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Rigid extensions of algebraic frames

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In memory of Paul F. Conrad

Abstract. An extension $G \leq H$ of lattice-ordered groups is said to be a *rigid extension* if for each $h \in H$ there exists a $g \in G$ such that $h^{\perp\perp} = g^{\perp\perp}$. In this paper, we will define rigid extensions and some other generalizations in the context of algebraic frames satisfying the FIP. One of the main results is a characterization of rigid extensions using d-elements of the frame. We also show that a rigid extension between two algebraic frames satisfying the FIP will induce a homeomorphism between their corresponding minimal prime spaces with respect to both the hull-kernel topology and the inverse topology. Moreover, basic open sets map to basic open sets.

1. Introduction

Rigid extensions were introduced some time ago for lattice-ordered groups and were studied mostly to analyze the interaction of the corresponding spaces of minimal prime convex ℓ-subgroups. The Conrad-Martinez paper [CM] discusses rigid extensions as well as weaker versions, for example r-extensions. Recall that $G \leq H$ is an r-extension if for each $0 < h \in H$ and each $P \in \mathrm{Min}(H)$ which does not contain h, there exists a $g \in G \setminus P$ so that $g^{\perp\perp} \subseteq h^{\perp\perp}$. In that article, the authors showed that $G \leq H$ is an r-extension if and only if the contraction map $P \to P \cap G$ is a homeomorphism of $\mathrm{Min}(H)$ onto $\mathrm{Min}(G)$ with respect to the hull-kernel topology. Later, McGovern, in the paper [12], introduced r^*-extensions for lattice-ordered groups: $G \leq H$ is an r^*-extension if for each $0 < h \in H$ and $P \in \mathrm{Min}(H)$ that contains h, there is a $0 < g \in G \cap P$ such that $h^{\perp\perp} \subseteq g^{\perp\perp}$. McGovern showed that $G \leq H$ is an r^*-extension if and only if the contraction map from $\mathrm{Min}(H)$ to $\mathrm{Min}(G)$ is a homeomorphism with respect to the inverse topology.

In this paper, we will define these types of extensions in the context of algebraic frames. We will prove that if $L \leq M$ is an r-extension or an r^*-extension of algebraic frames satisfying the FIP, then the contraction map is a homeomorphism between the minimal prime element spaces $\mathrm{Min}(M)$ and $\mathrm{Min}(L)$ with respect to the hull-kernel topology and the inverse topology, respectively. Also, we will establish two different characterizations for rigid extensions of algebraic frames, one in terms of minimal primes and the other using d-elements. Finally, we will provide an example which demonstrates that an r^*-extension and an r-extension are two different concepts.

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We recall some basic definitions from frame theory.

(1) A *frame* L is a complete lattice which satisfies the following distributive law: for each $a, b_i \in L$, and for all $i \in I$,

$$a \wedge \Big(\bigvee_{i\in I} b_i\Big) = \bigvee_{i\in I}(a \wedge b_i).$$

We denote the *top* and the *bottom* elements of L by 1 and 0, respectively.

(2) An element $c \in L$ is *compact* if $c \leq \bigvee_{i\in I} b_i$ implies that $c \leq \bigvee_{i\in I_0} b_i$ for some finite subset I_0 of I. We denote the collection of all compact elements of the frame L by $\mathfrak{K}(L)$.

(3) A frame L is *algebraic* if every element in L is the supremum of compact elements.

(4) A frame L is said to satisfy the *finite intersection property on compact elements* (the FIP for short) if $x, y \in \mathfrak{K}(L)$ implies that $x \wedge y \in \mathfrak{K}(L)$.

(5) A *frame homomorphism* is a map $f\colon L \to M$ that preserves arbitrary joins and finite meets. As a consequence, f also preserves the top and the bottom elements of the frame. Moreover, if f is injective, then by identifying $f(x)$ with x, L can be viewed as a *subframe* of M, and we will write $L \leq M$ in this situation.

A frame homomorphism $f\colon L \to M$ is called *coherent* if it maps the compact elements of L to the compact elements of M. We call $L \leq M$ a *coherent extension* when $\mathfrak{K}(L) \subseteq \mathfrak{K}(M)$.

(6) Let L be a frame, and for each $x \in L$ define

$$x^{\perp} = \bigvee\{y \in L \mid y \wedge x = 0\}.$$

The element $x^{\perp}$ is called the *polar* of x. Elements of the form $x^{\perp}$ are known as the *polars* of L. An element $x \in L$ is *complemented* if $x \vee x^{\perp} = 0$.

(7) An element x in a frame L is said to be *dense* if $x^{\perp} = 1$. A compact dense element of L is called a *unit*.

(8) An element $p \in L$ is *prime* if $p \neq 1$ and $x \wedge y \leq p$ implies that $x \leq p$ or $y \leq p$ for any $x, y \in L$.

A prime p is *minimal* if there does not exist any other prime element q (other than p itself) with the property that $q \leq p$. Such elements exist following the usual Zorn's Lemma argument.

We denote $\mathrm{Spec}(L)$ as the set of all prime elements of L and call it the *prime spectrum* of the frame; we denote $\mathrm{Min}(L)$ as the set of all minimal prime elements of L.

(9) Let L be an algebraic frame satisfying the FIP and let $x \in L$. We call x a *d-element* if $c \leq x$ implies $c^{\perp\perp} \leq x$ for all $c \in \mathfrak{K}(L)$. Any minimal prime element as well as any polar is a d-element. The reader should consult [11] for more information on d-elements.

2. Some known frame theory results

We start the second section with some well-known frame theory results which will be used throughout the paper.

Lemma 2.1. *Let L be a frame. For $x, y \in L$:*
(1) $(x \vee y)^{\perp} = x^{\perp} \wedge y^{\perp}$;
(2) $(x \wedge y)^{\perp} \geq x^{\perp} \vee y^{\perp}$.
Thus, $x^{\perp\perp} \vee y^{\perp\perp} \leq (x \vee y)^{\perp\perp}$.

The next lemma provides a test for an element of an algebraic frame to be prime.

Lemma 2.2. *Suppose L is an algebraic frame. An element $p \in L$ is prime if and only if $a \wedge b \leq p$ implies that $a \leq p$ or $b \leq p$ for all $a, b \in \mathfrak{K}(L)$.*

Finally, we state the Lemma on Ultrafilters which characterizes the minimal prime elements of an algebraic frame satisfying the FIP. An ultrafilter of compacts is a maximal filter of compacts. For the proof see [M].

Lemma 2.3 (Lemma on Ultrafilters)**.** *Let L be an algebraic frame satisfying the FIP. Then $p \in \mathrm{Spec}(L)$ is minimal if and only if*

$$F_p = \{c \in \mathfrak{K}(L) \mid c \nleq p\}$$

is an ultrafilter on $\mathfrak{K}(L)$. In this case, $p = \bigvee\{c^{\perp} \mid c \in F_p\}$.

We now prove the converse of the Lemma on Ultrafilters.

Lemma 2.4. *Suppose L is an algebraic frame satisfying the FIP and $\mathfrak{U}$ is an ultrafilter on $\mathfrak{K}(L)$. The element $p = \bigvee\{c^{\perp} \mid c \in \mathfrak{U}\}$ is a minimal prime element of L.*

Proof. It is sufficient to prove that p is prime in L and $\mathfrak{U} = F_p$. Let $x, y \in \mathfrak{K}(L)$ with $x \wedge y \leq p$. Since L satisfies the FIP, $x \wedge y \in \mathfrak{K}(L)$. Note that $x \wedge y \leq \bigvee\{c^{\perp} \mid c \in \mathfrak{U}\}$ implies that there exist finitely many elements $c_1, \ldots, c_n \in \mathfrak{U}$ such that

$$x \wedge y \leq c_1^{\perp} \vee \cdots \vee c_n^{\perp} \leq (c_1 \wedge \cdots \wedge c_n)^{\perp}.$$

Let us denote $c = c_1 \wedge \cdots \wedge c_n$; then $c \in \mathfrak{U}$ as $\mathfrak{U}$ is a filter. So $x \wedge y \leq c^{\perp}$ for some $c \in \mathfrak{U}$, and so $x \wedge y \wedge c = 0$. This says that $x \wedge y \notin \mathfrak{U}$ as $0 \notin \mathfrak{U}$. So either $x \notin \mathfrak{U}$ or $y \notin \mathfrak{U}$ (since $\mathfrak{U}$ is a filter). Without loss of generality, we assume that $x \notin \mathfrak{U}$. Since $\mathfrak{U}$ is an ultrafilter of compacts in L and $x \in \mathfrak{K}(L) \setminus \mathfrak{U}$, it follows that the filter generated by $\mathfrak{U}$ and x on $\mathfrak{K}(L)$, denoted by $\langle \mathfrak{U}, x \rangle$, is all of $\mathfrak{K}(L)$, that is,

$$\langle \mathfrak{U}, x \rangle = \{k \in \mathfrak{K}(L) \mid k \geq c \wedge x \text{ for some } c \in \mathfrak{U}\} = \mathfrak{K}(L).$$

Thus, there exists a $d \in \mathfrak{U}$ such that $d \wedge x = 0$. Consequently, $x \leq d^{\perp} \leq p$; thence, $p \in \mathrm{Spec}(L)$.

To complete the proof, we show that $\mathfrak{U} = F_p$. Suppose that $c \in \mathfrak{U}$; then $c^{\perp} \leq p$. This implies that $c \nleq p$, and so, $c \in F_p$. On the other hand, if $c \in F_p$,

then by definition, $c \nleq p$. If $c \notin \mathfrak{U}$, then by the similar argument as above, $\langle \mathfrak{U}, c\rangle = \mathfrak{K}(L)$. So there exists a $d \in \mathfrak{U}$ such that $d \wedge c = 0$, which implies that $c \leq d^{\perp} \leq p$, giving a contradiction. Hence, $c \in \mathfrak{U}$. □

The collection of minimal prime elements of an algebraic frame can be endowed with two different topologies: the hull-kernel topology and the inverse topology. A thorough discussion of the two topologies on $\mathrm{Min}(L)$ for algebraic frames L with the FIP can be found in [3]. We ought to also mention the article [8] which considered $\mathrm{Min}(R)$ for a commutative ring R with identity. We recite the definitions below.

Definition 2.5. Let L be an algebraic frame satisfying the FIP. The collection $\{U(x) \mid x \in \mathfrak{K}(L)\}$ forms a base for the open sets of a topology on $\mathrm{Min}(L)$, known as the *hull-kernel topology*, where $U(x) = \{p \in \mathrm{Min}(L) \mid x \nleq p\}$. The basic closed sets for the hull-kernel topology are each of the following form: $V(x) = \{p \in \mathrm{Min}(L) \mid x \leq p\}$ for $x \in \mathfrak{K}(L)$. Notice that for any $x \in L$, $U(x)$ and $V(x)$ are the set theoretic complements of each other.

On the other hand, the collection $\{V(x) \mid x \in \mathfrak{K}(L)\}$ also forms a base for the open sets of a topology on $\mathrm{Min}(L)$. This topology is known as the *inverse topology*, and we denote $\mathrm{Min}(L)$ with the inverse topology by $\mathrm{Min}(L)^{-1}$. In this case the basic closed sets are $U(x)$ for $x \in \mathfrak{K}(L)$.

It is interesting to note that for any algebraic frame L satisfying the FIP, $U(x) \cap U(y) = U(x \wedge y)$ for any $x, y \in L$. Moreover, the open sets of $\mathrm{Min}(L)$ are precisely the sets of the form $U(x)$ for some $x \in L$. The following lemma states another useful fact about minimal primes. Again, the reader is referred to [3] for proofs and details.

Lemma 2.6. *Let L be an algebraic frame satisfying the FIP and let $p \in \mathrm{Min}(L)$. For any $x \in \mathfrak{K}(L)$, either $x \nleq p$ or $x^{\perp} \nleq p$.*

3. Main Results On Rigid Extensions

From this point, we shall assume that L, M, and N are algebraic frames satisfying the FIP, and whenever we write $L \leq M$, we assume that this is a coherent extension. We begin with the definition of a rigid extension.

(1) *Rigid extension*: $L \leq M$ is a rigid extension if for each $k \in \mathfrak{K}(M)$, there exists $c \in \mathfrak{K}(L)$ such that $k^{\perp\perp} = c^{\perp\perp}$.
(2) *r-extension*: $L \leq M$ is an r-extension if for each $k \in \mathfrak{K}(M)$ and $p \in \mathrm{Min}(M)$ with $k \nleq p$, there exists $c \in \mathfrak{K}(L)$ with $c \nleq p$ and $c^{\perp\perp} \leq k^{\perp\perp}$.
(3) *r^*-extension*: M is an r^*-extension if for each $k \in \mathfrak{K}(M)$ and $p \in \mathrm{Min}(M)$ with $k \leq p$, there exists $c \in \mathfrak{K}(L)$ with $c \leq p$ and $k^{\perp\perp} \leq c^{\perp\perp}$.

The motivation for these definitions comes from the theory of lattice-ordered groups. Let us first recall some basic facts from the theory of ℓ-groups; we assume some familiarity with the theory. Our basic reference is [5].

Suppose G and H are two ℓ-groups with $G \leq H$. H is called a *major extension* of G (or, H *majorizes* G) if H is the convex ℓ-subgroup generated by G in H. By $H(K)$ we denote the convex ℓ-subgroup of H generated by K. The collection of all convex ℓ-subgroups of a group H is denoted by $\mathcal{C}(H)$, and it forms an algebraic frame satisfying the FIP. Moreover, we have $\mathfrak{K}(\mathcal{C}(H)) = \{H(h) \mid h \in H\}$.

Suppose $G \leq H$ is a major extension of ℓ-groups. We define a map from $\mathcal{C}(G)$ to $\mathcal{C}(H)$ by $K \mapsto H(K)$. Notice that this map is an injective, coherent frame homomorphism and so $\mathcal{C}(G) \leq \mathcal{C}(H)$ is a coherent extension. We will use $g^{\perp}$ to denote polars of the larger group H, where $g^{\perp} = \{h \in H \mid |h| \wedge |g| = 0\} \leq H$. Furthermore, $G \leq H$ is a *rigid* extension if for each $0 < h \in H$, there exists a $0 < g \in G$ such that $h^{\perp\perp} = g^{\perp\perp}$. $G \leq H$ is an *r-extension* (*r^*-extension*) if for each $0 < h \in H$ and $P \in \mathrm{Min}(H)$ with $h \notin P$ ($h \in P$), there exists a $g \in G \setminus P$ ($g \in G \cap P$) such that $g^{\perp\perp} \subseteq h^{\perp\perp}$ ($h^{\perp\perp} \subseteq g^{\perp\perp}$).

Theorem 3.1. *Suppose $G \leq H$ is a major extension of ℓ-groups. Then $G \leq H$ is a rigid, r- and r^*- extension of ℓ-groups if and only if $\mathcal{C}(G) \leq \mathcal{C}(H)$ is a rigid, r- and r^*-extension, respectively, of algebraic frames under the identification $K = H(K)$.*

Proof. Let $G \leq H$ be a major extension.
(1) ($\Rightarrow$): Suppose $G \leq H$ is a rigid extension. Let $h \in H$ with $H(h) \in \mathcal{C}(H)$. Since G is rigid in H, there exists a $g \in G$ such that $g^{\perp\perp} = h^{\perp\perp}$, that is, $H(g)^{\perp\perp} = H(h)^{\perp\perp}$. Hence, $\mathcal{C}(G) \leq \mathcal{C}(H)$ is a rigid extension.

($\Leftarrow$): Conversely, let $\mathcal{C}(G) \leq \mathcal{C}(H)$ be a rigid extension of algebraic frames and $h \in H$. Since $H(h) \in \mathcal{C}(H)$, using rigidity there exists a $g \in G$ such that $H(g)^{\perp\perp} = H(h)^{\perp\perp}$; in other words, $g^{\perp\perp} = h^{\perp\perp}$. Hence, G is rigid in H.

(2) ($\Rightarrow$): Next, suppose that $G \leq H$ is an r-extension. Let $h \in H$ with $H(h) \in \mathcal{C}(H)$ and $P \in \mathrm{Min}(H)$ such that $H(h) \nleq P$. So P does not contain h. Using the property of r-extension, there exists a $g \in G \setminus P$ such that $g^{\perp\perp} \subseteq h^{\perp\perp}$. So $H(g) \nleq P$ and $H(g)^{\perp\perp} \leq H(h)^{\perp\perp}$, satisfying the conditions for $\mathcal{C}(G) \leq \mathcal{C}(H)$ to be an r-extension.

($\Leftarrow$): To show the converse, let $\mathcal{C}(G) \leq \mathcal{C}(H)$ be an r-extension. Let $h \in H$ and let $P \in \mathrm{Min}(H)$ not contain h. Therefore, $H(h) \in \mathcal{C}(H)$ and $H(h) \nleq P$. Again, assuming r-extension for frames, there exists $g \in G$ with $H(g) \nleq P$ and $H(g)^{\perp\perp} \leq H(h)^{\perp\perp}$. These imply that $g \in G \setminus P$ and $g^{\perp\perp} \subseteq h^{\perp\perp}$. Hence, $G \leq H$ is an r-extension.

(3): Following a similar string of arguments as in (2), the result holds true for r^*-extensions also. □

To have a better understanding of rigid extensions, we start with a nice class of algebraic frames satisfying the FIP and specify their rigid subframes. Recall that a frame is *zero-dimensional* if every element is the supremum of complemented elements. An algebraic frame is zero-dimensional precisely when every compact element is complemented. Moreover such a frame satisfies

the FIP. A frame is called a *boolean* frame if every element of the frame is complemented.

Proposition 3.2. *Let $L \leq M$ be a coherent extension of algebraic frames and suppose that M is zero-dimensional. The extension is rigid if and only if it is an isomorphism. In particular, the only rigid subframe of an algebraic boolean frame is itself.*

Proof. It follows from rigidity that for each $k \in \mathfrak{K}(M)$, there exists a $c \in \mathfrak{K}(L)$ such that $k^{\perp\perp} = c^{\perp\perp}$. Since M is zero-dimensional, $k^{\perp\perp} = k$ for all $k \in \mathfrak{K}(M)$. Thus, we have the following:

$$c = c^{\perp\perp} = k^{\perp\perp} = k.$$

So $\mathfrak{K}(M) = \mathfrak{K}(L)$; thence $M = L$ since the two frames are algebraic.

The converse follows immediately. □

A frame is *spacial* if it is isomorphic to some topology on a set. Notice that an algebraic spacial frame is zero-dimensional if and only if it is isomorphic to a Hausdorff topology. Therefore, the above proposition can be framed in terms of topologies. The question then is what happens if we don't assume the topology is Hausdorff?

Example 3.3. Consider the set of all integers, $\mathbb{Z}$. Let τ_1 be the indiscrete topology on $\mathbb{Z}$ and let τ_2 be the cofinite topology on $\mathbb{Z}$. We notice that τ_2 is a T_1-topology which is not Hausdorff. In fact, every pair of distinct open sets in τ_2 intersects non-trivially and therefore every open set is dense, that is to say, $x^\perp = 0$ for every $x \in \tau_2$. Furthermore, every open set in τ_2 is compact; hence, τ_2 is an algebraic frame satisfying the FIP. It thus follows that for any $k \in \mathfrak{K}(\tau_2)$, we have $k^{\perp\perp} = 1 = 1^{\perp\perp}$ where $1 \in \mathfrak{K}(\tau_1)$. Hence, $\tau_1 \leq \tau_2$ is a coherent rigid extension but not a frame isomorphism.

Observe that in general if we have a coherent extension of algebraic frames satisfying the FIP, say $L \leq M$, and M is a coherent frame that has the property that every nonzero compact element is dense, then the extension is rigid.

We now turn our attention to the d-elements of a frame. A d-element $x \in L$ can be expressed as $x = \bigvee\{c^{\perp\perp} \mid c \in \mathfrak{K}(L), c \leq x\}$. The collection of all d-elements of an algebraic frame L is denoted by dL. Our standard reference is [11]. From now on, whenever $L \leq M$ for $x \in L$, we will use x' and $x^\perp$ to denote the polar of x in L and M, respectively. Observe that $x' \leq x^\perp$ for any $x \in L$.

Lemma 3.4. *Suppose that $L \leq M$, and let $x, y \in L$.*

(1) *$x \leq y^\perp$ if and only if $x \leq y'$.*
(2) *$x \leq y^{\perp\perp} \Rightarrow x \leq y''$.*

Proof. Let $x, y \in L$.

(1): Observe the following chain of implications:

$$x \leq y^{\perp} \Leftrightarrow x \wedge y = 0 \Leftrightarrow x \leq y'.$$

(2): If $x \leq y^{\perp\perp}$, then $x \wedge y^{\perp} = 0$. Therefore, $x \wedge y' \leq x \wedge y^{\perp} = 0$. Hence, $x \leq y''$. □

Proposition 3.5. *Let $L \leq M$ be a rigid extension. Every dense element in L is dense in M.*

Proof. Let $x \in L$ with $x' = 0$, and let $k \in \mathfrak{K}(M)$ be arbitrary with $k \leq x^{\perp}$. By rigidity, there exists $c \in \mathfrak{K}(L)$ with $c^{\perp\perp} = k^{\perp\perp}$. Since $k \leq x^{\perp}$, we have that $c \leq x^{\perp}$, and hence $c \leq x'$ (using Lemma 3.4). Consequently, $c = 0$ and therefore, $k = 0$. Hence, $x^{\perp} = 0$. □

As a consequence of the above proposition, we have the following result for rigid extensions.

Corollary 3.6. *Suppose $L \leq M$ is a rigid extension. Then $x'' \leq x^{\perp\perp}$ for any $x \in L$.*

Proof. We first notice that since $x'' \wedge x' = 0$, we have $x'' \leq (x')^{\perp}$. So

$$x'' \wedge x^{\perp} \leq (x')^{\perp} \wedge x^{\perp} = (x' \vee x)^{\perp} = 0$$

since $x' \vee x$ is dense in L. Hence, $x'' \leq x^{\perp\perp}$. □

The next lemma states some facts about d-elements.

Lemma 3.7. *Let $L \leq M$ and $x \in M$.*

(1) *The element $y = \bigvee\{c'' \mid c \in \mathfrak{K}(L), c \leq x\} \in dL$.*
(2) *The element $y = \bigvee\{c^{\perp\perp} \mid c \in \mathfrak{K}(L), c \leq x\} \in dM$*

Proof. Let $k \in \mathfrak{K}(L)$ with $k \leq y$; then $k \leq c_1'' \vee \cdots \vee c_n''$ for some compact elements $c_1, \ldots, c_n \leq x$. Using Lemma 2.1, we have $k \leq (c_1 \vee \cdots \vee c_n)''$. Setting $c = c_1 \vee \cdots \vee c_n$, we notice that $c \in \mathfrak{K}(L)$ and $c \leq x$, so $c'' \leq y$. Therefore, $k'' \leq c'' \leq y$, implying that $y \in dL$. This establishes (1), and (2) is similar. □

For $L \leq M$, we define a map $\phi\colon dM \to dL$ by

$$\phi(x) = \bigvee\{c'' \mid c \in \mathfrak{K}(L), c \leq x\}.$$

Lemma 3.7 implies that ϕ is well-defined. Furthermore, we notice that if $k \in \mathfrak{K}(L)$ with $k \leq \phi(x)$, then $k \leq k'' \leq c'' \leq \phi(x)$ for some $c \in \mathfrak{K}(L)$ and $c \leq x$.

Lemma 3.8. *Let L, M and ϕ be as above and $L \leq M$ a rigid extension. For any $k \in \mathfrak{K}(L)$ and $x \in dM$, we have $k \leq \phi(x) \Leftrightarrow k \leq x$.*

Proof. If $k \leq x$, then by the definition of $\phi(x)$, we have that $k \leq k'' \leq \phi(x)$. On the other hand, if $k \leq \phi(x)$, then $k \leq c'' \leq \phi(x)$ for some $c \in \mathfrak{K}(L)$ with $c \leq x$. Therefore, $k \leq c'' \leq c^{\perp\perp} \leq x$ using Corollary 3.6 and the fact that x is a d-element of M. □

We have arrived at the following characterization of a rigid extension:

Theorem 3.9. *$L \leq M$ is a rigid extension if and only if $\phi\colon dM \to dL$ is a frame isomorphism.*

Proof. ($\Rightarrow$): Suppose $L \leq M$ is a rigid extension. Let $x, y \in dM$ with $x \neq y$; then there exists some $k \in \mathfrak{K}(M)$ such that $k \leq x$, but $k \nleq y$. Therefore, $k^{\perp\perp} \leq x$, but $k^{\perp\perp} \nleq y$. Using rigidity, there exists $c \in \mathfrak{K}(L)$ with $c \leq x$, but $c \nleq y$. Thus, Lemma 3.8 implies that $c \leq \phi(x)$, but $c \nleq \phi(y)$. Consequently, $\phi(x) \neq \phi(y)$, implying that ϕ is injective.

Next, pick some $x \in dL$ so that $x = \bigvee\{c'' \mid c \in \mathfrak{K}(L), c \leq x\}$. Consider the element $x_0 = \bigvee\{c^{\perp\perp} \mid c \in \mathfrak{K}(L), c \leq x\}$; then $x_0 \in dM$ (Lemma 3.7) and $x \leq x_0$ (Corollary 3.6). We claim that $\phi(x_0) = x$, thereby proving that ϕ is a surjective map. By definition, $\phi(x_0) = \bigvee\{d'' \mid d \in \mathfrak{K}(L), d \leq x_0\}$. We first observe that $x \leq \phi(x_0)$ since for any $c \in \mathfrak{K}(L)$, $c'' \leq x$ implies that $c'' \leq \phi(x_0)$. For the reverse inequality, let $d \in \mathfrak{K}(L)$ with $d \leq x_0$. So $d \leq c^{\perp\perp} \leq x_0$ for some $c \in \mathfrak{K}(L)$ and $c \leq x$. From Lemma 3.4, we have $d \leq c'' \leq x$, which gives $d'' \leq x$. Therefore, $x = \phi(x_0)$.

Finally, it is easy to check that ϕ preserves order, proving that ϕ is a frame homomorphism, and hence, an isomorphism.

($\Leftarrow$): Suppose that ϕ is an isomorphism. We claim that for any $x \in dM$, we have $x = \bigvee\{c^{\perp\perp} \mid c \in \mathfrak{K}(L), c \leq x\}$. Let $y = \bigvee\{c^{\perp\perp} \mid c \in \mathfrak{K}(L), c \leq x\}$; then $y \in dM$ (by Lemma 3.7) and $y \leq x$, so $\phi(y) \leq \phi(x)$. On the other hand, if $d \in \mathfrak{K}(L)$ with $d \leq \phi(x)$, then $d \leq s'' \leq \phi(x)$ for some $s \in \mathfrak{K}(L)$ with $s \leq x$. Therefore, $s \leq s^{\perp\perp} \leq y$ and so $s'' \leq \phi(y)$. Thus, $d \leq \phi(y)$, proving that $\phi(x) = \phi(y)$. Since ϕ is an injective map, we have that $x = y = \bigvee\{c^{\perp\perp} \mid c \in \mathfrak{K}(L), c \leq x\}$. To show that $L \leq M$ is a rigid extension, pick $k \in \mathfrak{K}(M)$. Now, $k^{\perp\perp} \in dM$ and so $k^{\perp\perp} = \bigvee\{c^{\perp\perp} \mid c \in \mathfrak{K}(L), c \leq k^{\perp\perp}\}$. Since $k \leq k^{\perp\perp}$, there exists $c \in \mathfrak{K}(L)$ with $c \leq k^{\perp\perp}$ such that $k \leq c^{\perp\perp} \leq k^{\perp\perp}$. Hence, $k^{\perp\perp} = c^{\perp\perp}$. □

Observe the immediate corollary to Theorem 3.9, which uses the fact that the composition of maps $dM \to dL$ and $dN \to dM$, as in Theorem 3.9, is again the map $dN \to dL$ with the same properties. We leave the details for interested readers.

Corollary 3.10. *Let L, M and N be three algebraic frames with the FIP. If $L \leq M$ and $M \leq N$ are rigid extensions, then $L \leq N$ is a rigid extension. Hence, rigidity is a transitive property.*

We move on to discuss the relationship between the different types of extensions for algebraic frames. The first theorem in this context will show that the concept of rigidity is the strongest.

Theorem 3.11. *If $L \leq M$ is a rigid extension, then $L \leq M$ is both an r-extension and an r^*-extension.*

Proof. Suppose that $L \leq M$ is a rigid extension. So for each $k \in \mathfrak{K}(M)$, there exists $c \in \mathfrak{K}(L)$ with $k^{\perp\perp} = c^{\perp\perp}$. If $p \in \mathrm{Min}(M)$ with $k \nleq p$, then $c^\perp = k^\perp \leq p$ and so by Lemma 2.6, $c \nleq p$. Therefore, a rigid extension is an r-extension. On the other hand, if $p \in \mathrm{Min}(M)$ with $k \leq p$, then with a similar argument we have $c^\perp = k^\perp \nleq p$, and thence, $c \leq p$. □

We recall from the theory of ℓ-groups the notion of the contraction map: For ℓ-groups $G \leq H$, the contraction map $\mathcal{C}(H) \to \mathcal{C}(G)$ is defined by $K \mapsto H \cap G$. For algebraic frames, the definition goes like this. Suppose $L \leq M$ is a coherent extension. For any $x \in M$, we define $x^{\mathfrak{c}} = \bigvee\{c \in \mathfrak{K}(L) \mid c \leq x\}$. Then $x^{\mathfrak{c}}$ is the *contraction* of x to L, and the map is called the *contraction map* from M into L. We observe that the contraction map from M into L is always surjective and preserves order.

We first notice that the set $\{c \in \mathfrak{K}(L) \mid c \leq x\}$ is nonempty since $0 \in \mathfrak{K}(L)$ belongs to the set. Also, since $\{c \in \mathfrak{K}(L) \mid c \leq x\} \subseteq \{k \in \mathfrak{K}(M) \mid k \leq x\}$, we have the following inequality which we will use repeatedly:

$$x^{\mathfrak{c}} \leq x \text{ for any } x \in M.$$

Furthermore, notice that if $c \in \mathfrak{K}(L)$ with $c \leq x$, then by definition, $c \leq x^{\mathfrak{c}}$. Hence, for any $c \in \mathfrak{K}(L)$,

$$c \leq x^{\mathfrak{c}} \Leftrightarrow c \leq x.$$

Lemma 3.12. *Let $L \leq M$. The contraction map $x \mapsto x^{\mathfrak{c}}$ restricted to* $\mathrm{Spec}(M)$ *maps $p \in \mathrm{Spec}(M)$ to $p^{\mathfrak{c}} \in \mathrm{Spec}(L)$. Moreover, if $q \in \mathrm{Min}(L)$, then there exists some $p \in \mathrm{Min}(M)$ such that $p^{\mathfrak{c}} = q$.*

Proof. Let $x, y \in \mathfrak{K}(L)$ with $x \wedge y \leq p^{\mathfrak{c}}$; then $x \wedge y \leq p$. Since $p \in \mathrm{Spec}(M)$, it follows that $x \leq p$ or $y \leq p$. Therefore, $x \leq p^{\mathfrak{c}}$ or $y \leq p^{\mathfrak{c}}$. Using Lemma 2.2, we conclude that $p^{\mathfrak{c}} \in \mathrm{Spec}(L)$.

Finally, let $q \in \mathrm{Min}(L)$. By Lemma 2.3, the collection $\mathfrak{F} = \{c \in \mathfrak{K}(L) \mid c \nleq q\}$ is an ultrafilter of compact elements in L and $q = \bigvee\{c^\perp \mid c \in \mathfrak{F}\}$. $\mathfrak{F}$ is a filter base of compact elements of M which can be extended to a filter, and hence by using Zorn's Lemma, the filter can be further extended to an ultrafilter, namely $\overline{\mathfrak{F}}$, of compact elements in M. Consider the element $p = \bigvee\{k^\perp \mid k \in \overline{\mathfrak{F}}\}$. Then $p \in \mathrm{Min}(M)$ using Lemma 2.4. We claim that $p^{\mathfrak{c}} = q$. Let $c \in \mathfrak{K}(L)$ with $c \nleq q$; then $c \in \mathfrak{F} \subseteq \overline{\mathfrak{F}}$. By definition of p, we then have that $c^\perp \leq p$; hence, $c \nleq p$ using Lemma 2.6.

Therefore, $c \nleq q \Rightarrow c \nleq p$ for $c \in \mathfrak{K}(L)$, or in other words, $c \leq p \Rightarrow c \leq q$ for any $c \in \mathfrak{K}(L)$. Thus, $p^{\mathfrak{c}} = \bigvee\{d \in \mathfrak{K}(L) \mid d \leq p\} \leq q$. Since $q \in \mathrm{Min}(L)$ and we have proved that $p^{\mathfrak{c}}$ is prime, we have $p^{\mathfrak{c}} = q$. □

Remark 3.13. Observe that for any $c \in \mathfrak{K}(L)$ and $p \in \mathrm{Min}(M)$, we have $c \nleq p \Leftrightarrow c \nleq p^{\mathfrak{c}}$.

Let $L \leq M$ be a coherent extension of algebraic frames satisfying the FIP. $L \leq M$ is an *$r^{\flat}$-extension* if the contraction map $p \mapsto p^{\mathfrak{c}}$ is a bijective correspondence between $\mathrm{Min}(M)$ and $\mathrm{Min}(L)$. In general, a contraction map between M and L need not map a minimal prime element of M into a minimal prime element of L. Also, such a map may not be a bijection between $\mathrm{Min}(M)$ and $\mathrm{Min}(L)$. The next two examples will demonstrate these facts.

Example 3.14. Let L and M be as follows:

$$L = \begin{tikzcd} 1 \arrow[d, dash] \\ a \arrow[d, dash] \\ 0 \end{tikzcd} \qquad M = \begin{tikzcd} & 1 \arrow[dl, dash] \arrow[dr, dash] & \\ a \arrow[dr, dash] & & b \arrow[dl, dash] \\ & 0 & \end{tikzcd}$$

We notice that $L \leq M$ is a coherent subframe, $\mathrm{Min}(M) = \{a, b\}$ and $\mathrm{Min}(L) = \{0\}$. Under the contraction mapping, $a^{\mathfrak{c}} = a \notin \mathrm{Min}(L)$. Hence, the contraction map restricted to $\mathrm{Min}(M)$ does not map into $\mathrm{Min}(L)$.

Example 3.15. Let $L \leq M$ be as in the diagram below.

$$\begin{tikzcd} 1 \arrow[d, dash] \\ 0 \\ L \end{tikzcd} \qquad \begin{tikzcd} & 1 \arrow[dl, dash] \arrow[dr, dash] & \\ a \arrow[dr, dash] & & b \arrow[dl, dash] \\ & 0 & \\ & M & \end{tikzcd}$$

In this case, $\mathrm{Min}(M) = \{a, b\}$ and $\mathrm{Min}(L) = \{0\}$. Furthermore, $a^{\mathfrak{c}} = b^{\mathfrak{c}} = 0$, implying that the contraction map restricted to $\mathrm{Min}(M)$ is not a bijection.

Proposition 3.16. *If $L \leq M$ is either an r-extension or an r^*-extension, then $L \leq M$ is an $r^{\flat}$-extension.*

Proof. Let us first assume that $L \leq M$ is an r-extension. Let $p, q \in \mathrm{Min}(M)$ with $p \neq q$; then there exists a $k \in \mathfrak{K}(M)$ such that $k \leq p$, but $k \nleq q$. Since M is an r-extension of L, there exists a $c \in \mathfrak{K}(L)$ with $c \nleq q$ and $c^{\perp\perp} \leq k^{\perp\perp}$. Therefore, $c \leq c^{\perp\perp} \leq k^{\perp\perp} \leq p$ since p is a minimal prime element. So there exists $c \in \mathfrak{K}(L)$ such that $c \leq p$, but $c \nleq q$, thereby proving that $c \leq p^{\mathfrak{c}}$, but $c \nleq q^{\mathfrak{c}}$ using Remark 3.13. Hence, $p^{\mathfrak{c}} \neq q^{\mathfrak{c}}$, implying that the contraction mapping restricted to $\mathrm{Min}(M)$ is an injection.

The surjectivity of the contraction map restricted to $\mathrm{Min}(M)$ follows from Lemma 3.12. Consequently, $L \leq M$ is an $r^{\flat}$-extension.

The proof for r^*-extension is similar and is left to the interested reader. $\square$

Proposition 3.16 establishes the fact that an $r^\flat$-extension is the weakest of the types of extensions that we are considering. In the next theorem, we will prove that if we further assume that the contraction map restricted to the minimal prime space is a homeomorphism with respect to the hull-kernel topology and the inverse topology, then we will get back r- and r^*-extensions, respectively. In fact, r-extensions and r^*-extensions will be characterized by these properties. $\mathrm{Min}(M)$ denotes the minimal prime space of M with the hull-kernel topology and $\mathrm{Min}(M)^{-1}$ will denote the space with the inverse topology. To ensure unambiguity, we will use $U_M(x)$ (and $V_M(x)$) to denote the subsets of $\mathrm{Min}(M)$ (and $\mathrm{Min}(M)^{-1}$), respectively, while we use $U_L(x)$ (and $V_L(x)$) to denote the subsets of $\mathrm{Min}(L)$ (and $\mathrm{Min}(L)^{-1}$), respectively. Also, for $U_M(x)$ (and $V_M(x)$), we will denote by $U(x)^\mathfrak{c}$ (and $V(x)^\mathfrak{c}$) a subset of $\mathrm{Min}(L)$ obtained by taking the image of $U_M(x)$ (and $V_M(x)$),respectively, under the contraction map; that is, $U(x)^\mathfrak{c} = \{p^\mathfrak{c} \in \mathrm{Min}(L) \mid p \in U_M(x)\}$.

Theorem 3.17. *Let $L \leq M$ be as usual.*

(1) *$L \leq M$ is an r-extension if and only if $L \leq M$ is an $r^\flat$- extension and the contraction map restricted to $\mathrm{Min}(M)$ is a homeomorphism between $\mathrm{Min}(M)$ and $\mathrm{Min}(L)$.*

(2) *$L \leq M$ is an r^*-extension if and only if $L \leq M$ is an $r^\flat$-extension and the contraction map restricted to $\mathrm{Min}(M)$ is a homeomorphism between $\mathrm{Min}(M)^{-1}$ and $\mathrm{Min}(L)^{-1}$.*

Proof. We have already established in Proposition 3.16 the fact that both an r-extension and an r^*-extension yield an $r^\flat$-extension between L and M.

($\Rightarrow$): We show that we have a homeomorphism by proving that the contraction map $p \mapsto p^\mathfrak{c}$ is open and continuous with respect to the corresponding topologies on minimal prime spaces.

(1) Suppose that $L \leq M$ is an r-extension. Let $k \in \mathfrak{K}(M)$, and let $U_M(k) = \{p \in \mathrm{Min}(M) \mid k \nleq p\}$ be a basic open set of $\mathrm{Min}(M)$. We want to show that $U(k)^\mathfrak{c} = \{p^\mathfrak{c} \in \mathrm{Min}(L) \mid p \in U_M(k)\}$ is open in $\mathrm{Min}(L)$. For each $p \in U_M(k)$, by the definition of r-extension, there exists some $c_p \in \mathfrak{K}(L)$ such that $c_p \nleq p$ and $c_p^{\perp\perp} \leq k^{\perp\perp}$. Let us denote $c = \bigvee\{c_p \in \mathfrak{K}(L) \mid p \in U_M(k)\}$; then $c \in L$ and $U(k)^\mathfrak{c} = U_L(c) = \{q \in \mathrm{Min}(L) \mid c \nleq q\}$, and hence, is open in $\mathrm{Min}(L)$.

Observe that the inverse image of a basic open set $U_L(c)$ in $\mathrm{Min}(L)$ is the open set $U_M(c)$ of $\mathrm{Min}(M)$ where $c \in \mathfrak{K}(L)$. The reason being that the property of the contraction map that $c \nleq p^\mathfrak{c} \Leftrightarrow c \nleq p$ for any $c \in \mathfrak{K}(L)$, where $p \in \mathrm{Min}(M)$. We leave it to the reader to fill in the details. Hence, the contraction map is continuous.

(2) Next we assume that M is an r^*-extension of L. Let $k \in \mathfrak{K}(M)$, and let $V_M(k) = \{p \in \mathrm{Min}(M)^{-1} \mid k \leq p\}$ be a basic open set of $\mathrm{Min}(M)^{-1}$. As before, using the property of r^*-extension, for each $p \in V_M(k)$ there exists some $c_p \in \mathfrak{K}(L)$ such that $c_p \leq p$ and $c_p^\perp \leq k^\perp$. We consider the open set

$V = \bigcup_{p \in V_M(k)} V_L(c_p)$ of $\mathrm{Min}(L)^{-1}$ and claim that $V(k)^{\mathfrak{c}} = V$; consequently, the contraction of $V_M(k)$ is open in $\mathrm{Min}(L)^{-1}$. To show that $V \subseteq V(k)^{\mathfrak{c}}$, let $q \in V$; then $q = r^{\mathfrak{c}}$ for some $r \in \mathrm{Min}(M)^{-1}$ using $r^{\flat}$-extension. So $r^{\mathfrak{c}} \in V_L(c_p)$ for some $p \in V_M(k)$. Since $c_p \leq r^{\mathfrak{c}} \leq r$, it follows that $k \leq r$; hence, $r \in V_M(k)$. Therefore, $q = r^{\mathfrak{c}} \in V(k)^{\mathfrak{c}}$. The other inclusion follows immediately.

Using a similar argument (changing $\not\leq$ to $\leq$) as in the case of an r-extension, it follows that the contraction map from $\mathrm{Min}(M)^{-1}$ to $\mathrm{Min}(L)^{-1}$ is continuous and so a homeomorphism.

($\Leftarrow$): Conversely, suppose that $L \leq M$ is an $r^{\flat}$-extension.

(1) Suppose that the contraction map restricted to $\mathrm{Min}(M)$ is a homeomorphism from $\mathrm{Min}(M)$ onto $\mathrm{Min}(L)$. Let $k \in \mathfrak{K}(M)$ and $p \in \mathrm{Min}(M)$ with $k \not\leq p$; then $p^{\mathfrak{c}} \in U(k)^{\mathfrak{c}}$. Since $U(k)^{\mathfrak{c}}$ is an open set in $\mathrm{Min}(L)$, there exists $c \in \mathfrak{K}(L)$ such that $p^{\mathfrak{c}} \in U_L(c) \subseteq U(k)^{\mathfrak{c}}$. Therefore, $c \not\leq p^{\mathfrak{c}}$ and so $c \not\leq p$ using Remark 3.13. We finish the proof by showing that $k^{\perp} \wedge c = 0$, which will then imply $k^{\perp} \leq c^{\perp}$, thereby proving that M is an r-extension of L. Let q be any arbitrary element of $\mathrm{Min}(M)$. If $q \in U_M(k)$, then $k^{\perp} \wedge c \leq k^{\perp} \leq q$. If $q \notin U_M(k)$, then $q^{\mathfrak{c}} \notin U(k)^{\mathfrak{c}}$ and so $q^{\mathfrak{c}} \notin U_L(c)$. Therefore, $k^{\perp} \wedge c \leq c \leq q$. Thus, we have $k^{\perp} \wedge c \leq \bigwedge\{q \mid q \in \mathrm{Min}(M)\} = 0$.

(2) Similarly, the result holds for r^*-extensions. $\square$

To summarize the above results, we borrow the following diagram from [12] which demonstrates the relationship between the different rigid extensions of algebraic frames satisfying the FIP.

$$\begin{tikzcd}
 & r^*\text{-extension} \arrow[dr] & \\
\text{rigid extension} \arrow[ur] \arrow[dr] & & r^{\flat}\text{-extension} \\
 & r\text{-extension} \arrow[ur] &
\end{tikzcd}$$

A natural question arises as to whether any of the above arrows are reversible. Those familiar with [4] and [12] know that there are examples of ℓ-groups $G \leq H$ that demonstrate that the arrows are not reversible in the context of ℓ-groups. Unfortunately, none of these examples is useful in our context since the extensions are not majorizing and therefore $\mathcal{C}(G)$ is not a subframe of $\mathcal{C}(H)$. We provide an example of a majorizing extension of ℓ-groups $G \leq H$ which is an r^*-extension but not a rigid extension. Theorem 3.1 can then be applied to conclude that there exists an extension of algebraic frames which is an r^*-extension but not a rigid extension. Moreover, in this example r-extension implies rigid extension. Therefore, it is also an example which proves that r-extensions and r^*-extensions are two different concepts for algebraic frames satisfying the FIP.

We first give some preliminaries required for the following example. A topological space X is *zero-dimensional* if the topology has a base of clopen sets. We will assume that all of our spaces X are Tychonoff, that is, Hausdorff and completely regular. For a space X, $C(X)$ and $C(X,\mathbb{Z})$ denote respectively the rings of all real-valued and integer-valued continuous functions defined on X. For $p \in X$, O_p is the set of all functions in $C(X)$ that vanish in a neighborhood of p. It turns out that O_p is an ideal of $C(X)$. A space X is an *F-space* if every finitely generated ideal in $C(X)$ is principal. The assumption of X being an F-space guarantees that O_p is a prime ideal of $C(X)$.

Example 3.18. Let X be a compact, zero-dimensional F-space which is not basically disconnected, e.g. $X = \beta\mathbb{N} \smallsetminus \mathbb{N}$. Consider the ℓ-group $C(X)$ and its ℓ-subgroup $C(X,\mathbb{Z})$. Clearly, $C(X)$ majorizes $C(X,\mathbb{Z})$, and $C(X,\mathbb{Z}) \leq C(X)$ is not a rigid extension since X is not basically disconnected (see Proposition 3.5, [9]). Also notice that for a compact F-space X, the minimal primes of $C(X)$ are precisely the ideals O_p for $p \in X$. To show that $C(X,\mathbb{Z}) \leq C(X)$ is an r^*-extension, we consider an $f \in C(X)$ and $p \in X$ with $f \in O_p$. We have to show that there exists some $g \in C(X,\mathbb{Z})$ such that $g \in O_p$ and $f^{\perp\perp} \subseteq g^{\perp\perp}$. First, recall that

$$f^{\perp\perp} = \{h \in C(X) \mid cl_X \operatorname{coz}(h) \subseteq cl_X \operatorname{coz}(f)\}.$$

Therefore, we need to demonstrate that for all $p \in \operatorname{int}_X(Z(f))$, there exists some $g \in C(X,\mathbb{Z})$ with $p \in \operatorname{int}_X(Z(g))$ such that $cl_X(\operatorname{coz}(f)) \subseteq cl_X(\operatorname{coz}(g))$. Now, $cl_X(\operatorname{coz}(g))$ and $\operatorname{int}_X(Z(g))$ are clopen subsets of X for all $g \in C(X,\mathbb{Z})$. Therefore, for every $p \in \operatorname{int}_X(Z(f))$, we have to find a clopen set K of X satisfying the property that $p \in K \subseteq \operatorname{int}_X(Z(f))$. Since $p \in \operatorname{int}_X(Z(f))$ and X is zero dimensional, there exists some basic clopen set K such that $p \in K \subseteq \operatorname{int}_X(Z(f))$. Thus, $C(X,\mathbb{Z}) \leq C(X)$ is an r^*-extension which is not a rigid extension.

Proposition 3.19. *Suppose X is a compact, zero-dimensional F-space. The following statements are equivalent.*

(1) *X is basically disconnected.*
(2) *$C(X,\mathbb{Z}) \leq C(X)$ is a rigid extension.*
(3) *$C(X,\mathbb{Z}) \leq C(X)$ is an r-extension.*

Proof. (1) $\Rightarrow$ (2): Proposition 3.5 in [9].

(2) $\Rightarrow$ (3): Clear.

(3) $\Rightarrow$ (1): Suppose that X is not basically disconnected. Therefore, there exists a $f \in C(X)$ such that $cl_X(\operatorname{coz}(f))$ is not open. Let $p \in cl_X(\operatorname{coz}(f))$ be arbitrary; then $p \notin \operatorname{int}_X(Z(f))$, and therefore, $f \notin O_p$. If $C(X,\mathbb{Z}) \leq C(X)$ is an r-extension, then there will exist some $g_p \in C(X,\mathbb{Z})$ with $g_p \notin O_p$ and $g_p^{\perp\perp} \subseteq f^{\perp\perp}$. These reduce to saying that there will exist some $g_p \in C(X,\mathbb{Z})$ with $p \notin \operatorname{int}_X(Z(g_p)) = Z(g_p)$ and $\operatorname{coz}(g_p) \subseteq cl_X(\operatorname{coz}(f))$. In other words these would imply that $p \in \operatorname{coz}(g_p) \subseteq cl_X(\operatorname{coz}(f))$. Since p is chosen arbitrarily, the

preceding statement implies that $cl_X(\mathrm{coz}(f))$ is open, which is a contradiction. Hence, $C(X,\mathbb{Z}) \leq C(X)$ is not an r-extension. $\square$

Therefore, Example 3.18 provides us with an example of algebraic frames $L \leq M$ satisfying the FIP in which an r-extension and an r^*-extension are not equivalent.

A different question is whether an r-extension and an r^*-extension together gives a rigid extension? We give a partial answer to this question. Notice that if $L \leq M$ is a coherent extension, then it is both an r-extension and an r^*-extension if and only if $\mathrm{Min}(L)$ is homeomorphic to $\mathrm{Min}(M)$ with respect to both the hull-kernel topology and the inverse topology. Moreover, if the basic open sets of $\mathrm{Min}(M)$ are mapped to basic open sets of $\mathrm{Min}(L)$ with respect to both topologies, then $L \leq M$ will be a rigid extension. We denote a basic open set of $\mathrm{Min}(M)$ by $U_M(k)$ and a basic open set of $\mathrm{Min}(M)^{-1}$ by $V_M(k)$ for some $k \in \mathfrak{K}(M)$, and denote the set $\{p^{\mathfrak{c}} \mid p \in U_M(k)\}$ by $U_M(k)^{\mathfrak{c}}$.

Lemma 3.20. *Let $L \leq M$ be an $r^{\flat}$-extension. If $U_M(k)^{\mathfrak{c}} = U_L(c)$ for some $k \in \mathfrak{K}(M)$ and $c \in \mathfrak{K}(L)$, then $V_M(k)^{\mathfrak{c}} = V_L(c)$.*

Proof. Observe that for $k \in \mathfrak{K}(M)$ and $p \in \mathrm{Min}(M)$, we have $p \in U_M(k) \Leftrightarrow p^{\mathfrak{c}} \in U_M(k)^{\mathfrak{c}}$. $\square$

Theorem 3.21. *Let $L \leq M$ be a coherent extension of algebraic frames. Suppose that the contraction map $\mathrm{Min}(M) \to \mathrm{Min}(L)$ is a homeomorphism with respect to both the hull-kernel topology and the inverse topology. $L \leq M$ is a rigid extension if and only if the contraction map takes basic open sets of $\mathrm{Min}(M)$ ($\mathrm{Min}(M)^{-1}$) into basic open sets of $\mathrm{Min}(L)$ ($\mathrm{Min}(L)^{-1}$).*

Proof. ($\Rightarrow$): Let $L \leq M$ be a rigid extension. Let $k \in \mathfrak{K}(M)$ and consider $U_M(k) = \{p \in \mathrm{Min}(M) \mid k \not\leq p\}$. By rigidity, there exists $c \in \mathfrak{K}(L)$ with $k^{\perp} = c^{\perp}$. Notice that for any $q \in U_L(c)$, we have $q = p^{\mathfrak{c}}$ for some $p \in U_M(k)$. Hence, $U_M(k)^{\mathfrak{c}} = U_L(c)$. Using Lemma 3.20, it follows that $V_M(k)^{\mathfrak{c}} = V_L(c)$.

($\Leftarrow$): Let $k \in \mathfrak{K}(M)$ and consider $U_M(k) = \{p \in \mathrm{Min}(M) \mid k \not\leq p\}$. By the hypothesis, there exists some $c \in \mathfrak{K}(L)$ such that $U(k)^{\mathfrak{c}} = U_L(c)$. We show that $k^{\perp} \wedge c = 0 = k \wedge c^{\perp}$, proving that $k^{\perp} = c^{\perp}$. Suppose $p \in \mathrm{Min}(M)$ is arbitrary. If $p \in U_M(k)$, then $k^{\perp}, c^{\perp} \leq p$, which implies that $k^{\perp} \wedge c \leq p$ and $k \wedge c^{\perp} \leq p$. If $p \in V_M(k)$, then $k \leq p$ and so $c \leq p$, thereby proving that $k^{\perp} \wedge c \leq p$ and $k \wedge c^{\perp} \leq p$. Hence, $k^{\perp} \wedge c$ and $k \wedge c^{\perp}$ are below every minimal prime of M and so $k^{\perp} \wedge c = 0 = k \wedge c^{\perp}$. Thus, the extension is rigid. $\square$

We proved that an extension which is both an r-extension and an r^*-extension can be characterized as an extension for which the contraction map between $\mathrm{Min}(M)$ and $\mathrm{Min}(L)$ is a homeomorphism with respect to both the hull-kernel topology and the inverse topology. So together, an extension which is both an r-extension and an r^*-extension is "close" to being a rigid extension. We are unable to determine whether this extension is in fact rigid.

We finish the paper with a short discussion of a basis of an algebraic frame. As a reference see [5]. Let L be an algebraic frame. A nonzero element $b \in \mathfrak{K}(L)$ is called *basic* if $\{x \in L \mid x \leq b\}$ forms a chain. L is said to have a *basis* if it has a maximal set B of pairwise disjoint elements, where each $b \in B$ is basic. Notice that to check that an element $b \in L$ is basic, we only need to show that the set of all compact elements below b forms a chain. Also, any compact element below a basic element is again a basic element.

Proposition 3.22. *Let L be an algebraic frame and suppose that $b \in L$ is basic. If $x \leq b$, then either $x = 0$ or $x^\perp = b^\perp$.*

Proof. Let $x \leq b$ be nonzero. It follows immediately that $b^\perp \leq x^\perp$. To show the other inclusion, let $c \in \mathfrak{K}(L)$ with $c \leq x^\perp$; then $c \wedge x = 0$. Since $c \wedge b$ is compact and $c \wedge b \leq b$, by the definition of basic element, x and $c \wedge b$ are comparable. If $x \leq c \wedge b$, then $x \leq c$ and so $0 = c \wedge x = x$, which is a contradiction. Therefore, $c \wedge b \leq x$. In this case, $0 = (c \wedge x) \wedge b = (c \wedge b) \wedge x = c \wedge b$. Hence, $c \leq b^\perp$ for all $c \leq x^\perp$ and so $x^\perp \leq b^\perp$. □

Take note of the following corollary of Proposition 3.22.

Corollary 3.23. *Suppose L is an algebraic frame. For any polar $x \in L$ and basic element $b \in L$, either $x \wedge b = 0$ or $b \leq x$.*

Our goal is to answer the question: which extensions between $L \leq M$ will ensure that L has a basis if and only if M has a basis? We need to restrict our frames further in order to answer this question.

Suppose L is an algebraic frame. L is said to be a frame with *disjointification* if for each pair of compact elements x and y in L, there exists a disjoint pair of compacts c and d in L with $c \leq x$ and $d \leq y$ such that $c \vee y = d \vee x = x \vee y$. These frames are also known as *relatively normal*.

Theorem 3.24. *Suppose L is an algebraic frame with disjointification satisfying the FIP, and $b \in \mathfrak{K}(L)$. The following statements are equivalent:*

(1) *b is basic;*
(2) *$b^\perp \in \mathrm{Min}(L)$;*
(3) *$b^\perp \in \mathrm{Spec}(L)$;*
(4) *$\downarrow b^{\perp\perp}$ is a chain and $b^{\perp\perp}$ is maximal with respect to this property.*

Proof. The proof given in Theorem 19.1 in [5] can be easily adapted to our situation. □

Corollary 3.25. *Let L be an algebraic frame and $c, d \in L$. If c and d are basic elements in L, then either $c \wedge d = 0$ or c and d are comparable.*

Proof. Using Corollary 3.23, it follows that either $c \leq d^\perp$ or $c \wedge d^\perp = 0$. In the first case, $c \wedge d = 0$. Otherwise, $c \wedge d^\perp = 0$, which implies $c \leq d^{\perp\perp}$. Since c and d are both below $d^{\perp\perp}$, Theorem 3.24 tells us that c and d are comparable. □

Proposition 3.26. *Let L be an algebraic frame with a basis B. For every nonzero $c \in \mathfrak{K}(L)$, there exists a basic element $b \in L$ such that $b \leq c$.*

Proof. By definition, B is a maximal set of pairwise disjoint elements and each element in B is basic. Let $c \in \mathfrak{K}(L)$ with $c \notin B$. By maximality of B there exist some $b \in B$ such that $c \wedge b \neq 0$. So $c \wedge b$ is a nonzero compact element and $c \wedge b \leq b$; consequently, $c \wedge b$ is a basic element of L. Thus, c lies above a basic element $c \wedge b$. □

Therefore, for an algebraic frame with a basis, every nonzero element in the frame exceeds a supremum of basic elements.

Proposition 3.27. *Let $p \in \mathrm{Min}(L)$. The following statements are equivalent:*

(i) *p is isolated in $\mathrm{Min}(L)$ with respect to the hull-kernel topology;*
(ii) *for some $c \in \mathfrak{K}(L)$, $p = c^{\perp}$;*
(iii) *p is a polar.*

Proof. See Proposition 3.6 in [3]. □

Theorem 3.28. *Let L be an algebraic frame satisfying the FIP and with disjointification. L has a basis if and only if $\mathrm{Min}(L)$ has a dense discrete subspace.*

Proof. ($\Rightarrow$): Suppose L has a basis B. Consider the set $S = \{c^{\perp} \mid c \in B\}$. Theorem 3.24 implies that $S \subseteq \mathrm{Min}(L)$, and we conclude from Proposition 3.27 that each $c^{\perp} \in S$ is isolated in the hull-kernel topology; consequently, S is a discrete subspace of $\mathrm{Min}(L)$. We claim that S is dense in $\mathrm{Min}(L)$. Choose $p \in \mathrm{Min}(L) \setminus S$ and $d \in \mathfrak{K}(L)$ with $p \in U(d)$; hence, $d^{\perp} \leq p$. Since $d^{\perp}$ is not prime, $d \notin B$. Using the maximality of B and Theorem 3.24, there exists some $c \in B$ such that $c^{\perp} \in U(d)$. Thus, $U(d) \bigcap S \neq \emptyset$, concluding that S is dense in $\mathrm{Min}(L)$.

($\Leftarrow$): Conversely, suppose that $\mathrm{Min}(L)$ has a dense discrete subspace D. Then p is isolated for each $p \in D$, and so by Proposition 3.27, it follows that $p = c^{\perp}$ for some $c \in \mathfrak{K}(L)$. Let $B = \{c \in \mathfrak{K}(L) \mid c^{\perp} \in D\}$. We claim that B is a basis of L. First notice that each $c \in B$ is basic using Theorem 3.24. Also, for any two distinct elements c and d in B, $c \wedge d = 0$ using Corollary 3.25 and the fact that $c^{\perp}$ and $d^{\perp}$ are minimal prime elements of L. Finally, we need to show that B is a maximal set of pairwise disjoint elements. Let $k \in L$ such that $k \notin B$. If $k \wedge c = 0$ for all $c \in B$, then $k \leq c^{\perp}$ for all $c^{\perp} \in D$, whence $D \subseteq V(k)$. Since D is dense in $\mathrm{Min}(L)$, it follows that $\mathrm{Min}(L) = V(k)$; therefore, $k = 0$. Thus, for every nonzero element k in L, there exist some $c \in B$ such that $k \wedge c \neq 0$, or in other words, B is a maximal set of pairwise disjoint elements. Hence, L has a basis B. □

Corollary 3.29. *Suppose that L and M are two algebraic frames satisfying the FIP and with disjointification. If $L \leq M$ is an r-extension, then L has a basis if and only if M has a basis.*

Finally, observe from Example 3.14 that the converse may not be true, that is, there exist algebraic frames $L \leq M$ where both L and M have a basis, but the extension is not an r-extension.

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