THE CLASSICAL RING OF QUOTIENTS OF $C_c(X)$

PAPIYA BHATTACHARJEE[1], MICHELLE L. KNOX[2], AND WARREN WM. MCGOVERN[3]

Abstract. We construct the classical ring of quotients of the algebra of continuous real-valued functions with countable range. Our construction is a slight modification of the construction given in [3]. Dowker's example shows that the two constructions can be different.

1. Introduction

Our aim here is two-fold. We aim to add to the growing knowledge regarding the ring of continuous functions of countable range on the space X, denoted by $C_c(X)$, while also supplying a correction to representation of the classical ring of quotients of $C_c(X)$, denoted $q_c(X)$. In this section we supply the relevant definitions and concepts. In Section 2, we construct $q_c(X)$ in the same vein as the Representation Theorems of Fine, Gillman, and Lambek [2]. The third section is devoted to studying a specific space which shows why the construction in Section 2 is needed. We also include an example of Mysior that is peculiar in its own right.

Throughout this paper, X will denote a zero-dimensional Hausdorff space, that is, a Hausdorff space with a base of clopen sets. The ring of all real-valued continuous functions on X is denoted by $C(X)$, and the subring of $C(X)$ consisting of those functions with countable range is denoted by $C_c(X)$. We may restrict to the class of zero-dimensional spaces because, as it is argued in [3] and [12], for any space Y there is a zero-dimensional Hausdorff space X such that $C_c(Y)$ and $C_c(X)$ are isomorphic as rings.

One of the first results concerning $C_c(X)$ was by W. Rudin [14] who showed that a compact space X satisfies $C(X) = C_c(X)$ precisely when X is scattered. In [9] the authors studied general zero-dimensional spaces for which $C_c(X) = C(X)$, calling such a space *functionally countable*. Recently, there has been an interest in $C_c(X)$ as a ring in its own right.

Recall that for $f \in C(X)$, $Z(f)$ denotes its zero-set:

$$Z(f) = \{x \in X : f(x) = 0\}.$$

The set-theoretic complement of a zero-set is known as a cozero-set and we label this set by $\mathrm{coz}(f)$. We denote the collection of all cozero-sets by $\mathrm{coz}(X)$, and use $\mathrm{clop}(X)$ to denote the boolean algebra of clopen subsets of X. When we consider cozero-sets arising from functions in $C_c(X)$, we get what is denoted in [6] by $\mathrm{clop}(X)_\sigma$: the set of all countable unions of clopen subsets, i.e. the set of all σ-clopen sets.

One of the main differences between $C(X)$ and $C_c(X)$ is the realization of the maximal ideal spaces. The Gelfand-Kolmogorov Theorem states $\mathrm{Max}(C(X))$ is homeomorphic to

Key words and phrases. rings of continuous functions, ring of quotients, zero-dimensional space.

the Stone-Čech compactification of X, denoted βX. On the other hand, the maximal ideal space of $C_c(X)$ is homeomorphic to the Banaschewski compactification of X, denoted $\beta_0 X$. (A proof can be modeled after Theorem 5.1 of [6].) The most well-known way of realizing βX is as the collection of all z-ultrafilters of X. A nice way to view $\beta_0 X$ is as the Stone dual of $\mathrm{clop}(X)$. In general, βX and $\beta_0 X$ are not homeomorphic. The next theorem characterizing when they are is well-known (see Chapter 6.2 of [1]).

Theorem 1.1. *For a zero-dimensional space X the following statements are equivalent.*

(1) $\beta X = \beta_0 X$
(2) *Every cozero-set is a σ-clopen set.*
(3) *βX is zero-dimensional.*

Zero-dimensional spaces X for which βX is zero-dimensional are known as *strongly zero-dimensional spaces*. In this short article we are interested in zero-dimensional spaces which are not strongly zero-dimensional. The third section deals with one of the most well-known such examples.

As for references, the text [4] is still pivotal. We also mention [13] and [1] for topological considerations, definitions, and concepts not explicitly discussed here. We end this section with a remark about σ-clopens.

Remark 1.2. It should be apparent that if $U \in \mathrm{clop}(X)_\sigma$, then there is some $f \in C_c(X)$ such that $\mathrm{coz}(f) = U$. A sketch of this proof is as follows. First write $U = \bigcup_n K_n$ as a disjoint union of a countable number of clopen sets. Next, consider the function (call it f) that maps K_i to $\frac{1}{i}$ and the rest of X to 0. This function is continuous and is an element of $C_c(X)$. Finally, $\mathrm{coz}(f) = U$.

2. Classical Ring of Quotients

In [12] the authors constructed both the classical ring of quotients and the maximum ring of quotients of $C_c(X)$. They modeled their construction after the Representation Theorems of Fine, Gillman, and Lambek [2]. We recall these after we set up some notation. Let

$$\mathfrak{G}(X) = \{U \subseteq X : U \text{ is a dense open subset of } X\}$$

and

$$\mathfrak{G}_0(X) = \{O \subseteq X : O \text{ is a dense cozero-set of } X\}.$$

We let $q(X)$ and $q_c(X)$ denote the classical rings of quotients of $C(X)$ and $C_c(X)$, respectively. We let $Q(X)$ and $Q_c(X)$ denote the maximum rings of quotients of $C(X)$ and $C_c(X)$, respectively.

Theorem 2.1 ([2])**.** *Suppose X is a Tychonoff space. Then*

$$q(X) = \lim_{O \in \mathfrak{G}_0(X)} C(O) \text{ and } Q(X) = \lim_{U \in \mathfrak{G}(X)} C(U).$$

In the above theorem the use of the limit is to describe that the rings are direct limits of rings of continuous functions. This direct limit can be described as the union of the rings $C(U)$ modulo the equivalence that $f_1 \in C(U_1), f_2 \in C(U_2)$ are equivalent if they agree on $U_1 \cap U_2$.

In [12] the authors classified $q_c(X)$ and $Q_c(X)$ for zero-dimensional X as

$$q_c(X) = \lim_{O \in \mathfrak{G}_0(X)} C_c(O)$$

and

$$Q_c(X) = \lim_{U \in \mathfrak{G}(X)} C_c(U).$$

Unfortunately, we believe that the classification of $q_c(X)$ is incorrect. We now construct the classical ring of quotients of $C_c(X)$. Let

$$\mathfrak{G}_\sigma(X) = \{K \in X : K \text{ is a dense } \sigma\text{-clopen set of } X\}.$$

For the purpose of comparison, we will denote $q'(X) = \lim_{O \in \mathfrak{G}_0(X)} C_c(O)$.

Theorem 2.2. *Let X be a zero-dimensional space. Then*

$$q_c(X) = \lim_{U \in \mathfrak{G}_\sigma(X)} C_c(U).$$

Proof. Given that $X \in \mathfrak{G}_\sigma(X)$ we have an embedding $C_c(X) \leq q_c(X) \leq q'(X)$. First let $f \in C_c(X)$ be a non zero-divisor element of $C_c(X)$. We claim that $\operatorname{coz}(f)$ is a dense subset of X. If not, then $X \smallsetminus cl_X \operatorname{coz}(f)$ is a nonempty open subset and hence there is a nonempty clopen set of X which is disjoint from $cl_X \operatorname{coz}(f)$. The characteristic function on said clopen set is a non-zero function belonging to $C_c(X)$ and annihilates f, a contradiction. Thus, $\operatorname{coz}(f) \in \mathfrak{G}_\sigma(X)$. Restricting f to $\operatorname{coz}(f)$ produces an element which is invertible in $C_c(\operatorname{coz}(f))$ and hence also in $q_c(X)$. So every regular element of $C_c(X)$ is invertible in $q_c(X)$. It follows (by a straightforward ring theoretic argument) that the classical ring of quotients of $C_c(X)$ is embedded inside of $q_c(X)$.

As for the reverse inclusion, the proof follows mutatis mutandi from the proof of the Representation Theorem of [2] (Theorem 2.6). This was attempted in Theorem 2.12 of [12]. The only error made there was that when taking a dense cozeroset U of X there might not be a $d \in C_c(X)$ such that $\operatorname{coz}(d) = U$. In fact, the only change needed from their proof is the modification we have suggested in using $\mathfrak{G}_\sigma(X)$ instead of $\mathfrak{G}_0(X)$. □

Remark 2.3. Observe that $q_c(X) \leq q'(X)$ since $\mathfrak{G}_\sigma(X) \subseteq \mathfrak{G}_0(X)$. One might question whether both direct limits produce the same rings. In the next section we will exhibit an example of a zero-dimensional space for which $q_c(X) < q'(X)$. Of course, if X is strongly zero-dimensional, then $\mathfrak{G}_0(X) = \mathfrak{G}_\sigma(X)$ and hence $q_c(X) = q'(X)$. Remark 3.10 discusses the equality and the question of whether such a space need be strongly zero-dimensional.

We end this section with some remarks and results whose proofs are in the same vein as above.

Corollary 2.4. *Let $\mathbb{F}$ be a proper subfield of $\mathbb{R}$. For a zero-dimensional space X, the classical ring of quotients of $C(X,\mathbb{F})$ is $\lim_{U\in\mathfrak{G}_\sigma(X)} C(U,\mathbb{F})$. In particular, the classical ring of quotients of $C(X,\mathbb{Q})$ is $q(X,\mathbb{Q}) = \lim_{U\in\mathfrak{G}_\sigma(X)} C(U,\mathbb{Q})$.*

3. A Counterexample

We let X be the space defined in 4V of [13]. We give a brief sketch of the construction. Let $\mathbf{W}$ denote the space of countable ordinals equipped with the interval topology. For an ordinal σ, we use $W(\sigma)$ to denote the set of ordinals smaller than σ. Notice that $W(\omega_1) = \mathbf{W}$.

Let $J = \mathbb{R} \smallsetminus \mathbb{Q}$. For $x \in J$, let $J_x = \{x + r : r \in \mathbb{Q}\}$ and $\mathcal{J} = \{J_x : x \in J\}$. Re-index $\mathcal{J}$ by $\mathcal{J} = \{J_\alpha : \alpha \in \mathbb{W}\}$ so that $J_\alpha \cap J_\beta = \emptyset$ whenever $\alpha \neq \beta$. For $\alpha < \omega_1$, let $U_\alpha = \mathbb{R} \smallsetminus \bigcup\{J_\beta : \alpha < \beta < \omega_1\}$, and let $X = \bigcup\{\{\alpha\} \times U_\alpha : \alpha < \omega_1\}$. Equip X with the subspace topology from $\mathbf{W} \times \mathbb{R}$; X is a Tychonoff space. The space X is similar to Dowker's Example from [1]. X is a classical example of a zero-dimensional space that is not strongly zero-dimensional. Another reference for this space is Exercise 16M of [4].

For notational purposes, let $X_\sigma = \{(\tau, r) \in X : \tau < \sigma\}$. In other words $X_\sigma = X \cap (W(\sigma) \times \mathbb{R})$. We denote the X-complement of X_σ by X'_σ and call such a set a *cofinal band of* X since $X'_\sigma = \{(\tau, r) \in X : \sigma \leq \tau\}$.

Since X is not strongly zero-dimensional, we know that not every cozeroset of X is a σ-clopen. We aim to convince the reader of three things. First, that if C is a σ-clopen of X, then either C or $X \smallsetminus C$ is a subset of X_σ for some $\sigma \in \mathbf{W}$. Second, if C is a σ-clopen, then $X \smallsetminus cl_X C$ is also a σ-clopen. Third, $q_c(X) < q'(X)$.

We let $\pi : X \to \mathbf{W}$ be the continuous projection map. We recall some subsets of X defined in Section 3 of [8].

Definition 3.1. Let K be a clopen subset of X such that $\pi(K)$ is cofinal in $\mathbf{W}$. Define the following sets for $r \in \mathbb{R}$ and $\epsilon > 0$:

$$S_r^\epsilon = \{\sigma \in \mathbf{W} : (\{\sigma\} \times (r - \epsilon, r + \epsilon)) \cap X \subseteq K\}$$

and

$$T_K = \{r \in \mathbb{R} : S_r^\epsilon \text{ is cofinal in } \mathbf{W} \text{ for some } \epsilon > 0\}.$$

Proposition 3.18 of [8] states T_K is an unbounded subset of $\mathbb{R}$. Here we show more: that T_K is an open subset of $\mathbb{R}$.

Proposition 3.2. *Suppose K is a clopen subset of X such that $\pi(K)$ is cofinal in $\mathbf{W}$. Then*

$$T_K = \{r \in \mathbb{R} : \textit{ there is a } \sigma_r \in \mathbf{W} \textit{ such that } \{(\tau, r) \in X : \sigma_r \leq \tau\} \subseteq K\}.$$

Proof. For $r \in \mathbb{R}$ notice that by the construction of X, if $(\sigma, r) \in X$, then for all $\sigma \leq \tau \in \mathbf{W}$, $(\tau, r) \in X$.

If $r \in T_K$, then choose $\sigma \in \mathbf{W}$ such that $(\sigma, r) \in X$. Set $Y = \{(\tau, r) \in X : \sigma \leq \tau\}$ and observe that Y is homeomorphic to $\mathbf{W}$. Let $g = \chi_K \in C(X)$ be the characteristic function on K; $K = \mathrm{coz}(g)$. The restriction of g to Y is constant on a tail (see Chapter

5 [4]), and $r \in T_K$, so there is some $\sigma_r \in \mathbf{W}$ such that for all $\sigma_r \leq \tau \in \mathbf{W}$, $(\tau, r) \in \mathrm{coz}(g) = K$.

Next, let $r \in \mathbb{R}$ have the property that there is $\sigma_r \in \mathbf{W}$ such that $\{(\tau, r) \in X : \sigma_r \leq \tau\} \subseteq K$. Assume, by way of contradiction, that $r \notin T_K$, so for each $n \in \mathbb{N}$ we have that $S_r^{\frac{1}{n}}$ is not cofinal in $\mathbf{W}$. That is, for each $n \in \mathbb{N}$ there exists $\sigma_n \in \mathbf{W}$ where $(\{\alpha\} \times (r - \frac{1}{n}, r + \frac{1}{n})) \cap X$ is not a subset of K for all $\alpha > \sigma_n$. Let $\sigma = \sup\{\sigma_n : n \in \mathbb{N}\}$, then for all $\alpha > \sigma$, $(\{\alpha\} \times (r - \frac{1}{n}, r + \frac{1}{n})) \cap X$ is not a subset of K. However, choosing a $\beta \in \mathbf{W}$ for which $\sigma, \sigma_r \leq \beta$,then since K is open there must exist $n \in \mathbb{N}$ such that $(\{\beta\} \times (r - \frac{1}{n}, r + \frac{1}{n})) \cap X \subseteq K$, a contradiction. Therefore $r \in T_K$. □

Proposition 3.3. *Let K be a clopen subset of X such that $\pi(K)$ is cofinal in $\mathbf{W}$. The set T_K is a nonempty open subset of $\mathbb{R}$.*

Proof. As we did previously, let $g = \chi_K \in C(X)$ with $\mathrm{coz}(g) = K$. Let $t \in T_K$, and suppose, by means of contradiction, that there is no open neighborhood of t contained in T_K. Then there is (without loss of generality) an increasing sequence of rationals, say $\{q_n\}$, not belonging to T_K which converges to t. Note that g is eventually zero on a tail of $[\mathbf{W} \times \{q_n\}] \cap X$ for each $n \in \mathbb{N}$. Thus for each $n \in \mathbb{N}$ there is a $\sigma_n \in \mathbf{W}$ such that for all $\sigma > \sigma_n$ we have $g((\sigma, q_n)) = 0$, i.e. $(\sigma, q_n) \notin K$ for every $\sigma > \sigma_n$.

Let $\zeta = \sup\{\sigma_n : n \in \mathbb{N}\}$. Then consider an appropriate $\alpha > \zeta$ such that for all $\alpha \leq \beta$ $(\beta, t) \in K$. (Such an α exists because $t \in T_K$.) But then since K is open there is a rational $q_n < t$ such that $(\alpha, q_n) \in K$, a contradiction. □

Remark 3.4. We observe that the proof of Proposition 3.2 actually shows that if K is clopen subset of X such that $\pi(K)$ is cofinal in $\mathbf{W}$, then for any $r \in \mathbb{R}$ the set $\{\sigma \in \mathbf{W} : (\sigma, r) \in K\}$ is either bounded by a countable ordinal or contains a tail of $\mathbf{W}$. In other words,

$$T_K = \{r \in \mathbb{R} : \text{ there is a cofinal subset of } \mathbf{W}\text{, say W, such that } (\tau, r) \in K \text{ for all } \tau \in W\}.$$

This is pivotal in proving our next result.

Proposition 3.5. *Let K be a clopen subset of X such that $\pi(K)$ is cofinal in $\mathbf{W}$. Then $\pi(X \smallsetminus K)$ is not cofinal in $\mathbf{W}$. Thus K contains a cofinal band of X.*

Proof. We suppose that both $\pi(K)$ and $\pi(X \smallsetminus K)$ are cofinal in $\mathbf{W}$. By the previous remark it follows that $T_K \cap T_{X \smallsetminus K} = \emptyset$ and $T_K \cup T_{X \smallsetminus K} = \mathbb{R}$. By Proposition 3.3 both T_K and $T_{X \smallsetminus K}$ are nonempty open sets. This produces a disconnection of $\mathbb{R}$, the desired contradiction. □

Proposition 3.6. *Let K be a σ-clopen subset of X. Then $X \smallsetminus cl_X K$ is also a σ-clopen subset.*

Proof. Suppose K is a σ-clopen subset of X.

First consider the case where $\pi(K)$ is not cofinal in $\mathbf{W}$. Then $K \subseteq X_\tau$ for some $\tau \in \mathbf{W}$. X_τ is a separable zero-dimensional metrizable space and hence strongly zero-dimensional. Hence $X_\tau \smallsetminus cl_X K$ is a σ-clopen subset of X_τ. Since X_τ is a clopen subset of X, it follows that $X_\tau \smallsetminus cl_X K$ is a σ-clopen subset of X, thus $X \smallsetminus cl_X K = X'_\tau \cup (X_\tau \smallsetminus cl_X K)$ is σ-clopen subset of X.

Next consider the case where $\pi(K)$ is cofinal in $\mathbf{W}$. Then for some $\tau \in \mathbf{W}$ K contains the cofinal band X'_τ. Since $X \smallsetminus cl_X K$ is an open subset of X_τ. We mentioned in the previous paragraph that X_τ is a separable zero-dimensional metrizable space and hence strongly zero-dimensional. It follows that $X \smallsetminus cl_X K$ is a σ-clopen subset of X_τ and thus a σ-clopen subset of X. □

Corollary 3.7. *Suppose K is a dense σ-clopen subset of X. Then $\pi(K)$ is cofinal, and thus K contains a cofinal band of X.*

Theorem 3.8. *For the space X, $q_c(X) < q'(X)$.*

Proof. Let $T_1 = [\mathbf{W} \times (-\infty, 0)] \cap X$ and $T_2 = (\mathbf{W} \times (0, \infty)) \cap X$. Both T_1 and T_2 are cozero-sets of X and hence so is $T = T_1 \cup T_2$. Moreover, T is a dense subset of X. We note that $\pi(T)$ is cofinal in X, but T does not contain a cofinal band. Therefore, T is not a σ-clopen subset of X. It follows that $T \in \mathfrak{G}_0(X) \smallsetminus \mathfrak{G}_\sigma(X)$.

Let $f : T \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 1, & \text{if } x \in T_1 \\ 0, & \text{if } x \in T_2. \end{cases}$$

Then $f \in C_c(T)$ and so $f \in q'(X)$. We claim that $f \notin q_c(X)$. If it were, then there would exist a dense σ-clopen of X, say $V \in \mathfrak{G}_\sigma(X)$, and $g \in C_c(V)$ such that f and g agree on $T \cap V$. But since V is a dense σ-clopen set, V contains a cofinal band of X. Therefore, $V \cap T$ equals T on this band and so g sends T_1 to 1 and T_2 to 0. But g is defined on the whole band, contradicting continuity at points of the form $(\tau, 0)$ for large enough τ. □

Remark 3.9. Proposition 3.6 is interesting standing on its own. The proposition yields for a zero-dimensional space Y, both rings $q_c(Y)$ and $\lim_{U \in \mathfrak{G}_\sigma(Y)} C(U)$ are von Neumann regular rings. The proof would be modeled after the proofs of Proposition 1.2 [10] and Theorem 1.3 of [7]. Simply, you would need that X is σ-clopen complemented.

The ring $\lim_{U \in \mathfrak{G}_\sigma(X)} C(U)$ has not been studied except in the case that X is strongly zero-dimensional. We conjecture that the ring can be realized as the classical ring of quotients of the Alexandroff Algebra $A(X)$ (see [6] or [5] for more information).

Remark 3.10. Let X^* denote the finer topology on $\mathbb{R}^2$ defined by Mysior [11] using $D = \mathbb{Q} \times \mathbb{Q}$. X^* is another example of a zero-dimensional space that is not strongly zero-dimensional. The countable set D is precisely the set of all isolated points of X^*. Thus, D is the smallest dense σ-clopen subset of X^*. It follows that $q_c(X^*) = q'(X^*) = C(\mathbb{N}) = q(X^*) = Q(X^*)$. Therefore, in general it is not the case that the equality $q_c(X) = q'(X)$ forces X to be strongly zero-dimensional. We are unable to characterize in any nice way when $q_c(X) = q'(X)$.

References

[1] Engelking, R., General Topology, Sigma Series in Pure Mathematics. Vol. **6**, Heldermann Verlag, Berlin, 1989.

[2] Fine, N. J., L. Gillman, and J. Lambek, Rings of Quotients of Rings of Functions. Lecture Note Series, McGill University Press (Montreal), 1966.

[3] Ghadermazi, M., O.A.S. Karamzadeh, and M. Namdari, *On the functionally countable subalgebra of $C(X)$*. Rend. Sem. Mat. Univ. Padova, to appear.
[4] Gillman, L. and M. Jerison, Rings of Continuous Functions. Graduate Texts in Mathametics, Vol. **43**, Springer Verlag, Berlin-Heidelberg-New York, 1976.
[5] Hager, A., *Cozero fields*. Confer. Sem. Mat. Univ. Bari No. **175** (1980), 1–23.
[6] Hager, A., C. Kimber and W. Wm. McGovern, *Unique a-closure for some ℓ-groups of rational valued functions*. Czech. Math. J. **55** (2005), 409–421.
[7] Henriksen, M. and R. G. Woods, *Cozero complemented spaces; when the space of minimal prime ideals of a $C(X)$ is compact*. Topology Appl. **141** (2004), no. 1-3, 147-170.
[8] Knox, M. L. and W. Wm. McGovern, *Rigid extensions of ℓ-groups of continuous functions*. Czech. Math. Journal, **58**(133) (2008), 993–1014.
[9] Levy, R. and Rice, *Normal P-spaces and the G_δ-topology*. Colloq. Math. **44** (1981), no. 2, 227-240.
[10] Levy, R. and J. Shapiro, *Rings of quotients of rings of functions*. Topology Appl. **146/147** (2005), 253-265.
[11] Mysior, A, *Two easy examples of zero-dimensional spaces*. Proc. Amer. Math. Soc. **92** (1984), no. 4, 615-617.
[12] Namdari, M. and A. Veisi, *Rings of quotients of the subalgebra of $C(X)$ consisting of functions with countable image*. Inter. Math. Forum, **7** (2012), 561–571.
[13] Porter, J. and R.G. Woods, Extensions and Absolutes of Hausdorff Spaces. Springer-Verlag, New York, 1988.
[14] Rudin, W., *Continuous functions on compact spaces without perfect subsets*. Proc. Amer. Math. Soc. **8** (1957), 39-42.

[1]Penn State Erie, The Behrend College, Erie, PA 16563
E-mail address: pxb39@psu.edu

[2]Midwestern State University, Wichita Falls, TX 76308
E-mail address: michelle.knox@mwsu.edu

[3]H.L. Wilkes Honors College, Florida Atlantic University, Jupiter, FL 33458
E-mail address: warren.mcgovern@fau.edu