : q \in U(x)\} \cup V(x)$  is an open cover of  $\text{Min}(L)^{-1}$ . Since  $\text{Min}(L)^{-1}$  is compact this collection has a finite subcover. So there exists finitely many elements,  $q_1, q_2, \dots, q_n \in U(x)$ , such that

$$\text{Min}(L)^{-1} = K_{q_1} \cup K_{q_2} \cup \dots \cup K_{q_n} \cup V(x).$$

Letting  $K_1 = K_{q_1} \cup K_{q_2} \cup \dots \cup K_{q_n}$ , we observe that  $K_1$  is a clopen subset and  $\text{Min}(L)^{-1} = K_1 \cup V(x)$ . Also  $p \notin K_{q_i}$ , for all  $i$ , and hence  $p \notin K_1$ . If  $K = \text{Min}(L)^{-1} \setminus K_1$ , then  $K$  is a clopen subset and  $p \in K$ . Finally if  $q \in K$ , then  $q \notin K_i$ , for all  $i$  which implies that  $q \in V(x)$ . Hence,  $K \subseteq V(x)$ .

Thus we have shown (i)  $\Rightarrow$  (ii)  $\Rightarrow$  (iii)  $\Rightarrow$  (iv)  $\Rightarrow$  (i).

(i)  $\Rightarrow$  (v) Let  $a$  and  $b$  be two compact elements of  $L$  with  $a \wedge b = 0$ . We then have  $U(a) \cap U(b) = U(a \wedge b) = \emptyset$ . So,  $U(a)$  and  $U(b)$  are two disjoint closed subsets of the compact zero-dimensional space  $\text{Min}(L)^{-1}$  and hence can be separated by a clopen subset. That is, there exists a clopen subset  $K$  such that  $U(a) \subseteq K$  and  $U(b) \cap K = \emptyset$ . By applying Lemma 4.3 there exists  $x_1, y_1 \in \mathfrak{K}(L)$  such that  $K = U(x_1) = V(y_1)$ . Using Lemma 4.3 we then have  $x_1 \wedge y_1 = 0$  and  $x_1 \vee y_1$  is dense. Let us consider  $x = a \vee x_1$  and  $y = b \vee y_1$ . We notice that  $x, y \in \mathfrak{K}(L)$  and clearly  $a \leq x$  and  $b \leq y$ . It just remains to show that  $x \wedge y = 0$  and  $x \vee y$  is dense. To prove this we first observe  $U(a) \subseteq K = V(y_1)$  implies  $U(a) \cap U(y_1) = \emptyset$ , which says that  $a \wedge y_1 = 0$ . Similarly,  $U(b) \cap U(x_1) = \emptyset$  tells us  $b \wedge x_1 = 0$ . Thus, we have

$$x \wedge y = (a \vee x_1) \wedge (b \vee y_1) = (a \wedge b) \vee (x_1 \wedge b) \vee (a \wedge y_1) \vee (x_1 \wedge y_1) = 0,$$

since each term is zero. Finally, since  $x_1 \vee y_1$  is dense and  $x_1 \vee y_1 \leq x \vee y$ , it follows that  $x \vee y$  is also dense.

(v)  $\Rightarrow$  (iii) Let  $p, q \in \text{Min}(L)^{-1}$  with  $p \neq q$ . By Proposition 3.4 there exists  $a, b \in \mathfrak{K}(L)$  with  $a \wedge b = 0$  such that  $a \leq p$ ,  $a \not\leq q$ ,  $b \leq q$ , and  $b \not\leq p$ . Using (v) there exists  $x, y \in \mathfrak{K}(L)$ , with  $x \wedge y = 0$ ,  $x \vee y$  dense, such that  $a \leq x$  and  $b \leq y$ . We further notice that  $x \leq p$  since  $y \not\leq p$ , and  $x \not\leq q$  since  $a \not\leq q$ .  $\square$

**Corollary 4.7.**  $\text{Min}(L)^{-1}$  is a Boolean space if and only if  $L$  satisfies the condition of Theorem 4.6.

Since a zero-dimensional,  $T_1$ -space is necessarily a Hausdorff space it follows that if  $L$  satisfies the equivalent conditions of Theorem 4.6, then  $\text{Min}(L)^{-1}$  is Hausdorff. We will characterize this property for  $\text{Min}(L)^{-1}$ ; the topological property of normality shall play a pivotal role. First we prove an useful lemma.

**Lemma 4.8.** *Suppose  $L$  is an  $M$ -frame.  $\text{Min}(L)^{-1}$  is a normal space if and only if every pair of disjoint closed sets can be separated by disjoint basic open sets.*

**Proof.** ( $\Leftarrow$ ) It is clear.

( $\Rightarrow$ ) Let  $C$  and  $D$  be disjoint closed sets. There exist disjoint open sets  $V_1$  and  $V_2$  such that  $C \subseteq V_1$  and  $D \subseteq V_2$ . It follows that  $V_1 = \bigcup \{V(c_\alpha) : c_\alpha \in K_1 \subseteq \mathfrak{K}(L)\}$  and  $V_2 = \bigcup \{V(d_\beta) : d_\beta \in K_2 \subseteq \mathfrak{K}(L)\}$ . Furthermore,  $C$  and  $D$  are closed subsets of a compact space, and therefore are compacts. Since  $C \subseteq V_1 = \bigcup \{V(c_\alpha) : c_\alpha \in K_1 \subseteq \mathfrak{K}\}$ , there exists finitely many compact elements  $c_1, \dots, c_n \in K_1$  such that  $C \subseteq V(c_1) \cup \dots \cup V(c_n) = V(c) \subseteq V_1$ , where  $c = c_1 \wedge \dots \wedge c_n \in \mathfrak{K}(L)$ . Similarly,  $D \subseteq V(d) \subseteq V_2$ , for some  $d \in \mathfrak{K}(L)$ . Finally,  $V(c) \cap V(d) = \emptyset$ .  $\square$

**Theorem 4.9.** *The following are equivalent for an  $M$ -frame  $L$ .*

- (i)  $\text{Min}(L)^{-1}$  is Hausdorff.
- (ii)  $\text{Min}(L)^{-1}$  is normal.
- (iii) *Given any  $a, b \in \mathfrak{K}(L)$  with  $a \wedge b = 0$ , there exist  $c, d \in \mathfrak{K}(L)$  such that  $a \leq d^\perp, b \leq c^\perp$ , while  $c \vee d$  is a unit.*
- (iv) *Given any  $a, b \in \mathfrak{K}(L)$  with  $a \wedge b = 0$ , there exist  $c, d \in \mathfrak{K}(L)$  such that  $a \leq c, b \leq d, a \wedge d = 0 = c \wedge b$  while  $c \vee d$  is a unit.*
- (v)  $\text{Min}(L)^{-1}$  is regular.
- (vi) *Given any  $c \in \mathfrak{K}(L)$  and  $p \in \text{Min}(L)^{-1}$  with  $c \leq p$ , there exists  $x, y \in \mathfrak{K}(L)$  such that  $c \leq x^\perp \leq y^{\perp\perp} \leq p$ .*

**Proof.** (i)  $\Rightarrow$  (ii) It is clear, since  $\text{Min}(L)^{-1}$  is a compact, Hausdorff space.

(ii)  $\Rightarrow$  (iii) Let  $a, b \in \mathfrak{K}(L)$  with  $a \wedge b = 0$ . So,  $U(a) \cap U(b) = \emptyset$ , where  $U(a)$  and  $U(b)$  are closed subsets of  $\text{Min}(L)^{-1}$ . Since  $\text{Min}(L)^{-1}$  is assumed to be normal  $U(a)$  and  $U(b)$  can be separated by disjoint basic open sets, using Lemma 4.8. This means, there exist  $c, d \in \mathfrak{K}(L)$  such that  $U(a) \subseteq V(c)$ ,  $U(b) \subseteq V(d)$ , and  $V(c) \cap V(d) = \emptyset$ . Observe that  $U(a) \subseteq V(c)$  means  $U(a) \cap U(c) = \emptyset$ . Consequently  $a \wedge c = 0$ , or in other words,  $a \leq c^\perp$ . Similarly,  $b \leq d^\perp$ . Finally,  $c \vee d$  is dense (and thus a unit) due to the disjointness of  $V(c)$  and  $V(d)$ .

(iii)  $\Rightarrow$  (i) Let  $p$  and  $q$  be two distinct minimal prime elements of  $L$ . Using Proposition 3.4 there exist  $a, b \in \mathfrak{K}(L)$  with  $a \wedge b = 0$  such that  $p \in V(a) \setminus V(b)$  and  $q \in V(b) \setminus V(a)$ . Because of (iii) there exist  $c, d \in \mathfrak{K}(L)$  with  $a \wedge c = 0 = b \wedge d$ , and  $c \vee d$  is dense. Note that  $b \not\leq p$  implies that  $d \leq p$ , thence  $p \in V(d)$ . Similarly,  $a \not\leq q$  implies that  $q \in V(c)$ . Also,  $V(c) \cap V(d) = \emptyset$ . Hence,  $p$  and  $q$  are separated by disjoint open sets.

(iii)  $\Leftrightarrow$  (iv) Observe that (iv)  $\Rightarrow$  (iii) follows immediately. For (iii)  $\Rightarrow$  (iv), consider  $c' = c \vee a$  and  $d' = d \vee b$ . It will follow that  $a \leq c'$ ,  $b \leq d'$ ,  $a \wedge d' = 0 = b \wedge c'$ , and  $c' \vee d'$  is a unit.

We have proved the equivalence of conditions (i)–(iv).

(v)  $\Rightarrow$  (vi) Let  $p \in \text{Min}(L)^{-1}$  and  $c \in \mathfrak{K}(L)$  with  $c \leq p$ . So,  $p \notin U(c)$ , which is a closed subset with respect to the inverse topology. Since  $\text{Min}(L)^{-1}$  is a regular space, there exists basic open sets  $V(x)$  and  $V(y)$  for  $x, y \in \mathfrak{K}(L)$  such that  $p \in V(y)$ ,  $U(c) \subseteq V(x)$ , and  $V(x) \cap V(y) = \emptyset$ . It follows that  $y^{\perp\perp} \leq p$ , since  $y \leq p$ ;  $c \leq x^\perp$ , since  $U(c) \cap V(x) = \emptyset$ , and  $x \vee y$  is dense, concluding that  $x^\perp \leq y^{\perp\perp}$ . Combining, we have  $c \leq x^\perp \leq y^{\perp\perp} \leq p$ .

(vi)  $\Rightarrow$  (v) Let  $p \in \text{Min}(L)^{-1}$  and  $c \in \mathfrak{K}(L)$  with  $p \notin U(c)$ . Consequently  $c \leq p$ . There exists  $x, y \in \mathfrak{K}(L)$  such that  $c \leq x^\perp \leq y^{\perp\perp} \leq p$ . We leave it to the interested reader to show that  $p$  and  $U(c)$  can be separated by basic open sets  $V(x)$  and  $V(y)$ . Therefore,  $\text{Min}(L)^{-1}$  is a regular topological space.

(v)  $\Leftrightarrow$  (i) Notice that with  $T_1$  property, a normal space is a regular space and a regular space is Hausdorff.  $\square$

**Corollary 4.10.** *Suppose  $\text{Min}(L)^{-1}$  is Hausdorff. Then  $L$  possesses a unit.*

**Example 4.11.** Let  $X$  be an infinite set and set  $L = \mathcal{P}(X)$  be the Boolean frame of subsets of  $X$ . Then  $L$  is an  $M$ -frame which does not possess a unit, and therefore,  $\text{Min}(L)^{-1}$  is not Hausdorff.

We provide an example to show that the conditions on Theorems 4.6 and 4.9 are not equivalent.

**Example 4.12.** It is known that for an  $F$ -space  $X$ ,  $\text{Min}(C(X))^{-1}$  is zero-dimensional if and only if  $X$  is strongly zero-dimensional (see [15]). It is also known that for an  $F$ -space  $X$ ,  $\text{Min}(C(X))^{-1}$  is homeomorphic to  $\beta(X)$ . Therefore, if  $X$  is an  $F$ -space which is not strongly zero-dimensional, for example  $\beta\mathbb{R}^+ \setminus \mathbb{R}^+$ , then  $\text{Min}(C(X))^{-1}$  is Hausdorff but not zero-dimensional. Therefore, the frame of convex lattice-ordered subgroups of  $C(X)$  satisfies the equivalent conditions on Theorem 4.9, but does not satisfies the conditions on Theorem 4.6.

Recall from [14] that an  $M$ -frame  $L$  satisfies  $\text{Reg}(4)$  if and only if for any two disjoint compacts  $a, b \in \mathfrak{K}(L)$ ,  $a^\perp \vee b^\perp = 1$ . Moreover, this is equivalent to the statement that for distinct minimal primes  $p, q \in \text{Min}(L)$ ,  $p^\perp \vee q^\perp = 1$ . We now demonstrate that if  $L$  satisfies  $\text{Reg}(4)$ , then  $\text{Min}(L)^{-1}$  is Hausdorff.

**Proposition 4.13.** *Suppose that  $L$  is  $M$ -frame which possesses a unit. If  $L$  satisfies  $\text{Reg}(4)$ , then  $L$  satisfies the equivalent conditions of Theorem 4.9. Furthermore, if 1 is the only unit in  $L$ , then the converse follows.*

**Proof.** Let  $u \in L$  be a compact dense element of  $L$  and suppose  $a, b \in \mathfrak{K}(L)$  are disjoint compact elements. By  $\text{Reg}(4)$   $a^\perp \vee b^\perp = 1$  and so

$$(a^\perp \wedge u) \vee (b^\perp \wedge u) = u.$$

Set  $a_0 = a^\perp \wedge u$  and  $b_0 = b^\perp \wedge u$ . Writing  $a_0$  and  $b_0$  as joins of compact elements, it follows that there exists  $c, d \in \mathfrak{K}(L)$  such that  $c \leq a_0$ ,  $d \leq b_0$ , and  $c \vee d = u$ . Therefore,  $a \leq c^\perp$ ,  $b \leq d^\perp$ , and  $c \vee d$  is a unit.

To show that converse, assume further that 1 is the only unit in  $L$ . Let  $a, b \in \mathfrak{K}(L)$  with  $a \wedge b = 0$ . There exists  $c, d \in \mathfrak{K}(L)$  such that  $a \leq d^\perp$ ,  $b \leq c^\perp$ , and  $c \vee d = 1$ . Consequently,  $a^\perp \vee b^\perp \geq c \vee d = 1$ .  $\square$

The next theorem characterizes when  $\text{Min}(L)$  is a compact extremally disconnected space. This result is a translation of [16, Speed's Corollary 6.6]. Notice the strengthening of Theorem 4.2. We include a proof for completeness sake.

**Theorem 4.14.** *Suppose  $L$  is an  $M$ -frame.  $\text{Min}(L)$  is a Stone space (compact, Hausdorff, extremally disconnected) if and only if for each  $x \in L$  there exists  $y \in \mathfrak{K}(L)$  with  $x \wedge y = 0$  such that  $x \vee y$  is dense.*

**Proof.** Suppose that for each  $x \in L$  there exists  $y \in \mathfrak{K}(L)$  with  $w \wedge y = 0$  such that  $x \vee y$  is dense. By Theorem 4.2,  $\text{Min}(L)$  is compact. To show that  $\text{Min}(L)$  is extremally disconnected consider an arbitrary open set of  $\text{Min}(L)$ , say  $O$ . By Proposition 3.2  $O = U(x)$  for some  $x \in L$ , and by Lemma 3.5  $\text{cl}(O) = V(x^\perp)$ . Our hypothesis allows us to choose a compact  $y \in \mathfrak{K}(L)$  such that  $x \wedge y = 0$  and  $x \vee y$  is dense. We claim that  $V(x^\perp) = V(y)$  which since  $y$  is compact is a clopen subset of  $\text{Min}(L)$ .

Let  $p \in V(x^\perp)$ . Since  $y \leq x^\perp \leq p$  it follows that  $p \in V(y)$ . Next,  $x^\perp \wedge y^\perp = (x \vee y)^\perp = 0$  implies that  $\emptyset = U(x^\perp) \cap U(y^\perp) = U(x^\perp) \cap V(y)$  where the last equality holds by (7) of Lemma 3.1. This means that  $V(y) \subseteq V(x^\perp)$ , the desired reverse containment. Therefore,  $\text{Min}(L)$  is extremally disconnected.

Conversely, suppose that  $\text{Min}(L)$  is a Stone space. This means that given  $x \in L$ ,  $V(x^\perp) = \text{cl}(U(x))$  is a clopen subset of  $\text{Min}(L) = \text{Min}(L)^{-1}$ . Furthermore,  $U(x^\perp)$  is also a clopen subset. Applying Proposition 3.2, it follows that  $U(x^\perp) = U(y)$  for some compact  $y \in \mathfrak{K}(L)$ . We leave it to the interested reader to verify that  $x \wedge y = 0$  while  $x \vee y$  is dense.  $\square$

Finally, we will view  $\text{Min}(L)^{-1}$  from a different angle, considering a certain subframe of the  $M$ -frame  $L$ , which is described below.

Let  $L$  be an  $M$ -frame which possesses a unit. Let us define

$$L' = \left\{ l \in L : l = \bigvee_{\alpha} c_{\alpha}^{\perp}, \ c_{\alpha} \in K \subseteq \mathfrak{K}(L) \right\}.$$

Notice the following properties of  $L'$ :

- (1)  $1 = 0^\perp \in L'$ ;
- (2)  $0 = u^\perp \in L'$ , where  $u$  is an unit of  $L$ ;
- (3)  $L'$  is closed under arbitrary supremum;
- (4)  $L'$  is closed under finite infimum, since  $c^\perp \wedge d^\perp = (c \vee d)^\perp$ .

It follows then that  $L'$  is a subframe of  $L$ .

We want to make a remark that if  $L$  does not possess a unit, we can modify the definition of  $L'$  by  $L' = \{l \in L : l = \bigvee_\alpha c_\alpha^\perp, c_\alpha \in K \subseteq \mathfrak{K}(L)\} \cup \{0\}$ , thus having a subframe of  $L$ .

**Proposition 4.15.** *The open (closed) subsets of  $\text{Min}(L)^{-1}$  are precisely of the form  $U(l)$  ( $V(l)$ ) for some  $l \in L'$ .*

**Proof.** Let  $V$  be an open subset in the inverse topology. Then there exists  $x_\alpha \in \mathfrak{K}(L)$  such that  $V = \bigcup_\alpha V(x_\alpha)$ . We thus have,

$$V = \bigcup_\alpha V(x_\alpha) = \bigcup_\alpha U(x_\alpha^\perp) = U\left(\bigvee_\alpha x_\alpha^\perp\right) = U(l),$$

where  $l \in L'$ .

Conversely, let  $l \in L'$ . There exists  $c_\alpha \in \mathfrak{K}(L)$  such that  $l = \bigvee_\alpha c_\alpha^\perp$ . Consequently,

$$U(l) = U\left(\bigvee_\alpha c_\alpha^\perp\right) = \bigcup_\alpha U(c_\alpha^\perp) = \bigcup_\alpha V(c_\alpha),$$

which is an open subset of  $\text{Min}(L)^{-1}$ . □

From now on we will use  $L'$  to denote the subframe of an  $M$ -frame  $L$  as described above. Also, we will assume that  $L$  always possesses a unit.

We can restate Theorem 4.2 with some new equivalences:

**Theorem 4.16.** *Suppose  $L$  is an  $M$ -frame. The following statements are equivalent.*

- (i) *The Hull-kernel topology on  $\text{Min}(L)$  is compact.*
- (ii)  *$\text{Min}(L) = \text{Min}(L)^{-1}$ .*
- (iii) *For each  $x \in \mathfrak{K}(L)$  there exist  $y \in \mathfrak{K}(L)$  such that  $x \wedge y = 0$  and  $x \vee y$  is a unit.*
- (iv) *For every  $x \in \mathfrak{K}(L)$  there exists  $l \in L'$  such that  $x^\perp \wedge l = 0$  and  $x^\perp \vee l$  is dense. Furthermore,  $l = c^\perp$ , for some  $c \in \mathfrak{K}(L)$ .*
- (v) *For every  $x \in \mathfrak{K}(L)$  there exists  $c \in \mathfrak{K}(L)$  such that  $x^\perp = (c^\perp)^\perp$ , where  $(\cdot)^\perp$ s are with respect to the frame  $L$ .*

**Proof.** The equivalences of (i), (ii), and (iii) are already known (see Theorem 4.2).

(ii)  $\Rightarrow$  (iv) Let  $x \in \mathfrak{K}(L)$ . Since the inverse topology is equal to the Hull-kernel topology, it follows that  $U(x)$  is open in  $\text{Min}(L)^{-1}$ . Using Proposition 4.15, there exists some  $l \in L'$  such that  $V(x^\perp) = U(x) = U(l)$ . Therefore,  $V(x^\perp) \cap V(l) = \emptyset$  and  $V(x^\perp) \cup V(l) = \text{Min}(L)$ . Consequently,  $x^\perp \vee l$  is dense and  $x^\perp \wedge l = 0$ .

Furthermore, since  $l \in L'$ , we have  $l = \bigvee c_\alpha^\perp$ , for some  $c_\alpha \in \mathfrak{K}(L)$ . Therefore,  $U(x) = U(l) = \bigcup V(c_\alpha)$ . Since  $\text{Min}(L)$  is compact and  $U(x)$  is a closed subset in the Hull-kernel topology,  $U(x)$  is compact. Moreover,  $V(c_\alpha)$  are open subsets in the Hull-kernel topology. Hence,

$$U(l) = U(x) = V(c_1) \cup \cdots \cup V(c_n) = V(c_1 \wedge \cdots \wedge c_n) = V(c) = U(c^\perp),$$

for some  $c \in \mathfrak{K}(L)$ , since  $L$  satisfies the FIP. Thus, we can assume  $l = c^\perp$ , for some  $c \in \mathfrak{K}(L)$ .

(iv)  $\Rightarrow$  (v) Let  $x \in \mathfrak{K}(L)$  and  $c \in \mathfrak{K}(L)$  with  $x^\perp \wedge c^\perp = 0$  and  $x^\perp \vee c^\perp$  is dense. Therefore,  $x^{\perp\perp} \wedge c^{\perp\perp} = 0$ , which implies further that  $c^{\perp\perp} \leq x^\perp$ . Also,  $x^\perp \leq c^{\perp\perp}$  follows from the first part, concluding that  $x^\perp = (c^\perp)^\perp$ .

(v)  $\Rightarrow$  (ii) Let  $x \in \mathfrak{K}(L)$ . There exists  $c \in (L)$  such that  $x^\perp = c^{\perp\perp}$ . Notice that  $x \wedge c = 0$  and  $x \vee c$  is a unit. Consequently,  $U(x) = V(c) = U(c^\perp)$ , which is open in the inverse topology.  $\square$

We observe an interesting fact. If  $\mathfrak{K}(L') = \{c^\perp : c \in \mathfrak{K}(L)\}$ , then  $L'$  is an algebraic frame satisfying the FIP. In that case, a base for the Hull-kernel topology on  $\text{Min}(L')$  is

$$\mathfrak{B}_{\text{Min}(L')} = \{U'(c^\perp) : c \in \mathfrak{K}(L)\},$$

where  $U'(c^\perp) = \{p \in \text{Min}(L') : c^\perp \not\leq p\}$ .

Recall that if  $L$  and  $M$  are  $M$ -frames and  $L$  is a subframe of  $M$ , then  $L \hookrightarrow M$  is a *rigid extension* if for each  $x \in \mathfrak{K}(M)$  there exists  $c \in \mathfrak{K}(L)$  such that  $c^\perp = x^\perp$ , in the frame  $M$  (see [4]).

**Corollary 4.17.** *Let  $L'$  be the subframe of  $L$  and  $\mathfrak{K}(L') = \{c^\perp : c \in \mathfrak{K}(L)\}$ .  $L' \hookrightarrow L$  is a rigid extension if and only if  $\text{Min}(L) = \text{Min}(L')^{-1}$ .*

Next we give some examples.

**Example 4.18.** Let  $L$  be the discrete topology on  $\mathbb{N}$ , then  $L$  is an  $M$ -frame, but without any unit. By definition,  $L' = \{\emptyset\} \cup \{\text{cofinite topology on } \mathbb{N}\}$ . Therefore,  $L'$  is also an  $M$ -frame. Notice that  $\text{Min}(L)$  is not compact and  $L' \hookrightarrow L$  is not a rigid extension.

**Example 4.19.** Let  $L$  be the cofinite topology on  $\mathbb{N}$ .  $L$  is an  $M$ -frame with a unit and  $L' = \{\mathbb{N}, \emptyset\}$  is also an  $M$ -frame. Observe that in this case  $\text{Min}(L)$  is compact and  $L' \hookrightarrow L$  is a rigid extension.

Recall that a topological space is *extremally disconnected* if the closure of every open set is clopen. Notice that if every  $p \in \text{Min}(L)^{-1}$  satisfies the property in Theorem 4.5, then  $\text{Min}(L)^{-1}$  is an extremally disconnected space. We will characterize when  $\text{Min}(L)^{-1}$  is extremally disconnected.

We start with the following proposition.

**Proposition 4.20.** *Let  $l \in L'$  and  $p \in \text{Min}(L)^{-1}$ .  $p \in \text{cl}(U(l))$  if and only if for all  $x \in \mathfrak{K}(L)$  with  $x \vee l \leq p$ ,  $x^\perp \wedge l \neq 0$ .*

**Proof.** ( $\Rightarrow$ ) Let  $p \in \text{cl}(U(l))$  and  $x \in \mathfrak{K}(L)$  with  $x \vee l \leq p$ . Therefore,  $p \in V(x)$  and  $p \in \text{cl}(U(l)) \setminus U(l)$ . Since  $V(x)$  is an open neighborhood of  $p$ , it follows that  $U(x^\perp \wedge l) = U(x^\perp) \cap U(l) = V(x) \cap U(l) \neq \emptyset$ . Consequently,  $x^\perp \wedge l \neq 0$ .

( $\Leftarrow$ ) If  $p \notin \text{cl}(U(l))$ , then  $l \leq p$ . We claim that  $p \in \text{cl}(U(l))$ . To that end, let  $p \in V(x)$  for some  $x \in \mathfrak{K}(L)$ . Therefore,  $x \vee l \leq p$ . Using the hypotheses it follows that  $x^\perp \wedge l \neq 0$ . Hence,

$$V(x) \cap U(l) = U(x^\perp) \cap U(l) \neq \emptyset,$$

thereby proving that every basic open neighborhood of  $p$  intersects  $U(l)$ , hence  $p \in \text{cl}(U(l))$ .  $\square$

**Corollary 4.21.** *Let  $l \in L'$ .  $U(l)$  is a dense open subset in the inverse topology if and only if for all  $x \in \mathfrak{K}(L)$ ,  $x^\perp \wedge l \neq 0$ .*

**Proof.** Observe that  $U(l)$  is dense if and only if every basic open set  $V(x)$ , for  $x \in \mathfrak{K}(L)$ , in  $\text{Min}(L)^{-1}$  intersects  $U(l)$ . That is if and only if  $x^\perp \wedge l \neq 0$ , for every  $x \in \mathfrak{K}(L)$ .  $\square$

Let  $l \in L$ . Define  $l^* = \bigvee \{y \in L' : l \wedge y = 0\}$ , then  $l^* \in L'$  and  $l^* \leq l^\perp$ . Notice that  $l \wedge l^* = 0$ .

**Theorem 4.22.** *For every  $l \in L'$ ,  $\text{cl}(U(l)) = V(l^*)$  with respect to the inverse topology.*

**Proof.** Since  $l^* \in L'$ , it follows that  $V(l^*)$  is a closed subset with respect to the inverse topology. Let  $p \in \text{Min}(L)^{-1}$  with  $p \in U(l)$ , then  $l \not\leq p$ . Since  $p$  is prime,  $l^\perp \leq p$ . Therefore,  $l^* \leq l^\perp \leq p$ , concluding that  $p \in V(l^*)$ . Since  $U(l) \subseteq V(l^*)$  and  $V(l^*)$  is closed,  $\text{cl}(U(l)) \subseteq V(l^*)$  in  $\text{Min}(L)^{-1}$ . To show the reverse inclusion let  $p \in V(l^*)$  and  $c \in \mathfrak{K}(L)$  with  $p \in V(c)$ , which is a basic open set in  $\text{Min}(L)^{-1}$ . We claim that  $V(c) \cap U(l) \neq \emptyset$ . Suppose by contradiction,  $V(c) \cap U(l) = \emptyset$ . So,  $U(c^\perp \wedge l) = U(c^\perp) \cap U(l) = V(c) \cap U(l) = \emptyset$ . Consequently,  $c^\perp \wedge l = 0$ , using Lemma 3.1. Since  $c^\perp \in L'$ , we have  $c^\perp \leq l^* \leq p$ . This is a contradiction by Lemma 2.4. Therefore,  $\text{cl}(U(l)) = V(l^*)$  in  $\text{Min}(L)^{-1}$ .  $\square$



**Theorem 4.23.** *Let  $L$  be an  $M$ -frame and  $L'$  be the subframe of  $L$  as defined above. The following are equivalent with respect to the inverse topology,  $\text{Min}(L)^{-1}$ .*

- (1)  $\text{Min}(L)^{-1}$  is an extremally disconnected space.
- (2) For every  $l \in L'$ ,  $V(l^*)$  is a clopen subset.
- (3) For every  $l \in L'$  there exists a component  $x \in \mathfrak{K}(L)$  such that  $V(l^*) = V(x)$
- (4) For every  $l \in L'$  there exists a component  $y \in \mathfrak{K}(L)$  such that  $l^* \wedge y = 0$  and  $l^* \vee y$  is dense in  $L$ .

At this point we are unable to provide a characterization similar to Theorem 4.14, for  $\text{Min}(L)^{-1}$  to be a Stone space. We speculate that restating every property of  $\text{Min}(L)^{-1}$  in terms of  $L'$  might help establish the characterization. We can only say that  $\text{Min}(L)^{-1}$  is a Stone space precisely when  $L$  satisfies the properties of Theorem 4.9 and Theorem 4.14.

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## References

- [1] B. Banachewski, Radical ideals and coherent frames, *Comment. Math. Univ. Carolin.* **37**(2) (1996) 349–370.
- [2] B. Banaschewski, Gelfand and exchange rings: Their spectra in pointfree topology, *Arab. J. Sci. Eng. Sect. C Theme Issues* **25**(2) (2000) 3–22.
- [3] B. Banaschewski, Archimedean lattices, *Algebr. Univ.* **3** (1973) 247–260.
- [4] P. Bhattacharjee, Rigid extensions of algebraic frames, *Algebr. Univ.* **62** (2009) 133–149.
- [5] A. Bigard, K. Keimel and S. Wolfenstein, *Groupes et Anneaux Reticulés*, Lecture Notes in Mathematics, Vol. 608 (Springer-Verlag, 1977).
- [6] P. Conrad and J. Martinez, Complemented lattice-ordered groups, *Indagt. Math. (N.S.)* **1**(3) (1990) 281–298.
- [7] M. Darnel, *Theory of Lattice-Ordered Groups*, Monographs and Textbooks in Pure and Applied Mathematics, Vol. 187 (Marcel Dekker, New York, 1995).
- [8] L. Gillman and M. Jerison, *Rings of Continuous Functions*, Graduate Texts in Mathematics, Vol. 43 (Springer-Verlag, Berlin, 1976).
- [9] A. Hager and J. Martinez, Fraction-dense algebras and spaces, *Canadian J. Math.* **45**(5) (1993) 977–996.
- [10] M. Henriksen and M. Jerison, The space of minimal prime ideals of a commutative ring, *Trans. Amer. Math. Soc.* **115** (1965) 110–130.
- [11] W. Iberkleid and W. McGovern, A natural equivalence for the category of coherent frames, *Algebra Universalis* **62**(2–3) (2009) 247–258.
- [12] M. Knox, R. Levy, W. McGovern and J. Shapiro, Generalizations of complemented rings and applications to rings of continuous functions, *J. Algebra Appl.* **8**(1) (2009) 17–40.

- [13] J. Martinez, Archimedean lattices, *Algebr. Univ.* **3** (1973) 247–260.
- [14] J. Martinez and E. Zenk, When an algebraic frame is regular, *Algebr. Univ.* **50** (2003) 231–257.
- [15] W. McGovern, Neat rings, *J. Pure Appl. Algebra* **205** (2006) 243–265.
- [16] T. Speed, Spaces of ideals of distributive lattices II. Minimal prime ideals, *J. Austral. Math. Soc.* **18** (1974) 54–72.