Minimal Prime Element Space of an Algebraic Frame

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ABSTRACT

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The following dissertation investigates algebraic frames. Formally speaking, a frame is a complete lattice which satisfies a strengthened distributive law where finite infima distribute over arbitrary suprema. In particular, we are interested in focussing on a certain space associated with an algebraic frame: the space of minimal prime elements. In the first half of the dissertation we will investigate different interesting properties of these topological spaces in terms of the algebraic properties of the frame. In one of our main results we state internal conditions of an algebraic frame which will ensure its minimal prime element space is compact.

In Chapter 5 we will describe the radical of an algebraic frame. This is a generalization in context to the frame of radical ideals of a commutative ring with identity. We will demonstrate that the radical of an algebraic frame is an algebraic frame.

The last part of the dissertation focuses on extensions of algebraic frames. We will generalize the notions of rigid extension, r-extension and r^*-extension which are known in the theory of lattice-ordered groups. Our main result will characterize rigid extensions in several ways. We will answer the following question: "Which type of extensions between two algebraic frames will ensure a homeomorphism between their corresponding minimal prime element spaces?" This question had been looked at and answered for lattice-ordered groups by Conrad and Martinez in [4] and later by McGovern in [17]. We will also provide an important example from the theory of rings of continuous functions. In this example, we will construct an extension of algebraic frames which will demonstrate that an r^*-extension and an r-extension are two different concepts. In the end we will provide several open questions which may lead to future study.

Ma, I dedicate my dissertation to you, for all your love, care, and sacrifices for me.

- From your loving daughter, Sonai.

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CHAPTER 1

Preliminaries

INTRODUCTION

The study of frames began in the 1930s by the likes of A. Heyting and R. Brouwer. It is known that Hausdorff was one of the mathematicians who used the notion of open sets as the main tool in the study of topological properties. Since 1914 a topological space has been known to be something which possessed a lattice structure (see [13] and [12]). However, the study of topological spaces just as a lattice of open sets, ignoring the elements of the set itself, took some time to emerge. During the 1930s several mathematicians started looking at the topological spaces from an algebraic point of view, and considered just the structure of the topology without any mention of the points of the space. These generalized spaces were often referred to as a 'pointless' or 'point-free' topology. G. Birkhoff called them complete Brouwerian lattices in his book [3]. The application of lattice theory to topology became prominent in the middle 1930s with the works of the mathematicians M. Stone and H. Wallman, and later the study was undertaken by J.C.C. McKinsey and A. Tarski. In the late 1950s quite a number of results from topology were generalized to frames. It was not until 1970s that the subject began to emerge with the help of J.R. Isbell [11] and B. Banaschewski as an independent area irrespective of topology. J.R. Isbell pointed out that the category of frames behave differently than the category of topological spaces. Nowadays

the study of frames has become an ultimate goal, irrespective of whether these are the lattice of open sets of a space. To understand frames in the point of view of generalized topology, we shall first develop the theories of lattices and topological spaces.

LATTICE THEORY

A *partial order*, say $\leq$ on a set S is a relation that is reflexive, antisymmetric, and transitive. A *partially ordered set* (a *poset* for short) is a set S together with a partial order relation $\leq$; we denote a poset by $(S, \leq)$. An *upper bound* of a subset P of S (if it exists) is an element $u \in S$ which satisfies the property that $p \leq u$ for all $p \in P$. Similarly, a *lower bound* of P is an element l of S which satisfies the property that $l \leq p$, for all $p \in P$. An upper bound u of P is the *least upper bound* (or *supremum*) of P if u is a lower bound for the set of all upper bounds of P. If a least upper bound exists for a set, then it is unique. We write $\bigvee\{x, y\}$ to mean the supremum of x and y, and will always denote it by $x \vee y$. Also, we write $\bigvee P$ to denote the supremum of P, which is defined as $\bigvee P = \bigvee\{x : x \in P\}$. Likewise, a lower bound l of P is the *greatest lower bound* (or *infimum*) of P if l is an upper bound for the set of all lower bounds. If a greatest lower bound of a set exists, then it is unique. We denote the infimum of x and y by $x \wedge y$, and the infimum of a set P by $\bigwedge P$, which is equal to $\bigwedge\{x : x \in P\}$.

A poset $(L, \leq)$ is a *lattice* if each nonempty finite subset of L has a least upper bound and a greatest lower bound. In the following proposition we state the relationship between the partial order $\leq$ of a lattice L and the operations of infimum and supremum.

Proposition 1.1. *Let $(L, \leq)$ be a lattice and $a, b \in L$.*

$$a \leq b \Leftrightarrow a = a \wedge b \Leftrightarrow b = a \vee b.$$

Proof. We recall that by definition, $a \wedge b \leq a, b$ and $a, b \leq a \vee b$. Now, if $a \leq b$ then a is a lower bound of b and b is an upper bound of a. Since $a \wedge b$ is the greatest lower bound

of a and b, we have that $a \leq a \wedge b$. Similarly, $a \vee b$ is the least upper bound of a and b, and threfore $a \vee b \leq b$. Thus we have, $a \leq a \wedge b \leq a$ and $b \leq a \vee b \leq b$, which proves that $a = a \wedge b$ and $b = a \vee b$. To complete the proof we observe that $a = a \wedge b$ implies that a is a lower bound of a and b and so $a \leq b$. Similarly, $b = a \vee b$ concludes that b is an upper bound of a and b and so $b \geq a$. $\square$

A lattice L is said to be *complete* if every subset of L has a least upper bound and a greatest lower bound. When a lattice $(L, \leq)$ has a largest element it is referred to as the *top* element and is denoted by 1. Dually, when $(L, \leq)$ has a least element it is referred to as the *bottom* element and is denoted by 0. We notice that a complete lattice has both a top and a bottom element, since the supremum and the infimum of all the elements of the lattice exist and will give, respectively, the top element and the bottom element for the lattice. We state some basic properties of $\vee$ and $\wedge$ in a lattice L.

Proposition 1.2. *Let L be a lattice and $x, y, z \in L$. The following are true:*

(a) $x \vee x = x$ *and* $x \wedge x = x$.

(b) $x \vee 0 = x$ *and* $x \wedge 0 = 0$.

(c) $x \vee 1 = 1$ *and* $x \wedge 1 = x$.

(d) $x, y \leq x \vee y$ *and* $x \wedge y \leq x, y$.

(e) $x \wedge (y \wedge z) = (x \wedge y) \wedge z$ *and* $x \vee (y \vee z) = (x \vee y) \vee z$.

(f) $x \leq y$ *implies that* $x \wedge z \leq y \wedge z$ *and* $x \vee z \leq y \vee z$.

(g) $x \wedge (x \vee y) = x$ *and* $x \vee (x \wedge y) = x$.

Proof. The first six properties $(a) - (f)$ follow immediately from the definition of supremum and infimum of a set. To discuss (g) we notice the following. First of all, $x \wedge (x \vee y)$ is a lower bound of x and $x \vee y$, concluding that $x \wedge (x \vee y) \leq x$. On the other hand $x \leq x \vee y$

implies that $x = x \wedge x \leq x \wedge (x \vee y)$. Consequently, $x = x \wedge (x \vee y)$. The other equality follows similarly. $\square$

A lattice L is *distributive* if for all $x, y, z \in L$,

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \text{ and } x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z).$$

It turns out that a lattice is distributive if and only if it satisfies only one of the two equations.

Lemma 1.3. *Let L be a lattice. L is distributive if and only if $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ (or $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$).*

Proof. Suppose first that L is a distributive lattice. By definition, L satisfies both equalities. On the other hand, suppose that $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ holds for all $a, b, c \in L$. Let x, y, z be arbitrary elements of L. We claim that $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$. Let us denote $x \vee y = a$, then $a \in L$. We observe the following:

$$\begin{aligned}
(x \vee y) \wedge (x \vee z) &= a \wedge (x \vee z) \\
&= (a \wedge x) \vee (a \wedge z), \text{ by hypothesis} \\
&= ((x \vee y) \wedge x) \vee ((x \vee y) \wedge z) \\
&= x \vee ((x \vee y) \wedge z), \text{ using } (g) \text{ of Proposition 1.2} \\
&= x \vee ((x \wedge z) \vee (y \wedge z)), \text{ by hypothesis} \\
&= (x \vee (x \wedge z)) \vee (y \wedge z), \text{ by } (e) \text{ of Proposition 1.2} \\
&= x \vee (y \wedge z), \text{ using } (g) \text{ of Proposition 1.2}
\end{aligned}$$

Hence, L is a distributive lattice. $\square$

A partially ordered set $(L, \leq)$ is *totally ordered* if for every pair of elements $x, y \in L$, either $x \leq y$ or $y \leq x$. As an example, we notice that [0,1] is a totally ordered set. It can be verified easily that a totally ordered set is distributive.

A *frame* is a complete lattice L which satisfies a stronger distributive law; for every $x, y_\alpha \in L$,

$$x \wedge \left(\bigvee_\alpha y_\alpha \right) = \bigvee_\alpha (x \wedge y_\alpha).$$

It is evident from the definition and Lemma 1.3 that a frame L is a distributive lattice. The original source of frames came from the theory of topology. Therefore, we now take some time to remind the reader of the fundamental concepts of topological spaces.

GENERAL TOPOLOGY

Recall the definition of a topology. Let X be a set. A *topology* on X is a collection of subsets of X, say τ, satisfying the following properties:

1. $\emptyset, X \in \tau$,
2. if $U_1, U_2 \in \tau$, then $U_1 \cap U_2 \in \tau$, and
3. if $\{U_\alpha\}_{\alpha \in I} \subseteq \tau$, then $\bigcup_{\alpha \in I} U_\alpha \in \tau$, for any index set I.

The pair (X, τ) is called a *topological space*. We will, in the future, write X (instead of (X, τ)) to mean a topological space when the topology is understood. The elements of the topology τ are called the *open* sets in X. The complements of open sets in X are the *closed* sets. We observe that since the union of an arbitrary collection of open sets is open, the intersection of an arbitrary collection of closed sets is closed.

Suppose that τ_1 and τ_2 are two topologies on a given set X. If $\tau_1 \subseteq \tau_2$, we say that τ_1 is *coarser* than τ_2 or, that τ_2 is *finer* than τ_1.

Let X be a set. A *base* (or *basis*) for a topology on X is a collection $\mathfrak{B}$ of subsets of X such that

(a) for each $x \in X$, there is a $B \in \mathfrak{B}$ containing x, and

(b) if $x \in B_1 \cap B_2$, for some $B_1, B_2 \in \mathfrak{B}$ and $x \in X$, then there exists a $B_3 \in \mathfrak{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

The elements of a basis are called the *basis elements* or the *basic open sets* of X. Conversely, given a collection of subsets of X, say $\mathfrak{B}$, satisfying (a) and (b) there is a unique topology on X, say τ, for which $\mathfrak{B}$ is a base for τ. In this case we define the *topology* τ *generated by* $\mathfrak{B}$ to be the collection of subsets $U \subseteq X$ for which whenever $p \in U$ there is a $B \in \mathfrak{B}$ such that $p \in B \subseteq U$. On the other hand, given a topological space (X, τ) and a collection of subsets $\mathfrak{B}$ of X that satisfies (a) and (b), the topology generated by $\mathfrak{B}$ is τ. Here are examples of some standard topologies.

Example 1.4.

1. Consider the set of real numbers $\mathbb{R}$. The collection of bounded open intervals of $\mathbb{R}$, $\{(a, b) : a, b \in \mathbb{R}\}$, forms a base for a topology on $\mathbb{R}$, known as the *usual* topology on $\mathbb{R}$ and is denoted by u.

2. Suppose X is a set. The power set of X, $\mathcal{P}(X)$, is a topology on X. In this topology every subset of X is open (and closed), and $\mathcal{P}(X)$ is called the *discrete* topology on X.

3. Let X be any arbitrary set and let us consider $\tau = \{X, \emptyset\}$. τ is a topology on X, known as the *indiscrete* topology on X.

4. Let X be a set, let τ be the collection of subsets of X defined as follows: $U \in \tau$ if and only if either $U = \emptyset$ or $X \backslash U$ is finite. It follows that τ is a topology on X, known as the *finite complement* topology on X. We notice here that if X is a finite set, then the finite complement topology on X is the discrete topology on X.

At this point we provide our first example of a frame.

Example 1.5. Let (X, τ) be a topological space. The collection of open sets of X, τ, is a lattice when partially ordered by inclusion relation. The supremum and infimum of the lattice τ coincides with the set-theoretic union and intersection, respectively, which preserves 'open-ness'. Whereas, the infima of arbitrary elements of τ coincides with the interior of the intersection. With these operations, $(\tau, \subseteq)$ is a complete lattice and moreover, the strong distributive law holds. Hence, τ is a frame.

An element x in a topological space X is called an *isolated point* if $\{x\}$ is an open set. It thus follows that in a discrete topological space X, every element $x \in X$ is an isolated point and vice versa. Moreover, $\{\{x\} : x \in X\}$ is a base for the discrete topology on X.

Let X and Y be two topological spaces. A function $f : X \to Y$ is *continuous* if for every open set V in Y, $f^{-1}(V)$ is an open set in X. In other words, f^{-1} carries open sets of Y into open sets of X.

Example 1.6. Let $f : X \to Y$, where either X is equipped with the discrete topology or Y is equipped with the indiscrete topology, then f is continuous. Also, the identity map on a space X, $id_X : X \to X$, is a continuous map with respect to any topology.

A function $f : X \to Y$ is called an *open* map if f maps open sets of X into open sets of Y. Similarly f is a *closed* map if f maps closed sets of X into closed sets of Y. A function is called a *homeomorphism* if it is bijective and bicontinuous (that is, both the functions $f : X \to Y$ and $f^{-1} : Y \to X$ are continuous). If there is a homeomorphism between two spaces X and Y, then X and Y are said to be *homeomorphic*. We have the following equivalences in regards to open and closed maps when dealing with homeomorphisms.

Proposition 1.7. *Suppose that X and Y are two topological spaces. For any continuous bijection $f : X \to Y$ the following conditions are equivalent:*

1. *f is a homeomorphism.*

2. *f is an open map.*

3. *f is a closed map.*

Proof. Refer to Proposition 1.4.18 in [7]. □

For a topological space X and a subset V of X we define the *closure* of V in X, $\mathrm{cl}_X(V)$ (or $\overline{V}$, when the space is obvious), as the smallest closed set containing V. So, $\mathrm{cl}_X(V) = \bigcap\{K \mid K \text{ is closed and } V \subseteq K\}$. Clearly, the collection of closed sets containing V is nonempty

since X itself is a closed set and so is in the collection. Notice that $\mathrm{cl}_X(V)$ is a closed set in X as it is the intersection of closed sets.

Some well known properties of closure of a subset Y of a topological space X are as follows:

1. $Y \subseteq \mathrm{cl}_X(Y)$, since $\mathrm{cl}_X(Y)$ is the intersection of all closed sets that contains Y.

2. Suppose Y and Z are two subsets of X with $Y \subseteq Z$. If V is a subset of X containing Z, then V also contains Y. So, the collection of closed sets containing Y is a larger collection than those containing Z, and thus $\mathrm{cl}_X(Y) \subseteq \mathrm{cl}_X(Z)$.

3. Let Y be a subset of X, then by (2), $\mathrm{cl}_X(\mathrm{cl}_X(Y)) \subseteq \mathrm{cl}_X(Y)$. On the other hand, $Y \subseteq \mathrm{cl}_X(Y)$ (by (1)), and so applying (2) we have $\mathrm{cl}_X(Y) \subseteq \mathrm{cl}_X(\mathrm{cl}_X(Y))$. Hence, $\mathrm{cl}_X(\mathrm{cl}_X(Y)) = \mathrm{cl}_X(Y)$.

Dually, for any subset V of X, the *interior* of V in X, denoted $\mathrm{int}_X(V)$, is the union of all open sets contained in V. It thus follows that

$$\mathrm{int}_X(V) = \bigcup\{U \mid U \text{ open in } X, U \subseteq V\}.$$

Similarly, we observe that $\mathrm{int}_X(V)$ is an open set as it is the union of open sets. It is thus evident, from the definition, that $\mathrm{cl}_X(V) = V$, if V is closed in X, and $\mathrm{int}_X(V) = V$, if V is open in X. Next we state an interesting result on closure of open sets.

Proposition 1.8. *Let X be a topological space and U, V are open sets of X.*

$$U \cap V = \emptyset \Rightarrow U \cap cl_X(V) = \emptyset.$$

Proof. Suppose that $U \cap V = \emptyset$. If possible, let $x \in U \cap \mathrm{cl}_X(V)$. Therefore, $x \in U$ and $x \in \mathrm{cl}_X(V) \setminus V$. $x \in \mathrm{cl}_X(V)$ implies that every open set O of X containing x intersects V. We notice that U is an open set of X such that $x \in U$ and $U \cap V = \emptyset$. This is a contradiction. Hence, $U \cap \mathrm{cl}_X(V) = \emptyset$. $\square$

We frequently deal with some local properties of a topological space and for that it is necessary to introduce neighborhood of a point. Let X be a topological space and $p \in X$. A subset N is a *neighborhood* of p if there is an open set U of X such that $p \in U \subseteq N$. Thus it follows that any open set is a neighborhood of each of its points. It is sometime more convenient to use an equivalent definition of the closure of a subset of X in terms of neighborhood of points.

Proposition 1.9. *Suppose (X, τ) is a topological space and $Y \subseteq X$. An element $p \in cl_X(Y)$ if and only if every neighborhood of p intersects Y at a point other than p itself.*

Proof. Suppose (X, τ) is a topological space and Y is a subset of X. If $p \notin \mathrm{cl}_X(Y)$, then $p \in X \backslash \mathrm{cl}_X(Y)$ which is open in X. So, there exists a neighborhood U of p in X such that $U \subseteq X \backslash \mathrm{cl}_X(Y)$; thence, $U \cap Y = \emptyset$.

On the other hand, suppose $p \in X$ and U is a neighborhood of p in X that does not intersect Y. Therefore, $p \notin X \backslash U$, which is a closed subset of X containing Y. Hence, $p \notin \mathrm{cl}_X(Y)$. □

The following theorem will state some equivalent conditions for a function being continuous.

Theorem 1.10. *Let X and Y be topological spaces and let $f : X \to Y$ be a function. Then the following are equivalent:*

(a) f is continuous.

(b) For every subset V of X, $f(cl_X(V)) \subseteq cl_Y(f(V))$.

(c) For every closed set W of Y, $f^{-1}(W)$ is closed in X that is, f takes closed sets of Y into closed sets of X (under inverse image).

(d) For each $x \in X$ and each neighborhood V of $f(x)$, there is a neighborhood U of x such that $f(U) \subseteq V$.

Proof. Refer to Theorem 18.1 of [18].

□

Next we introduce the separation axioms on a topological space. Let X be a topological space. X is said to be:

1. T_0, if given two distinct points $p, q \in X$, there exists an open set U containing one of them but not the other.

2. T_1, if for any two distinct points $p, q \in X$, there exist two open sets U and V such that $p \in U \backslash V$ and $q \in V \backslash U$.

3. T_2 or *Hausdorff*, if for each pair of distinct points $p, q \in X$, there exist disjoint open sets containing p and q, respectively.

The following string of implications are true for a topological space X:

$$T_2 \Rightarrow T_1 \Rightarrow T_0.$$

In the following example we show that not every topological space is Hausdorff.

Example 1.11. Suppose X is an infinite set. Let τ be the collection of all subsets of X whose complement is finite, together with $\emptyset$. τ is a topology on X referred to as the *finite complement topology*. This topology fails to be Hausdorff, since given any two nonempty open sets U and V, $X \backslash U$ and $X \backslash V$ are finite. So that,

$$X \backslash (U \cap V) = (X \backslash U) \cup (X \backslash V) \text{ is finite .}$$

This in turn says that $U \cap V$ is infinite since X is infinite, and so in particular $U \cap V \neq \emptyset$. Thus, any two nonempty open sets of X intersect. So, no two distinct points of X can be separated by disjoint open sets of X.

Let X be a space. A collection of subsets of X is said to be a *covering* for X or said to *cover* X if X is the union of all the sets in the collection. If the cover contains only open sets of X then it is called an *open cover* of X. A sub collection of a covering of X which also covers X is called a *subcover* for X. A topological space X is a *compact* space if every open cover of X has a finite subcover.

Facts 1.12. Let X be a topological space.

1. X is T_1 if and only if every singleton set is closed in X.

2. A closed subset of a compact space is compact.

3. A compact subset of a Hausdorff space is closed.

4. Let $f : X \to Y$ be a continuous function, where Y is a topological space. If X is compact then $f(X)$ is compact in Y, that is, continuous functions take compact sets to compact sets.

Proof. For the proof of the above theorem we refer the reader to [18]. □

Suppose that (X, τ) is a topological space and $Y \subseteq X$. The *subspace* (or *induced*) *topology* on Y is the collection $\tau_Y = \{U \cap Y \mid U \in \tau\}$. With this inherited topology Y is called a *subspace* of X.

Lemma 1.13. *If $\mathfrak{B}$ is a basis for the topology of X, then the collection $\{B \cap Y : B \in \mathfrak{B}\}$ is a basis for the subspace topology on Y inherited from X.*

Proof. Refer to Lemma 16.1 in [18]. □

A topological property is said to be (*closed or open*)*hereditary* if whenever a space X has the property so does every (closed or open) subspace of it. When using the concept of "open set", we need to be careful about which space we are speaking of. It is not always the case that an open set of a subspace Y of X is open in X.

Lemma 1.14. *Let X be a topological space and Y is a subspace of X. If Y is open in X, then every open set in Y is also open in X.*

Proof. Refer to Lemma 16.2 in [18]. □

It can be verified that every subspace of a Hausdorff space is Hausdorff.

We will finish the chapter with a discussion of connectedness properties of a topological space. Let X be a topological space with the topology τ. A *separation* of X is a pair of disjoint nonempty open subsets U and V of X whose union is X, that is,

$$X = U \cup V, \text{ where, } U \text{ and } V \text{ are open with } U \cap V = \emptyset.$$

In this case X is said to be separated by the two open sets U and V. The space X is *connected* if there does not exist any separation of X. In other way, a space X is *disconnected* if there is a separation of X by two nonempty disjoint open subsets of X. A subset of a topological space X is *clopen* if it is both closed and open. Another characterization of connected spaces is the following: A topological space X is disconnected if and only if there exist a nonempty proper clopen subset of X.

Proposition 1.15. *The image of a connected space under a continuous map is connected.*

Proof. Refer to Theorem 23.5 in [18]. □

Finally we define several types of disconnectedness on a topological space. We will find these concepts useful later in this dissertation. Let (X, τ) be a topological space. The space X is called

1. *totally disconnected* if the only nonempty proper connected subsets of X are the singletons.

2. *zero-dimensional* if the topology τ has a base consisting of clopen sets.

3. *extremally disconnected* if the closure of every open set is clopen in the topology.

4. *basically disconnected* if the closure of every cozero set is clopen in the topology.

It is an easy fact to check that for any topological space X, basically disconnectedness implies extremally disconnectedness and an extremally disconnected space X is zero dimensional.

Lemma 1.16. *Let X be a zero-dimensional topological space. For every basic open set B in the topology and each $x \in B$, there exist a clopen set K such that $p \in K \subseteq B$.*

Proof. Straightforward and is left to the reader. □

Example 1.17. A few well-known examples of zero dimensional space are $\mathbb{N}$ and $\mathbb{Z}$, with the usual topology. Whereas, $\mathbb{R}$ with the usual topology is not a zero-dimensional space. Also, every discrete space is zero dimensional.

GROUPS AND RINGS

A *binary operation* on a set S is a function $\phi : S \times S \to S$. A *group* is a set G with a binary operation $+$ which satisfies the following:

1. $(x + y) + z = x + (y + z)$, for all $x, y, z \in G$.

2. There exists an element $0_G \in G$ such that $0_G + x = x + 0_G = x$, for all $x \in G$; 0_G is known as an *identity* of G.

3. For each $x \in G$ there exists a $x' \in G$ such that $x' + x = x + x' = 0_G$.

It is straightforward to show that in a group an identity is unique and such a x' satisfying (3) is also unique; we usually write $-x$ for such an x'.

A group G is *abelian* if it also satisfies that $x + y = y + x$, for all $x, y \in G$. A subset H of a group G which is also a group with respect to the same operation $+$ on G is called a *subgroup* of G. In this case we use the notation $H \leq G$.

A *ring* is an abelian group R with a second binary operation $\cdot$ which satisfies the following:

1. $(x \cdot y) \cdot z = x \cdot (y \cdot z)$, for all $x, y, z \in R$.

2. $x \cdot (y + z) = x \cdot y + x \cdot z$ and $(x + y) \cdot z = x \cdot z + y \cdot z$, for all $x, y, z \in R$.

The additive identity of the ring R is the identity element, 0_R, of the group R. If there is an element $1_R \in R$ such that $x \cdot 1_R = 1_R \cdot x = x$, for all $x \in R$, then 1_R is called the (multiplicative) identity of R and we say R is a ring with identity. A ring R is *commutative* if $x \cdot y = y \cdot x$, for all $x, y \in R$. An element $x \in R$ is a *unit* if there exists an element $y \in R$ such that $x \cdot y = y \cdot x = 1_R$. A *subring* S is a subset of the ring R, with 1_R, which is also a ring with the same binary operations '+' and '$\cdot$'. A *division ring* is a ring R where every element is a unit. A commutative division ring is called a *field*.

Throughout our discussion we will assume all rings are commutative with a multiplicative identity which is different from the additive identity.

Example 1.18. The set of all integers, $\mathbb{Z}$, under the usual operations of addition and multiplication is a commutative ring with the identity element 1. Moreover, the set of all rational numbers, $\mathbb{Q}$, and the set of all real numbers, $\mathbb{R}$, are fields under the usual addition and multiplication. However, the set $\mathbb{N}$ is not a ring under usual addition and multiplication since $\mathbb{N}$ is not a group under addition.

Let R be a commutative ring with identity. An nonzero element $x \in R$ is called a *zero divisor* if there is a nonzero element $y \in R$ such that $x \cdot y = 0_R$. An *integral domain* is a ring which has no non-zero zero divisors. We notice that the ring $\mathbb{Z}$ is an integral domain, as well as any field.

A subset I of a ring R is an *ideal* of R if it satisfies the following properties:

(i) $x - y \in I$, for all $x, y \in I$, and

(ii) $r \cdot x \in I$, for all $r \in R$ and $x \in I$.

Every ring R possesses at least two ideals, namely, R and $\{0_R\}$. These are called the trivial ideals. A *prime* ideal is a proper ideal I such that if $x \cdot y \in I$, then $x \in I$ or $y \in I$.

Notice that R is an integral domain if and only if the ideal $\{0_R\}$ is prime. A prime ideal I is *minimal* if there does not exist any other prime ideal properly contained in I, that is, if there exist a prime ideal J of R such that $J \leq I$, then $J = I$. We denote the collection of prime ideals of R by $Spec(R)$, and the collection of minimal prime ideals by $Min(R)$.

Proposition 1.19. *Let I be an ideal of a commutative ring R with identity.*

1. *$I = R$ if and only if I contains a unit.*

2. *If R is commutative, then R is a field if and only if its only ideals are trivial.*

Proof. For the proof see Chapter 7, Section 4, Proposition 9 of [6]. □

An ideal M of a ring R is called a *maximal* ideal if $M \neq R$ and the only ideals containing M are M and R. A commutative ring R is called a *local ring* if it has a unique maximal ideal. Using the usual Zorn's Lemma argument it follows that every proper ideal in a ring R with identity is contained in a maximal ideal (see Proposition 11, Chapter 7.4 of [6]).

Proposition 1.20. *Assume R is a commutative ring. Every maximal ideal of R is prime.*

Proof. We refer to Corollary 14 of Chapter 7.4 in [6]. □

Let I be an ideal in a commutative ring R.

(a) The *radical* of I, denoted by rad I, is defined as follows:

$$\text{rad } I = \{x \in R : x^n \in I \text{ for some } n \in \mathbb{N}\}.$$

(b) The radical of the ideal $\{0\}$ is called the *nilradical* of R.

(c) I is called a *radical* ideal if $I = \text{rad } I$.

An element $x \in R$ is called *nilpotent* if $x^m = 0$, for some $m \in \mathbb{N}$. Note that $x \in R$ is in the nilradical of R if and only if some power of x is 0. So, the nilradical of R is the set of all nilpotent elements of R.

Proposition 1.21. *Suppose that R is a commutative ring with identity. The radical of a proper ideal I is the intersection of all prime ideals containing I. In particular, the nilradical is the intersection of all the prime ideals in R.*

Proof. We refer to Chapter 7.4 of [6]. □

LATTICE-ORDERED GROUP

In this section we will develop the theory of lattice-ordered groups and generalize some of its notions to the theory of frames. Excellent references for the theory of lattice-ordered groups are [1] and [5]

A group $(G, +)$ is said to be *partially ordered* if it is equipped with a partial order, $\leq$, which is compatible with $+$; that is, if $g \leq h$, then $g + k \leq h + k$ and $k + g \leq k + h$, for any $g, h, k \in G$. When the partial order on G is a lattice, then G is called a *lattice-ordered group* or *ℓ-group* for short. If the lattice order of an ℓ-group G is a total order, we call G an *o-group*. The set $G^+ = \{g \in G : 0 \leq g\}$ of positive elements is called the *positive cone* of G.

Remark 1.22. Let G be a partially ordered group. The partial order of G is determined by G^+, that is, $g \leq h$ if and only if $h - g \in G^+$.

Suppose that G is a partially ordered group and G^+ is the positive cone of G. We notice the following important properties of G^+:

1. G^+ is closed under addition,
2. G^+ is normal, that is, $g + G^+ - g = G^+$, for all $g \in G$, and
3. $G^+ \cap -G^+ = \{0\}$.

We state below a few basic properties of ℓ-groups. For the proof we refer to Section 1.1, Proposition 1.1.1 and 1.1.2 of [1].

Proposition 1.23. *Suppose G is an ℓ-group.*

1. *The group operation distributes over both meet and join; that is,*

$$x + (y \vee z) + w = (x + y + w) \vee (x + z + w),$$

for all $x, y, z, w \in G$*, and likewise for* $\wedge$*.*

2. *For all* $x, y \in G$*,* $-(x \vee y) = -x \wedge -y$ *and* $-(x \wedge y) = -x \vee -y$*.*

3. $x - (x \wedge y) + y = x \vee y$*, for all* $x, y \in G$*.*

For $g \in G$, we define the *positive part* of g, $g^+ = g \vee 0$; and the *negative part* of g, $g^- = (-g) \vee 0$. So, $g^+, g^- \in G^+$ and

$$g + g^- = g + (-g \vee 0) = 0 \vee g = g^+;$$

and hence we have that $g = g^+ - g^-$. Also,

$$g^+ \wedge g^- = (g + g^-) \wedge g^- = (g \wedge 0) + g^- = -g^- + g^- = 0.$$

We say two positive elements g and h are *disjoint* if $g \wedge h = 0$. Thus, g^+ and g^- are disjoint. So, every element of an ℓ- group can be written as a difference of two disjoint positive elements and furthermore, this representation is unique.

Proposition 1.24. *Suppose* G *is an* ℓ*-group. If* $x, y \in G$ *are disjoint, then* x *and* y *commute in* G*.*

Proof. Let G be an ℓ-group and $x, y \in G$ with $x \wedge y = 0$. Using property (3) of Proposition 1.23 we have that $x + y = x \vee y$. Hence, $x + y = x \vee y = y \vee x = y + x$. □

We define the *absolute value* of an element $g \in G$, denoted $|g|$, as $g^+ + g^-$. Similar to ordinary absolute value, we have, $g = |g|$ precisely when $0 \leq g$.

Proposition 1.25. *Suppose* G *is an* ℓ*-group and* $g \in G$*.* $|g| = g^+ \vee g^- = g \vee -g$*.*

The natural question is whether the absolute value satisfies the triangle inequality. As it turns out, the triangle inequality holds for ℓ-groups which are abelian.

Theorem 1.26. *Suppose G is an ℓ-group. For all $g, h \in G$, $|g+h| \leq |g| + |h| + |g|$. If G is abelian, then the usual triangle inequality holds.*

Proof. For the proof we again refer to [1], Section 1.1, Proposition 1.1.3. □

Another nice property satisfied by ℓ-groups is given by our next theorem.

Theorem 1.27 (Riesz Decomposition Property)**.** *Suppose G is an ℓ-group and let $h_1, \ldots, h_n \in G^+$. If $0 \leq g \leq h_1 + \ldots + h_n$, then there exists $g_1, \ldots, g_n \in G$ with $0 \leq g_i \leq h_i$, for all i, such that $g = g_1 + \ldots + g_n$.*

Proof. For the proof we refer to Proposition 1.1.4 of [1] □

As a consequence of the Riesz Decomposition property we have that for $g, h, k \in G^+$;

$$g \wedge (h+k) \leq (g \wedge h) + (g \wedge k).$$

The main purpose of our introduction of the theory of lattice-ordered groups is to study a particular class of subgroups of an ℓ-group, namely, the convex ℓ-subgroups. A subgroup H of an ℓ-group G is an *ℓ-subgroup* if H is also a sublattice of G. An ℓ-subgroup K of an ℓ-group G is called *convex* if $h, k \in K$ and $h \leq g \leq k$ for some $g \in G$ implies that $g \in K$.

Remark 1.28. To check the convexity of an ℓ-subgroup K of G, it is enough to verify that if $k \in K$, $g \in G$, and $|g| \leq |k|$, then $g \in K$.

If G is an ℓ-group, $\mathcal{C}(G)$ denotes the set of all convex ℓ-subgroups of G. When partially ordered by inclusion, $\mathcal{C}(G)$ is a complete lattice since the intersection of an arbitrary family of convex ℓ-subgroups is again a convex ℓ-subgroup. We may speak of the convex ℓ-subgroup generated by a set. For any $g \in G$, $G(g)$ denotes the convex ℓ-subgroup of G generated by g. In other words, $G(g)$ is the smallest convex ℓ-subgroup of G containing g. Likewise, for any

$X \subseteq G$, the convex ℓ-subgroup generated by X is denoted by $G(X)$ and is defined as the smallest convex ℓ-subgroup of G containing X. In general, for any $C \in \mathcal{C}(G)$, $C = \bigvee_{c \in C} G(c)$.

Proposition 1.29. *Let G be an ℓ-group and $g, h \in G$.*

$$G(g) = \{k \in G : |k| \leq n|g|, \text{ for some positive integer } n\}.$$

Consequently, $G(g) = G(|g|)$. Furthermore, if $0 < g, h$, then

$$G(g \vee h) = G(g) \vee G(h) \text{ and } G(g \wedge h) = G(g) \cap G(h).$$

Proof. Refer to Proposition 1.2.3 of [1] □

Proposition 1.30. *Let G be an ℓ-group and $g \in G$. If $C \in \mathcal{C}(G)$, then $g \in C$ if an only if $G(g) \subseteq C$.*

Proof. Suppose G is an ℓ-group and $g \in G$. Let C be a convex ℓ-subgroup of G. If $G(g) \subseteq C$, then $g \in C$ since $g \in G(g)$. On the other hand, suppose $g \in C$. Since $G(g)$ is the smallest convex ℓ-subgroup of G containing g, so $G(g) \subseteq C$. □

Let G be an ℓ-group and $C \in \mathcal{C}(G)$. Define $C' = \bigvee\{D \in \mathcal{C}(G) : D \cap C = \{0\}\}$; C' is the unique maximum convex ℓ-subgroup for which $C \cap C' = \{0\}$, known as the *pseudo-complement* of C. If $C = C''$, then C is called a *polar* subgroup of G. We denote the collection of all polar subgroups of G by $\mathfrak{P}(G)$.

Remark 1.31. Let G be an ℓ-group and $B, C \in \mathcal{C}(G)$, then the following holds:

1. $C \subseteq C''$.
2. If $B \subseteq C$, then $C' \subseteq B'$.
3. $C'' = C'''$; that is, $C' \in \mathcal{P}(G)$.
4. $(B \vee C)' = B' \cap C'$.

5. $(B \cap C)'' = B'' \cap C''$.

Suppose that X is a subset of an ℓ-group G, then $\{g \in G : |g| \wedge |x| = 0, \text{ for all } x \in X\}$ is a polar subgroup of G. Furthermore, if $X \in \mathcal{C}(G)$, then the above subgroup is precisely X'. From now on we will use the notation X' to denote the polar of any subset X of G without any hesitation, even if X is not a convex ℓ-subgroup of G. In particular, for any $g \in G$, we shall understand by g' the polar subgroup $\{g\}' = \{h \in G : |h| \wedge |g| = 0\}$.

Proposition 1.32. *Suppose G is an ℓ-group and $g \in G$, then $g' = G(g)'$.*

Proof. Let G be an ℓ-group and $g \in G$. By definition, $g' = \{h \in G : |h| \wedge |g| = 0\}$ and $G(g)' = \{h \in G : |h| \wedge |k| = 0, \text{ for all } k \in G(g)\}$. Suppose $h \in g'$, then $|h| \wedge |g| = 0$ and so $|h| \wedge |k| = 0$ for all $k \in G$ with $|k| \leq n|g|$. Thus, $|h| \wedge |k| = 0$ for all $k \in G(g)$. Therefore $h \in G(g)$. Hence, $g' \subseteq G(g)'$. To show the other inclusion we notice that $g \in G(g)$. So, if $h \in G(g)$, then $|h| \wedge |g| = 0$ and hence $h \in g'$. □

Suppose G and H are two ℓ-groups with G an ℓ-subgroup of H. If K is an ℓ-subgroup of H, then $K \cap G$ is an ℓ-subgroup of G. Furthermore, if $K \in \mathcal{C}(H)$, then $K \cap G \in \mathcal{C}(G)$ for; $k \in K \cap G$ and $g \in G$ with $|g| \leq |k|$ implies that $g \in K$, since K is convex in H and $g \in H$.

Theorem 1.33. *Let G and H be two ℓ-groups with $G \leq H$. For every $g \in G$, $H(g) \cap G = G(g)$. In general, for any subset $X \subseteq G$, $H(X) \cap G = G(X)$.*

Proof. Suppose that G and H are ℓ-groups with $G \leq H$. Let $g \in G$. First we notice that $H(g) \cap G$ is a convex ℓ-subgroup of G and so $G(g) \subseteq H(g) \cap G$. To show the other inclusion, we let $k \in H(g) \cap G$. So $k \in G$ and $|k| \leq n|g|$, for some positive integer n. Thus $k \in G(g)$, by definition.

To finish the proof, suppose $X \subseteq G$; then

$$H(X) \cap G = \left(\bigvee_{x \in X} H(x) \right) \cap G = \bigvee_{x \in X} (H(x) \cap G) = \bigvee_{x \in X} G(x) = G(X).$$

□

It follows from the above theorem that for any convex ℓ-subgroup K of G, $H(K) \cap G = K$.

A proper convex ℓ-subgroup P of the ℓ-group G is *prime* if $a \wedge b \in P$ implies that $a \in P$ or $b \in P$ $(a, b \in G)$. Equivalently, $P \in \mathcal{C}(G)$ is prime if and only if $a \wedge b = 0$ implies that $a \in P$ or $b \in P$. It is a standard Zorn's Lemma result that for any $0 \neq g \in G$ there is a convex ℓ-subgroup V maximal with respect to $g \notin V$. Such an object is prime. For notational convenience we will denote by $Spec(G)$ the collection of all prime subgroups of G, and by $Min(G)$ the collection of all minimal prime elements of $\mathcal{C}(G)$. We now endow $Min(G)$ with the hull-kernel topology, where the basic open sets are of the form $M(a) = \{P \in Min(G) : a \notin P\}, a \in G$. As it turns out, $Min(G)$ with the hull-kernel topology is Hausdorff, since, given any two distinct minimal primes P and Q, there are distinct elements $a, b \in G$ such that $a \in P \setminus Q$ and $b \in Q \setminus P$. Thus it follows that $M(a) \cap M(b) = \emptyset$ with $P \in M(a)$ and $Q \in M(b)$.

An attempt to find the conditions for $Min(G)$ to be compact with respect to the hull-kernel topology lead to defining another topology on $Min(G)$, known as the inverse topology. The basic open sets of this topology are given by, $N(a) = \{P \in Min(G) : a \in P\}$, for arbitrary $a \in G$. $Min(G)$ with the inverse topology is a compact, T_1 space. A detailed discussion about this topology can be found in [17].

We will end the section with a discussion of rigid subgroups of ℓ-groups. Suppose G and H are two ℓ-groups with $G \leq H$. We say that G is *rigid* in H or H is a *rigid extension* of G if for each $h \in H$ there exists a $g \in G$ so that $g'' = h''$, where the polars are defined in the larger group H. It turns out that if G is rigid in H, then $Min(G) = Min(H)$. In fact, we need a weaker version of rigid extension to prove the result; G is called an *r-subgroup* of H or H is an *r-extension* of G if for each $0 < h \in H$ and each $P \in Min(H)$ which does not contain h, there exists a $0 < g \in G \setminus P$ so that $g'' \subseteq h''$. So, a rigid subgroup is an r-subgroup as well. The fact that G is an r-subgroup of H if and only if the contraction map $P \to P \cap G$ is a homeomorphism of $Min(H)$ onto $Min(G)$ with respect to the hull-kernel topology is proved in Proposition 2.3 of [4]. Finally, we define an r^*-extension of ℓ-groups: $G \leq H$ is

an r^*-*subgroup* or H is an r^*-*extension* of G if for every $0 < h \in H$ and $P \in Min(H)$ which contains h, there exists a $0 < g \in G \cap P$ such that $h'' \subseteq g''$. As it turns out, a rigid subgroup is also an r^*-subgroup and an r^*-extension $G \leq H$ implies homeomorphism of $Min(H)$ onto $Min(G)$ with respect to the inverse topology, the homeomorphism being the contraction map $P \to P \cap G$. We refer [17] for detailed proof of the above result.

As an ending comment of the first chapter we will like to mention that in this dissertation we intend to generalize several notions of the theory of lattice-ordered groups to the theory of algebraic frames. One of these being the rigid extensions of algebraic frames, which will be the main topic of discussion in Chapter 6. Moreover, in Chapter 4 we will provide the information of when the minimal prime element space of an algebraic frame is compact. This result goes parallely to the result for $\ell-$groups which was proved in [4].

CHAPTER 2

Algebraic Frame

We begin with the definition of a compact element of a complete lattice. This definition is analogous to the definition of a compact subset of a topological space.

Let $(L, \leq)$ be a complete lattice. An element $c \in L$ is said to be *compact* if $c \leq \bigvee_{i \in I} b_i$ implies that $c \leq \bigvee_{i \in I_0} b_i$ for some finite subset I_0 of I ($b_i \in L$). Denote the set of compact elements of L by $\mathfrak{K}(L)$. We say L is *compact* if 1 is compact. L is called an *algebraic* lattice if every element in L is the supremum of compact elements. If L is a frame which is an algebraic lattice, we say that L is an *algebraic frame*. A frame L is said to satisfy the *finite intersection property on compact elements* (the FIP for short) if $x, y \in \mathfrak{K}(L)$ implies that $x \wedge y \in \mathfrak{K}(L)$.

As an example of an algebraic lattice, we recall the theory of lattice-ordered groups.

Example 2.1. Let G be a lattice-ordered group and $\mathcal{C}(G)$ denotes the collection of all convex ℓ-subgroups of G. $\mathcal{C}(G)$ is a complete lattice under inclusion and is, in fact, algebraic. It can be easily verified that the compact elements of $\mathcal{C}(G)$ are precisely the ones of the form $G(g)$, for all $g \in G$. Also, for any $C \in \mathcal{C}(G)$, $C = \bigvee_{x \in C} G(x)$. Hence, $\mathcal{C}(G)$ is a complete algebraic lattice.

The following example describes a complete non-algebraic lattice.

Example 2.2. Let us consider the usual topology on the real numbers, $\mathbb{R}$. Let $\mathfrak{O}(\mathbb{R})$ denote the collection of all open sets of $\mathbb{R}$. It is a complete frame under inclusion (see Example 1.5). $\mathfrak{O}(\mathbb{R})$ is not an algebraic lattice since nonempty open sets are *not compact* in the usual topology, which says that $\mathfrak{O}(\mathbb{R})$ has no compact elements at all except for the $\emptyset$.

Proposition 2.3. *Let L be a complete lattice, then $\mathfrak{K}(L)$ is closed under finite joins.*

Proof. Suppose that L is a complete lattice and $c, d \in \mathfrak{K}(L)$. Let $c \vee d \leq \bigvee_{i \in I} b_i$, where $b_i \in L$ for all i in some index set I. It follows that $c, d \leq c \vee d \leq \bigvee_{i \in I} b_i$. Since c, d are compact, there exist finite subsets I_0 and I_1 of I such that $c \leq \bigvee_{i \in I_0} b_i$ and $d \leq \bigvee_{i \in I_1} b_i$. Thus, $c \vee d \leq \left(\bigvee_{i \in I_0} b_i\right) \bigvee \left(\bigvee_{i \in I_1} b_i\right) = \bigvee_{i \in I'} b_i$ where $I' = I_0 \cup I_1$ is a finite subset of I. Hence, $c \vee d \in \mathfrak{K}(L)$. $\square$

Let $(P, \leq)$ be a poset and $\gamma : P \to P$ is a function. γ is called a *closure operator* on the poset P if it satisfies the following conditions:

(i) $x \leq y$ implies that $\gamma(x) \leq \gamma(y)$, for all x, y in P.

(ii) $x \leq \gamma(x)$, for all x in P.

(iii) $\gamma(\gamma(x)) = \gamma(x)$, for all x in P.

The subset $\{x \in P : \gamma(x) = x\}$ of P is the set of all γ-*closed* elements of P.

We observe the fact that if L is a complete lattice then a closure operator γ on L maps the top element to the top element, whereas it does not necessarily take the bottom element to the bottom element. As an example of closure operator we consider a topological space X. The power set of X, $\mathfrak{P}(X)$, is a complete lattice under the set-theoretic inclusion relation. The function $\mathrm{cl} : \mathfrak{P}(X) \to \mathfrak{P}(X)$ given by, $\mathrm{cl}(Y) =$ the closure of Y in X, for all subset Y of X, is a closure operator, since:

(i) $Y_1 \subseteq Y_2$ implies that $\mathrm{cl}(Y_1) \subseteq \mathrm{cl}(Y_2)$.

(ii) $Y \subseteq \mathrm{cl}(Y)$, for every subset Y of X, and

(iii) $\mathrm{cl}(\mathrm{cl}(Y)) = \mathrm{cl}(Y)$, for every subset Y of X.

Let L be a frame, for each $x \in L$, we define

$$x^{\perp} = \bigvee\{y \in L : y \wedge x = 0\}.$$

The element $x^{\perp}$ is called the *pseudocomplement* of x. An element of the form $x^{\perp}$ is known as a *pseudocomplement* or *polar* of L. Due to the presence of the strong distributive law in a frame, it follows that for any $x \in L$,

$$x \wedge x^{\perp} = x \wedge \bigvee\{y \in L : y \wedge x = 0\} = \bigvee\{x \wedge y : y \wedge x = 0\} = 0$$

Therefore, $x^{\perp}$ is the largest element of L disjoint from x. It is evident from the definition of pseudo-complement that for any element x in a frame L, $x \leq x^{\perp\perp}$, since $x^{\perp}$ has the property that $x^{\perp} \wedge x^{\perp\perp} = 0$. Furthermore, we observe that for any $x, y \in L$, $x \leq y$ implies that $y^{\perp} \leq x^{\perp}$. The reason behind this fact is the following: any element of L that is disjoint from y, is also disjoint from x. Hence, the supremum of all the elements of L that are disjoint from y is smaller than the supremum of those that are disjoint from x. An element $x \in L$ is *complemented* if $x \vee x^{\perp} = 1$. L is said to be a *zero-dimensional* frame if each element in L can be written as a supremum of complemented elements.

For our convenience we will define some more concepts which will be used frequently in this dissertation. An element x in a frame L is called *dense* if $x^{\perp} = 0$. A compact dense element of L is called a *unit*. We observe that for any element x in L, $x \vee x^{\perp}$ is dense, which will follow as a consequence of the next lemma. In particular $1 \in L$ is dense, as $1^{\perp} = 0$.

Lemma 2.4. *Let L be a frame and $x, y \in L$. The following are true:*

1. $(x \vee y)^{\perp} = x^{\perp} \wedge y^{\perp}$ *and*

2. $(x \wedge y)^{\perp} \geq x^{\perp} \vee y^{\perp}$.

Proof. Assume that L is a frame.

1. Since $x, y \leq x \vee y$, we have $(x \vee y)^{\perp} \leq x^{\perp}$ and $(x \vee y)^{\perp} \leq y^{\perp}$. Hence, $(x \vee y)^{\perp} \leq x^{\perp} \wedge y^{\perp}$. To show the other direction, we observe the following equalities:

$$\begin{aligned}
(x \vee y) \wedge (x^{\perp} \wedge y^{\perp}) &= (x \wedge (x^{\perp} \wedge y^{\perp})) \vee (y \wedge (x^{\perp} \wedge y^{\perp})) \\
&= ((x \wedge x^{\perp}) \wedge y^{\perp}) \vee ((y \wedge y^{\perp}) \wedge x^{\perp}) \\
&= (0 \wedge y^{\perp}) \vee (0 \wedge x^{\perp}) \\
&= 0 \vee 0 \\
&= 0
\end{aligned}$$

   Using the definition of $(x \vee y)^{\perp}$ it follows that $(x \vee y)^{\perp} \geq x^{\perp} \wedge y^{\perp}$. Therefore, the equality holds.

2. We observe that $(x \wedge y) \wedge (x^{\perp} \vee y^{\perp}) = (x \wedge y \wedge x^{\perp}) \vee (x \wedge y \wedge y^{\perp}) = 0$. Once again using the definition of the polar of an element it follows that $x^{\perp} \vee y^{\perp} \leq (x \wedge y)^{\perp}$.

□

As a corollary to the preceding lemma it follows that for any x in L,

$$(x \vee x^{\perp})^{\perp} = x^{\perp} \wedge x^{\perp\perp} = 0,$$

establishing the fact that $x \vee x^{\perp}$ is dense for any element x in L.

Proposition 2.5. *Let L be an algebraic lattice. L is distributive if and only if it is an algebraic frame.*

Proof. If L is an algebraic frame, then L is distributive, by definition. On the other hand, let L be a distributive, algebraic lattice. By the definition of an algebraic lattice L is complete.

To prove that L is a frame, it remains to show that L satisfies the strong distributive law of a frame. Let $x, y_\alpha \in L$, for some α in an index set I. We first observe that $x \wedge \left(\bigvee_\alpha y_\alpha\right) \geq \bigvee_\alpha (x \wedge y_\alpha)$ always holds, for:

$$\begin{aligned} x \wedge y_\alpha \ \leq \ x,\ y_\alpha,\ \text{for all } \alpha \in I \ &\Rightarrow \ x \wedge y_\alpha \ \leq x,\ \bigvee_{\alpha \in I} y_\alpha,\ \text{for all } \alpha \in I \\ &\Rightarrow \ x \wedge y_\alpha \ \leq \ x \wedge \left(\bigvee_{\alpha \in I} y_\alpha\right),\ \text{for all } \alpha \in I \\ &\Rightarrow \ \bigvee_{\alpha \in I} (x \wedge y_\alpha) \ \leq x \wedge \left(\bigvee_{\alpha \in I} y_\alpha\right) \end{aligned}$$

To show the other inequality, we consider any arbitrary compact element c of L with $c \leq x \wedge \left(\bigvee_{\alpha \in I} y_\alpha\right)$. Since L is an algebraic lattice, we only need to show that $c \leq \bigvee_\alpha (x \wedge y_\alpha)$. Let us consider the following string of implications:

$$\begin{aligned} c \leq x \wedge \left(\bigvee_{\alpha \in I} y_\alpha\right) \ &\Rightarrow \ c \leq x \text{ and } c \leq \bigvee_{\alpha \in I} y_\alpha \\ &\Rightarrow \ c \leq y_{\alpha_1} \vee \ldots \vee y_{\alpha_n},\ \text{for finitely many } \alpha_1, \ldots, \alpha_n \in I,\ \text{since } c \in \mathfrak{K}(L) \\ &\Rightarrow \ c \leq x \wedge (y_{\alpha_1} \vee \ldots \vee y_{\alpha_n}) \\ &\Rightarrow \ c \leq (x \wedge y_{\alpha_1}) \vee \ldots \vee (x \wedge y_{\alpha_n}),\ \text{since } L \text{ is a distributive lattice} \\ &\Rightarrow \ c \leq \bigvee_\alpha (x \wedge y_\alpha). \end{aligned}$$

□

Like group or ring homomorphism, a frame homomorphism is a map between two frames which preserves the frame operations. To state it formally, let L and M be two frames. A function $f : L \to L$ is said to be a *frame homomorphism* if the following holds:

1. $f(0_L) = 0_M$.

2. $f(1_L) = 1_M$.

3. $f(x \wedge y) = f(x) \wedge f(y)$, for all $x, y \in L$ and

4. $f\left(\bigvee_{\alpha \in I} x_\alpha\right) = \bigvee_{\alpha \in I} f(x_\alpha)$, for all $x_\alpha \in L$ and any index set I.

Therefore, a frame homomorphism f preserves the partial order of the lattice L, that is, for all $x, y \in L$, $x \leq y$ implies that $f(x) \leq f(y)$.

As an example, suppose that X and Y are two topological spaces and $f : X \to Y$ is a continuous function. The map $f^{-1} : \mathfrak{O}(Y) \to \mathfrak{O}(X)$ is a frame homomorphism between the frames of open sets of Y and X.

Lemma 2.6. *Let L and M be algebraic frames and $f : L \to M$ is a frame homomorphism. For each $x \in L$, $f(x^\perp) \leq f(x)^\perp$.*

Proof. Suppose that L and M are algebraic frames and $x \in L$. We notice that

$$f(x^\perp) \wedge f(x) = f(x^\perp \wedge x) = f(0_L) = 0_M,$$

by the properties of a frame homomorphism. Hence, $f(x^\perp) \leq f(x)^\perp$.

□

A frame homomorphism $f : L \to M$ is called *coherent* if f maps the compact elements of L to the compact elements of M; that is, $f(\mathfrak{K}(L)) \subseteq \mathfrak{K}(M)$. We notice that a coherent, frame homomorphism preserves the 'algebraic' property of a frame; that is, if $f : L \to M$ is a coherent frame homomorphism and L is algebraic, then $f(L)$ is an algebraic frame.

CHAPTER 3

The Minimal Prime Element Space, $Min(L)$

Finally we are ready to talk about the minimal prime element space of an algebraic frame. From now on the finite intersection property of a frame will be a minimum requirement for the rest of the dissertation, unless otherwise stated. Also, we will assume the existence of a unit element in the frame. Since in this chapter we will talk about the prime elements, we start the chapter with the definition of prime elements in a complete lattice. The notion of prime elements of a complete lattice is a generalization of the prime ideals of a commutative ring with identity, in fact, the definition also goes parallely.

Let L be a complete lattice. An element $p < 1$ is called *prime* if for any $x, y \in L$, $x \wedge y \leq p$ implies that $x \leq p$ or $y \leq p$. A prime p is *minimal* if there does not exist any other prime element q (other than p itself) with the property that $q \leq p$. We denote $Spec(L)$ as the set of all prime elements of L and call it the prime spectrum of the frame. We denote $Min(L)$ as the set of all minimal prime elements of L. We notice here that for an algebraic frame L the set $Spec(L)$ is nonempty. This follows from the fact that with the existence of a compact element, say c, the usual Zorns Lemma argument assures us of the existence of elements m maximal with respect to $c \nleq m$. Such elements turn out to be primes. We will prove this

result formally in Chapter 5. The same argument can be used to prove that an algebraic frame is semiprime.

If L is a distributive lattice to start with, then the definition of a prime element can be replaced by a stronger condition. Recall the relationship between a partial order of a lattice and the operations of infimum and supremum. Given a lattice L and $x, y \in L$,

$$x \leq y \Leftrightarrow x \wedge y = x \Leftrightarrow x \vee y = y.$$

Proposition 3.1. *Let L be a distributive lattice.*

$$p \in Spec(L) \Leftrightarrow (x \wedge y = p \Rightarrow x = p \text{ or } y = p),$$

for all $x, y \in L$

Proof. Suppose that L is a distributive lattice.

($\Rightarrow$) Let $p \in Spec(L)$ and $x, y \in L$ with $x \wedge y = p$. We have the following string of implications:

$$\begin{aligned} x \wedge y = p &\Rightarrow x \wedge y \leq p \\ &\Rightarrow x \leq p \text{ or } y \leq p, \text{ by the definition of } p \\ &\Rightarrow x \leq x \wedge y \text{ or } y \leq x \wedge y, \text{ by assumption} \\ &\Rightarrow x = x \wedge y \text{ or } y = x \wedge y \\ &\Rightarrow x = p \text{ or } y = p. \end{aligned}$$

($\Leftarrow$) Let $x, y \in L$ with $x \wedge y \leq p$. In this case we consider the following string of

arguments:

$$\begin{aligned} x \wedge y \leq p &\Rightarrow (x \wedge y) \vee p = p \\ &\Rightarrow (x \vee p) \wedge (y \vee p) = p, \text{ since } L \text{ is distributive} \\ &\Rightarrow x \wedge p = p \text{ or } y \wedge p = p, \text{ by assumption} \\ &\Rightarrow x \leq p \text{ or } y \leq p \end{aligned}$$

Hence, $p \in Spec(L)$.

□

Moreover, it turns out that to check the primeness of an element in an algebraic frame we just need check the prime condition with respect to compact elements.

Proposition 3.2. *Let L be an algebraic frame and $p \in L$. $p \in Spec(L)$ if and only if for any compact elements c, d in L, $c \wedge d \leq p$ implies that $c \leq p$ or $d \leq p$.*

Proof. Suppose that L is an algebraic frame and $p \in L$. Let $p \in Spec(L)$. Using the definition of prime element it follows that if $c, d \in \mathfrak{K}(L)$, then $c, d \in L$ and so, $c \wedge d \leq p$ implies that $c \leq p$ or $d \leq p$.

Conversely, we assume that for each $c, d \in \mathfrak{K}(L)$, $c \wedge d \leq p$ implies that $c \leq p$ or $d \leq p$. We want to show that $p \in Spec(L)$. For that let $x, y \in L$ with $x \wedge y \leq p$. Since L is algebraic, $x = \bigvee_{\alpha \in I} c_\alpha$ and $y = \bigvee_{\beta \in J} d_\beta$, where $c_\alpha, d_\beta \in \mathfrak{K}(L)$ for all $\alpha \in I, \beta \in J$, I and J being some index sets. So,

$$\left(\bigvee_{\alpha \in I} c_\alpha\right) \wedge \left(\bigvee_{\beta \in J} d_\beta\right) = \bigvee_{\alpha \in I} \bigvee_{\beta \in J} (c_\alpha \wedge d_\beta) \leq p.$$

Thus we get, $c_\alpha \wedge d_\beta \leq p$, for all $\alpha \in I, \beta \in J$. If possible, let $x \nleq p$ and $y \nleq p$. Then there exist some $\alpha \in I$ and some $\beta \in J$ such that $c_\alpha \nleq p$ and $d_\beta \nleq p$. In this case the element $c_\alpha \wedge d_\beta \nleq p$, by assumption, which is a contradiction. Hence, $x \leq p$ or $y \leq p$, proving that p is prime in L. □

The Lemma on Ultrafilters is a fundamental result that connects the minimal primes and ultrafilters of algebraic frames. This result has been used several times in the literature. As for references we suggest the reader consult [2], [4] and [16]. Before stating the result we remind the reader of the notions of filters and ultrafilters.

Let L be a frame and $\mathfrak{F}$ be a nonempty subset of L. $\mathfrak{F}$ is called a *filter* if:

(i) $0 \notin \mathfrak{F}$,

(ii) for all $x, y \in L$, $x, y \in \mathfrak{F}$ implies $x \wedge y \in \mathfrak{F}$, and

(iii) if $x \in \mathfrak{F}$ and $y \in L$ with $x \leq y$ then $y \in \mathfrak{F}$.

A maximal filter is called an *ultrafilter*. Again, Zorn's Lemma argument can be applied to show that every filter is contained in an ultrafilter.

Lemma 3.3 (Lemma on Ultrafilter)**.** *Let L be an algebraic frame satisfying the FIP and $p \in Spec(L)$. p is a minimal prime element of L if and only if*

$$F_p = \{c \in \mathfrak{K}(L) : c \nleq p\}$$

is an ultrafilter on $\mathfrak{K}(L)$. In this case, $p = \bigvee\{c^{\perp} : c \in F_p\}$.

For the proof of the above lemma we refer to [15]. We will prove a partial converse to the above result.

Lemma 3.4. *Suppose that L is an algebraic frame satisfying the FIP and $\mathfrak{U}$ is an ultrafilter on $\mathfrak{K}(L)$. The element $p = \bigvee\{c^{\perp} : c \in \mathfrak{U}\}$ is a minimal prime element of L.*

Proof. Let us assume that L is an algebraic frame satisfying the FIP and $\mathfrak{U}$ is an ultrafilter of compact elements in L. Let $p = \bigvee\{c^{\perp} : c \in \mathfrak{U}\}$. To prove the lemma we only need to show that p is a prime element in L and $p = \bigvee\{c^{\perp} : c \in \mathfrak{K}(L), c \nleq p\}$; because, using Lemma 3.3 on p with $F_p = \mathfrak{U}$, an ultrafilter, we can conclude that $p \in Min(L)$. We start by showing that $p \in Spec(L)$. For that let $x, y \in \mathfrak{K}(L)$ with $x \wedge y \leq p$. Since L has the

FIP, $x \wedge y \in \mathfrak{K}(L)$. Therefore, $x \wedge y \leq \bigvee\{c^{\perp} : c \in \mathfrak{U}\}$ implies that there exist finitely many elements $c_1, \ldots, c_n \in \mathfrak{U}$ such that

$$x \wedge y \leq c_1^{\perp} \vee \ldots \vee c_n^{\perp} \leq (c_1 \wedge \ldots \wedge c_n)^{\perp},$$

where the second inequality follows from Lemma 2.4. Let us denote $c = c_1 \wedge \ldots \wedge c_n$, then $c \in \mathfrak{U}$ as $\mathfrak{U}$ is a filter. Thus, $x \wedge y \leq c^{\perp}$, for some $c \in \mathfrak{U}$ and so $x \wedge y \wedge c = 0$. This concludes that $x \wedge y \notin \mathfrak{U}$, as $0 \notin \mathfrak{U}$. By the definition of a filter it follows that either $x \notin \mathfrak{U}$ or $y \notin \mathfrak{U}$. Without loss of generality we assume that $x \notin \mathfrak{U}$. Since $\mathfrak{U}$ is an ultrafilter of compacts in L and $x \in \mathfrak{K}(L) \setminus \mathfrak{U}$, it follows that the filter generated by $\mathfrak{U}$ and x on $\mathfrak{K}(L)$, denoted by $\langle \mathfrak{U}, x \rangle$, will give the entire $\mathfrak{K}(L)$, that is,

$$\langle \mathfrak{U}, x \rangle = \{k \in \mathfrak{K}(L) : k \geq c \wedge x \text{ for some } c \in \mathfrak{U}\} = \mathfrak{K}(L).$$

Since $0 \in \mathfrak{K}(L)$ there exists some element $d \in \mathfrak{U}$ such that $d \wedge x = 0$ and so $x \leq d^{\perp}$. d being an element of the ultrafilter $\mathfrak{U}$, $x \leq d^{\perp} \leq p$. Hence, $p \in Spec(L)$.

To complete the proof we have to show that $\mathfrak{U} = F_p = \{c \in \mathfrak{K}(L) : c \nleq p\}$. First we suppose that $c \in \mathfrak{U}$. If $c \leq p$, then $c \leq \bigvee\{d^{\perp} : d \in \mathfrak{U}\}$. Since c is a compact element in L,

$$c \leq d_1^{\perp} \vee \ldots \vee d_n^{\perp} \leq (d_1 \wedge \ldots \wedge d_n)^{\perp},$$

the last inequality following from Lemma 2.4. Letting $d = d_1 \wedge \ldots \wedge d_n$ we get that $d \in \mathfrak{U}$ (since $\mathfrak{U}$ is a filter) and $c \leq d^{\perp}$. Therefore, $c \wedge d = 0$ and so $0 \in \mathfrak{U}$, as both c and d are elements in $\mathfrak{U}$. This leads us to a contradiction, concluding that $c \nleq p$. Consequently, $\mathfrak{U} \subseteq F_p$. For the converse inclusion let $c \in \mathfrak{K}(L)$ with $c \nleq p$. If $c \notin \mathfrak{U}$, then $\mathfrak{U}$ being an ultrafilter of compacts,

$$\langle \mathfrak{U}, c \rangle = \{k \in \mathfrak{K}(L) : k \geq d \wedge c, \text{ for some } d \in \mathfrak{U}\} = \mathfrak{K}(L).$$

Since $0 \in \mathfrak{K}(L)$ there exists some $d \in \mathfrak{U}$ with $c \wedge d = 0$. Thence, $c \leq d^{\perp} \leq p$ (as $d \in \mathfrak{U}$), which is a contradiction. Thus $c \in \mathfrak{U}$, concluding that $F_p \subseteq \mathfrak{U}$. $\square$

A frame L is called *semiprime* if $\bigwedge\{p : p \in Min(L)\} = 0$. We observe that, in general, a non-algebraic frame might not be semiprime; however, an algebraic frame is always semiprime, which will be proved eventually. We need another definition before proving the next result. Let L be a frame and $x \in L$. We define the upward directed set of x, denoted by $\uparrow x$, to be the set $\{y \in L : x \leq y\}$. In other words, $\uparrow x = \{x \vee y : y \in L\}$.

Proposition 3.5. *Suppose that L is a frame (not necessarily algebraic) and $n = \bigwedge\{p : p \in Min(L)\}$. $Min(L) \cong Min(\uparrow n)$.*

Proof. Let L be a frame. We first notice as a simple exercise that $n = \bigwedge\{p : p \in Min(L)\} = \bigwedge\{p : p \in Spec(L)\}$. We consider two cases:

Case 1: $n = 0$. In this case $\uparrow n = L$, concluding that $Min(L) \cong Min(\uparrow n)$.

Case 2: $n \neq 0$. We want to show that $p \in Min(L)$ if and only if $p \in Min(\uparrow n)$. For that, we first prove that $p \in Spec(L)$ if and only if $p \in Spec(\uparrow n)$. Let $p \in Spec(L)$. Observe that $p = p \vee n$ is an element in $\uparrow n$. Let $x, y \in L$ with $(x \vee n) \wedge (y \vee n) \leq p$. We have the following:

$$\begin{aligned}
(x \vee n) \wedge (y \vee n) \leq p \quad &\Rightarrow \quad (x \wedge y) \vee n \leq p, \text{ by the property of distributivity} \\
&\Rightarrow \quad x \wedge y \leq p, \text{ since } x \wedge y \leq (x \wedge y) \vee n \\
&\Rightarrow \quad x \leq p \text{ or } y \leq p, \text{ since } p \in Min(L) \\
&\Rightarrow \quad x \vee n \leq p \vee n = p \text{ or } y \vee n \leq p \vee n = p.
\end{aligned}$$

Hence, $p \in Spec(\uparrow n)$. To prove the other implication, let $q \vee n \in Spec(\uparrow n)$ for some

$q \in L$. Let $x, y \in L$ with $x \wedge y \leq q \vee n$. We again consider the following:

$$\begin{aligned} x \wedge y \leq q \vee n &\Rightarrow (x \wedge y) \vee n \leq q \vee n \\ &\Rightarrow (x \vee n) \wedge (y \vee n) \leq q \vee n, \text{ using distributive property of } L \\ &\Rightarrow (x \vee n) \leq q \vee n \text{ or } (y \vee n) \leq q \vee n, \text{ since } q \vee n \in Spec(\uparrow n) \\ &\Rightarrow x \leq q \vee n \text{ or } y \leq q \vee n. \end{aligned}$$

Therefore, $q \vee n \in Spec(L)$. Finally, let $p \in Min(L)$ and $q \in Spec(\uparrow n)$, such that $q \leq p$. Since $q \in Spec(L)$ also, it follows that $p = q$. So, there does not exist any prime element of $\uparrow n$ that is strictly below p. Hence, $p \in Min(\uparrow L)$. With a similar argument it follows that if $q \in Min(\uparrow n)$, then $q \in Min(L)$. Hence, $Min(L) \cong Min(\uparrow n)$.

□

It thus follows from the above result that when dealing with $Min(L)$ of a frame L, we can assume, without loss of generality, that L is semiprime. In the following theorem we prove that an algebraic frame is always semiprime.

Theorem 3.6. *For an algebraic frame L,* $\bigwedge_{p \in Min(L)} p = 0$.

Proof. Suppose that L is an algebraic frame. If possible, let $\bigwedge_{p \in Min(L)} p \neq 0$. Since L is algebraic there exist some $c \in \mathfrak{K}(L)$ with

$$0 < c \leq \bigwedge_{p \in Min(L)} p.$$

Consequently, $c \leq p$ for all $p \in Min(L)$. Let us consider the set $S = \{x \in L \mid c \nleq x\}$. $S \neq \emptyset$ since $0 \in S$. Let $\{x_\alpha\}$ be a chain in S and let $x = \bigvee x_\alpha$. Since $c \nleq x_\alpha$ for all α, $c \nleq x$ which means that $x \in S$. Thus, Zorn's Lemma concludes that S has a maximal element, say m; so $c \nleq m$. We claim that m is prime. For that let $a, b \in L$ with $a \wedge b = m$. If $a > m$ and $b > m$, then the maximality of m says that $c \leq a$ and $c \leq b$. So, $c \leq a \wedge b = m$, which is a

contradiction. Therefore, m is prime. Since every prime element contains a minimal prime element, there exist some $p \in Min(L)$ with $p \leq m$. Since $c \nleq m$, $c \nleq p$, which is again a contradiction. Hence, $\bigwedge\{p \in L \mid p \in Min(L)\} = 0$. $\square$

Let L be an algebraic frame satisfying the FIP. We define a topology on $Min(L)$ as follows: the collection $\{U(a) \mid a \in \mathfrak{K}(L)\}$ forms a base of open sets for a topology on $Min(L)$, known as the *hull-kernel* topology (or *Zariski* topology), where $U(a) = \{p \in Min(L) \mid a \nleq p\}$. The basic closed sets for the hull-kernel topology are of the form $V(a) = \{p \in Min(L) \mid a \leq p\}$, $a \in \mathfrak{K}(L)$. We notice that for any $x \in L$, $U(x)$ and $V(x)$ are the set theoretic complement of each other. In the following lemma we state some basic properties satisfied by the sets $U(x)$ and $V(x)$.

Lemma 3.7. *Suppose that L is an algebraic frame satisfying the FIP. The following are true:*

1. $\bigcup_{\alpha} U(x_\alpha) = U\left(\bigvee_{\alpha} x_\alpha\right)$, $x_\alpha \in L$.
2. $\bigcap_{\alpha} V(x_\alpha) = V\left(\bigvee_{\alpha} x_\alpha\right)$, $x_\alpha \in L$.
3. *For any $x, y \in L$, $U(x) \cap U(y) = U(x \wedge y)$ and similarly, $V(x) \cup V(y) = V(x \wedge y)$.*
4. $U(x) = \emptyset \Leftrightarrow x = 0 \Leftrightarrow V(x) = Min(L)$.

Proof. 1. Let $p \in \bigcup_{\alpha} U(x_\alpha)$. Then $p \in U(x_\alpha)$, for some $x_\alpha \in L$. So $x_\alpha \nleq p$, which implies that $\bigvee_{\alpha} x_\alpha \nleq p$. Hence, $p \in U\left(\bigvee_{\alpha} x_\alpha\right)$.

Conversely, let $p \in U\left(\bigvee_{\alpha} x_\alpha\right)$. By definition we have $\bigvee_{\alpha} x_\alpha \nleq p$. This implies that there exist some $x_\alpha \in L$ such that $x_\alpha \nleq p$. Therefore, $p \in U(x_\alpha) \subseteq \bigcup_{\alpha} U(x_\alpha)$.

2. Let $p \in \bigcap_{\alpha} V(x_\alpha)$. Then $p \in V(x_\alpha)$ for all α. By definition, $x_\alpha \leq p$ for all α which concludes that $\bigvee_{\alpha} x_\alpha \leq p$. Thus, $p \in V\left(\bigvee_{\alpha} x_\alpha\right)$.

To prove the other inclusion we notice that $\bigvee_{\alpha} x_\alpha \leq p$ implies that $x_\alpha \leq p$, for all α. Thence, $V\left(\bigvee_{\alpha} x_\alpha\right) = \bigcap_{\alpha} V(x_\alpha)$.

3. Suppose that $p \in U(x) \cap U(y)$, then $x \nleq p$ and $y \nleq p$. Since p is prime in L, this implies that $x \wedge y \nleq p$; that is, $p \in U(x \wedge y)$. So, $U(x) \cap U(y) \subseteq U(x \wedge y)$. On the other hand if $x \wedge y \nleq p$, then $x \nleq p$ and $y \nleq p$. This means that $U(x \wedge y) \subseteq U(x) \cap U(y)$.

   The proof for $V(x)$ follows similarly.

4. Let $x \in L$ with $U(x) = \emptyset$. This means that $x \leq p$ for all $p \in Min(L)$. Consequently $x = 0$, using Theorem 3.6 and hence, $V(x) = V(0) = Min(L)$. With a similar string of arguments it follows that if $V(x) = Min(L)$, then $x \leq p$ for all minimal prime elements of L. Therefore, $x = 0$ and $U(x) = U(0) = \emptyset$.

□

In the following proposition we will completely describe the open and closed sets of $Min(L)$ with respect to the hull-kernel topology.

Proposition 3.8. *Suppose that L is an algebraic frame satisfying the FIP. The open (closed) sets of $Min(L)$ endowed with the hull-kernel topology are precisely of the form $U(x)$ ($V(x)$) for some $x \in L$.*

Proof. Let L be an algebraic frame satisfying the FIP and let $x \in L$. Since L is algebraic, $x = \bigvee_{\alpha \in I} c_\alpha$, where $\{c_\alpha \mid \alpha \in I\} \subseteq \mathfrak{K}(L)$. Thus, $U(x) = U\left(\bigvee_{\alpha \in I} c_\alpha\right) = \bigcup_{\alpha \in I} U(c_\alpha)$ (by lemma 3.7), which is open with respect to the hull-kernel topology.

Conversely, suppose that U is an open set of $Min(L)$ endowed with the hull-kernel topology. Since $\{U(c) \mid c \in \mathfrak{K}(L)\}$ forms a base of open sets for the hull-kernel topology, $U = \bigcup_{\alpha \in I} U(c_\alpha)$, for some subset $\{c_\alpha \mid \alpha \in I\}$ of $\mathfrak{K}(L)$. Therefore by Lemma 3.7, $U = U(c)$ where, $c = \bigvee_{\alpha \in I} c_\alpha$. □

The next lemma will state an important relationship between the minimal primes and the compact elements of an algebraic frame. We will use this lemma several times in this dissertation.

Lemma 3.9. *Suppose that L is an algebraic frame satisfying the FIP and $p \in Min(L)$. For any $x \in \mathfrak{K}(L)$ either $x \nleq p$ or $x^{\perp} \nleq p$.*

Proof. Let L be an algebraic frame satisfying the FIP and $p \in Min(L)$. By Lemma 3.3 we have $p = \bigvee\{d^{\perp} \mid d \in \mathfrak{K}(L), d \nleq p\}$. Let $x \in \mathfrak{K}(L)$ with $x \leq p$, then $x \leq \bigvee\{d^{\perp} \mid d \in \mathfrak{K}(L), d \nleq p\}$. Since x is a compact element of L, there exist a finite number of compact elements of L, say $d_1, d_2, .., d_n$, with $d_i \nleq p$ for all $i = 1, 2, .., n$ such that $x \leq {d_1}^{\perp} \vee .. \vee {d_n}^{\perp} \leq (d_1 \wedge .. \wedge d_n)^{\perp}$, where the last inequality stems from Lemma 2.4. Let $d = d_1 \wedge .. \wedge d_n$. Since L satisfies the FIP, d is compact and $x \leq d^{\perp}$. Since $d_i \nleq p$ for all $i = 1, 2, .., n$, it follows that $d \nleq p$. Now, $x \leq d^{\perp}$ implies that $x^{\perp} \geq d^{\perp\perp} \geq d$. Therefore, $x^{\perp} \nleq p$; because if $x^{\perp} \leq p$, then $d \leq x^{\perp} \leq p$, which is a contradiction. □

An useful fact stems as a consequence of the above lemma. We recall that an element $x \in L$ is a dense element if $x^{\perp} = 0$.

Lemma 3.10. *Let L be an algebraic frame and let $x \in \mathfrak{K}(L)$. If x is a dense element of L, then $V(x) = \emptyset$.*

Proof. Suppose that L is an algebraic frame and x is a compact dense element of L. Since x is dense, $x^{\perp} = 0$, which says that $x^{\perp} \leq p$ for all minimal prime elements p of L. So by Lemma 3.9, $x \nleq p$ for all minimal primes p. Hence for all $p \in Min(L)$, $p \notin V(x)$. Consequently, $V(x) = \emptyset$. □

We notice some immediate properties of the hull-kernel topology on $Min(L)$. Firstly, for

any $x \in \mathfrak{K}(L)$, $U(x) = V(x^{\perp})$ and $U(x^{\perp}) = V(x)$, since:

$$\begin{aligned} p \in V(x^{\perp}) &\Leftrightarrow x^{\perp} \leq p \\ &\Leftrightarrow x \nleq p, \text{ using Lemma 3.9} \\ &\Leftrightarrow p \in U(x) \end{aligned}$$

Hence, $\{U(x) \mid x \in \mathfrak{K}(L)\}$ is a base of clopen sets for the hull-kernel topology on $Min(L)$. Therefore, $Min(L)$ endowed with the hull-kernel topology is zero-dimensional. Furthermore, if $p \in Min(L)$, then every other $q \in Min(L)$ different than p has the property that $q \nleq p$. Hence, $V(p) = \{p\}$, saying that points are closed in the hull-kernel topology on $Min(L)$. Therefore, $Min(L)$ is a T_1 space. Recall that a topological space is Hausdorff if every pair of distinct points in the space can be separated by disjoint open subsets of the space. As it turns out, $Min(L)$ endowed with the hull-kernel topology is a Hausdorff space.

Lemma 3.11. *Let L be an algebraic frame satisfying the FIP. The hull-kernel topology on $Min(L)$ is a Hausdorff topology.*

Proof. Suppose that L is an algebraic frame satisfying the FIP. Let $p, q \in Min(L)$ with $p \neq q$. Since L is algebraic, we have $p = \bigvee\{c : c \in S \subseteq \mathfrak{K}(L)\}$. Since $p \neq q$, there exist some compact c in the subcollection S such that $c \nleq q$; for, if $c \leq q$ for all $c \in S$, then $p \leq q$, which is a contradiction. So we have that $q \in U(c)$, $p \in U(c^{\perp})$ (by lemma 3.9) and $U(c) \cap U(c^{\perp}) = U(c \wedge c^{\perp}) = U(0) = \emptyset$. Hence, p and q are separated by disjoint open sets $U(c)$ and $U(c^{\perp})$ respectively, proving that $Min(L)$ is Hausdorff. $\square$

We observed that $Min(L)$ endowed with the hull-kernel topology is a zero-dimensional, Hausdorff space. As a consequence we have the following proposition.

Proposition 3.12. *Let L be an algebraic frame. For any two distinct minimal prime elements p and q of L there exist compact elements c and d of L with $c \wedge d = 0$ such that*

(i) $c \leq p$, $c \nleq q$ *and*

(ii) $d \leq q$, $d \nleq p$.

Proof. Suppose that L is an algebraic frame. Let $p, q \in Min(L)$ with $p \neq q$. Since $Min(L)$ is a Hausdorff space there exist $c, d \in \mathfrak{K}(L)$ such that $p \in U(d)$, $q \in U(c)$, and $U(c) \cap U(d) = \emptyset$. So, $d \nleq p$, $c \nleq q$, and $c \wedge d = 0$, the last conclusion being a consequence of Lemma 3.7. Finally, $p \notin U(c)$ implies that $p \in V(c)$ and so $c \leq p$. Similarly, $d \leq q$. □

We have seen that the hull-kernel topology on $Min(L)$ is a zero dimensional space. A natural question as a consequence of this fact is, when is $Min(L)$ extremally disconnected or even discrete? (Recall that a space is called extremally disconnected if the closure of every open set is open.) To answer this we need a lemma.

Lemma 3.13. *Suppose $x \in L$, then cl$(U(x)) = V(x^{\perp})$ with respect to the hull-kernel topology on $Min(L)$. In particular, if $x \in \mathfrak{K}(L)$, $U(x) = V(x^{\perp})$.*

Proof. We first prove that $U(x) \subseteq V(x^{\perp})$. Let $p \in U(x)$; that is, $x \nleq p$. Since $x \wedge x^{\perp} = 0$ and p is prime, $x^{\perp} \leq p$. Thus $p \in V(x^{\perp})$, concluding the $U(x) \subseteq V(x^{\perp})$. So, $\mathrm{cl}(U(x)) \subseteq V(x^{\perp})$ since $V(x^{\perp})$ is closed (using Proposition 3.8).

To show the other inclusion, let $p \in V(x^{\perp})$. So, $x^{\perp} \leq p$. We will show that every open set containing p will intersect $U(x)$, thus proving that $p \in \mathrm{cl}(U(x))$. Let $y \in \mathfrak{K}(L)$ satisfying $p \in U(y)$. Suppose, by way of contradiction, that $U(y) \cap U(x) = \emptyset$; then, $U(y \wedge x) = \emptyset$ which implies that $x \wedge y = 0$ (by Lemma 3.7). It thus follows that $y \leq x^{\perp} \leq p$, that is, $p \in V(y)$ which is a contradiction. Therefore, every neighborhood around p intersects $U(x)$, thence $p \in \mathrm{cl}(U(x))$.

□

Theorem 3.14. *Let L be an algebraic frame satisfying the FIP. The hull-kernel topology on $Min(L)$ is extremally disconnected if and only if $V(x^{\perp})$ is open for all $x \in L$.*

Proof. Let L be an algebraic frame satisfying the FIP. Suppose that $Min(L)$ is extremally disconnected. Let U be an open set in $Min(L)$; then $U = U(x)$ for some $x \in L$ (by

Proposition 3.8). By hypothesis, $\mathrm{cl}(U) = \mathrm{cl}(U(x))$ is open. Using Lemma 3.13 we have that $\mathrm{cl}(U(x)) = V(x^{\perp})$. Hence $V(x^{\perp})$ is open.

Conversely, suppose that $V(x^{\perp})$ is open for all $x \in L$. Let U be an open set in $Min(L)$; then $U = U(x)$ for some $x \in L$. Once again using Lemma 3.13 we have that $\mathrm{cl}(U(x)) = V(x^{\perp})$ which is open. Thus, the closure of every open set is clopen and so $Min(L)$ is extremally disconnected. □

We state a different characterization for when the space $Min(L)$ is an extremally disconnected space, in terms of the internal characteristics of the frame L.

Theorem 3.15. *Let L be an algebraic frame satisfying the FIP. The hull-kernel topology on $Min(L)$ is extremally disconnected if and only if for each $a \in L$ there exists $c \in L$ such that $U(a) \subseteq U(c)$ and the following holds:*

1. *For all $b \in L$, $a \wedge b = 0$ implies that $c \wedge b = 0$.*

2. *For all $p \in V(c)$ there exists $c_p \in \mathfrak{K}(L)$ with $p \in U(c_p)$ such that $a \wedge c_p = 0$.*

Proof. Let the hull-kernel topology on $Min(L)$ be extremally disconnected. Let $a \in L$ and consider the subset $U(a)$ of $Min(L)$. By hypothesis, $\mathrm{cl}(U(a))$ is open. Therefore, there exist some $c \in L$ such that $\mathrm{cl}(U(a)) = U(c)$. Thence, $U(a) \subseteq U(c)$. To prove the remaining conditions, let $b \in L$ with $a \wedge b = 0$. Thus $U(a) \cap U(b) = \emptyset$, using Lemma 3.7 and therefore $\mathrm{cl}(U(a)) \cap U(b) = \emptyset$, by Proposition 1.8. It thus follows that $p \in U(b)$ implies that $p \notin \mathrm{cl}(U(a))$. Consequently, $p \notin U(c)$. Thence, $U(c) \cap U(b) = \emptyset$, concluding that $c \wedge b = 0$. Finally, if $p \in V(c)$, then $p \notin U(c) = \mathrm{cl}(U(a))$. So, there exist a compact $c_p \in L$ such that $p \in U(c_p)$ and $U(a) \cap U(c_p) = \emptyset$. This implies that $a \wedge c_p = 0$.

Conversely, let U be an open set of $Min(L)$. So, $U = U(a)$ for some $a \in L$, by Proposition 3.8. By hypothesis, there exist some $c \in L$ with the properties given in (1) and (2) such that $U(a) \subset U(c)$. This implies that $\mathrm{cl}(U(a)) \subseteq \mathrm{cl}(U(c))$. We first show that $U(c)$ is closed. Notice that if $p \notin U(c)$, then $c \leq p$. By (2), there exists a compact element $c_p \in X$ such

that $p \in U(c_p)$ and $a \wedge c_p = 0$. By (1), $c \wedge c_p = 0$. Thus, $U(c \wedge c_p) = U(c) \cap U(c_p) = \emptyset$. Hence, $p \notin \mathrm{cl}(U(c))$. So, $U(c) = \mathrm{cl}(U(c))$. Thus, $U(c)$ is a closed set containing $U(a)$. Finally we show that $U(c)$ is the smallest closed set containing $U(a)$. Let V be a closed set containing $U(a)$. So, there exist $b \in L$ such that $V = V(b)$. $U(a) \subseteq V(b)$ implies that $U(a) \cap U(b) = U(a \wedge b) = \emptyset$. So, $V(a \wedge b) = Min(L)$. This implies that $a \wedge b = 0$. Therefore, using (1), it follows that $c \wedge b = 0$. Consequently, $U(c \wedge b) = U(c) \cap U(b) = \emptyset$. Hence, $U(c) \subset V(b) = V$. Combining the above arguments we conclude that $\mathrm{cl}(U(a)) = U(c)$ is open. So $Min(L)$ is extremally disconnected. $\square$

We finish the chapter with a theorem which states that equivalent conditions for $Min(L)$ to be a discrete space. Recall that a in a discrete space X, every element of X is an isolated point.

Theorem 3.16. *Suppose that L is an algebraic frame satisfying the FIP and let $p \in Min(L)$. The following statements are equivalent:*

(i) p is an isolated point of $Min(L)$.

(ii) For some $c \in \mathfrak{K}(L)$, $p = c^{\perp}$.

(iii) p is a polar of L.

Proof. Let L be an algebraic frame satisfying the FIP and $p \in Min(L)$.

$(i) \Rightarrow (ii)$ Suppose that p is an isolated point of $Min(L)$, that is, $\{p\} = U(x)$ for some $x \in L$. Since $p \in U(x)$ and $\{U(c) : c \in \mathfrak{K}(L)\}$ forms a base for the topology on $Min(L)$, there exists a $c \in \mathfrak{K}(L)$ such that $p \in U(c) \subseteq U(x)$. Therefore, $\{p\} = U(c)$. Since $c \nleq p$ and p is a prime element of L it follows that $c^{\perp} \leq p$. We claim that $p \leq c^{\perp}$. We notice that $\{p\} = U(c)$ means that the only ultrafilter of L containing c is F_p. Let $d \in \mathfrak{K}(L)$ satisfy $d \leq p$. We observe that if $d \wedge c \neq 0$, then there would be an ultrafilter containing the compact element $d \wedge c$. Any such ultrafilter would necessarily contain c and therefore would be equal to F_p.

Thus, $d \in F_p$, a contradiction since $d \leq p$. the only recourse we have is that $d \wedge c = 0$ and so $d \leq c^{\perp}$. Since L is an algebraic frame it follows that $p \leq c^{\perp}$, whence $p = c^{\perp}$.

$(ii) \Rightarrow (iii)$ This implication is clear since $c^{\perp}$ is a polar and hence p is a polar.

$(iii) \Rightarrow (i)$ Suppose that p is a polar, then $p = p^{\perp\perp}$. Since $p < 1$ it follows that $p^{\perp} \neq 0$. Therefore, we can choose a compact element c satisfying $0 < c \leq p^{\perp}$. First off we observe that this implies $p = p^{\perp\perp} \leq c^{\perp}$. On the other hand we also know that $c \nleq p$, since $c \neq 0$, and so $c \in F_p$. This means that $c^{\perp} \leq p$. It thus follows that $p = c^{\perp}$. To finish the proof we show that $\{p\} = U(c)$. We have already explained why $\{p\} \subseteq U(c)$. As to the reverse inclusion we observe that if $q \in U(c)$, then $p = c^{\perp} \leq q$ and so since they are both minimal prime elements we conclude that $p = q$, whence $U(c) = \{p\}$. Thus, p is an isolated point of $Min(L)$. $\square$

CHAPTER 4

The Inverse Topology On $Min(L)$, $Min(L)^{-1}$

In this chapter we will discuss a different topology on the minimal prime element space of an algebraic frame satisfying the FIP, namely, the inverse topology. We will like to mention here that the notion of inverse topology was first introduced by McGovern in [17] for the theory of the lattice-ordered groups. We begin with the definition of the inverse topology. Suppose that L is an algebraic frame satisfying the FIP. Recall that for any $x \in L$, $V(x) = \{p \in Min(L) : x \leq p\}$ is the collection of all closed sets for the hull-kernel topology on $Min(L)$. Since for any $c, d \in \mathfrak{K}(L)$, $V(c) \cap V(d) = V(c \vee d)$ and $c \vee d \in \mathfrak{K}(L)$, the collection of sets $\{V(x) : x \in \mathfrak{K}(L)\}$ forms a base for a topology on $Min(L)$ known as the *inverse topology* on $Min(L)$. We will denote $Min(L)$ endowed with the inverse topology by $Min(L)^{-1}$, and by $Min(L)$ we will mean the set $Min(L)$ is endowed with the hull-kernel topology. The first observation we make is that if $V(x)$ is a basic open set in $Min(L)^{-1}$, then $V(x) = U(x^{\perp})$ is open in $Min(L)$. Thus, $Min(L)^{-1}$ is coarser than $Min(L)$. In contrast to the hull-kernel topology we will see that $Min(L)^{-1}$ is not a Hausdorff nor a zero-dimensional space, generally speaking. However, $Min(L)^{-1}$ satisfies another important topological property: compactness.

Lemma 4.1. *Suppose L is an algebraic frame satisfying the FIP. $Min(L)^{-1}$ is a compact, T_1 space.*

Proof. We first show that $Min(L)^{-1}$ is a T_1 space. Let $p, q \in Min(L)^{-1}$ with $p \neq q$. Since L is algebraic, $p = \bigvee_{\alpha \in I} c_\alpha$ and $q = \bigvee_{\beta \in J} d_\beta$, for some index I and J where, $\{c_\alpha \mid \alpha \in I\}$ and $\{d_\beta \mid \beta \in J\}$ are some subcollections of $\mathfrak{K}(L)$. It follows that there exist at least one c_{α_0} and one d_{β_0} from the two respective subcollections such that $c_{\alpha_0} \nleq q$ and $d_{\beta_0} \nleq p$; since otherwise, $p \leq q$ and $q \leq p$ respectively, which is a contradiction. Hence, $q \in U(c_{\alpha_0}) \bigcap V(d_{\beta_0})$ and $p \in U(d_{\beta_0}) \bigcap V(c_{\alpha_0})$. So, $p \in V(c_{\alpha_0}) \setminus V(d_{\beta_0})$ and $q \in V(d_{\beta_0}) \setminus V(c_{\alpha_0})$. Hence, $Min(L)^{-1}$ is T_1.

To show compactness suppose that $Min(L)^{-1} = \bigcup \{V(c) \mid c \in K, K \subseteq \mathfrak{K}(L)\}$ where $\{V(c) \mid c \in K, K \subseteq \mathfrak{K}(L)\}$ is a basic open cover of the space $Min(L)^{-1}$. We want to show that a finite number of members of the collection also covers the space. Let us consider the set

$$S = \{c_1 \wedge c_2 \wedge \ldots \wedge c_n : c_i \in K, \text{ for all } i = 1, \ldots, n; n \in \mathbb{N}\}.$$

We first notice that S is closed under finite meets. To complete the proof it suffices to show that 0 is a member of the set S since if $0 \in S$, then $0 = c_1 \wedge c_2 \wedge \ldots \wedge c_n$, for some n with $\{c_i\}_{i=1}^n \in K$. Thus we will have,

$$Min(L)^{-1} = V(0) = V(c_1 \wedge c_2 \wedge \ldots \wedge c_n) = V(c_1) \cup V(c_2) \cup \ldots \cup V(c_n).$$

Suppose, by way of contradiction, that $0 \notin S$. Let us construct the set $\overline{S} = \{x \in \mathfrak{K}(L) : s \leq x \text{ for some } s \in S\}$. We observe that $\overline{S}$ is a proper filter of compact elements of the lattice L since,

(i) $0 \notin \overline{S}$.

(ii) If $x, y \in \overline{S}$, then $s_1 \leq x$ and $s_2 \leq y$, for some $s_1, s_2 \in S$. So we have $s_1 \wedge s_2 \leq x \wedge y$, with $s_1 \wedge s_2 \in S$ and thus, $x \wedge y \in \overline{S}$.

(iii) If $x \in \overline{S}$ and $y \in \mathfrak{K}(L)$ with $x \leq y$, then there exist $s \in S$ such that $s \leq x \leq y$; whence $y \in \overline{S}$.

Applying Zorn's Lemma, the filter of compact elements $\overline{S}$ can be extended to an ultrafilter, say $\mathfrak{F}$, of compact elements of L, so that we have, $\overline{S} \subseteq \mathfrak{F}$. Now, let us consider the element $p = \bigvee\{c^{\perp} : c \in \mathfrak{F}\}$ of L. Using Lemma 3.3 and Lemma 3.4 we can conclude that p is a minimal prime element of L and furthermore, $p = \bigvee\{c^{\perp} : c \in \mathfrak{K}(L), c \nleq p\}$. Since $p \in Min(L)^{-1}$ there exist some $c \in K$ such that $p \in V(c)$, that is, $c \leq p$. However, $c \in S$, which in turn says that $c \in \mathfrak{F}$. Thus, $c^{\perp} \leq p$ and hence $c \nleq p$, which is a contradiction. So, the assumption that $0 \notin S$ leads us to a contradiction. Hence $0 \in S$ and the result follows. $\square$

It is clear from the construction that, in general, the inverse topology is not zero-dimensional. Therefore, our next question is, when is $Min(L)^{-1}$ zero-dimensional? To answer the question we first need to know about the clopen subsets of the space $Min(L)^{-1}$. Our next two lemmas will give us the required insight.

Lemma 4.2. *Let L be an algebraic frame satisfying the FIP. A subset K of $Min(L)^{-1}$ is clopen if and only if $K = V(y) = U(z)$ for some compact elements y and z in L.*

Proof. Let L be an algebraic frame satisfying the FIP. Suppose that K is a clopen subset of $Min(L)^{-1}$. Since K is open, $K = \bigcup\{V(c) : c \in \mathfrak{K}(L)\}$. Also K is a closed subset of the compact space, $Min(L)^{-1}$, and so is compact. Therefore, $K = V(c_1) \cup V(c_2) \cup \ldots \cup V(c_n)$ for some n. Thus we have $K = V(c_1 \wedge \ldots \wedge c_n) = V(y)$, where $y = c_1 \wedge \ldots \wedge c_n$ is a compact element in L (using the FIP). Again $Min(L)\backslash K$ is clopen; by a similar argument as above we gather that $Min(L)\backslash K = V(z)$ for some compact $z \in L$. So, $K = V(y)$ and $K = Min(L)\backslash V(z) = U(z)$.

On the other hand, if K is a subset of $Min(L)^{-1}$ with $K = V(y) = U(z)$, for some compact $y, z \in L$ then K is both closed and open and hence is clopen in $Min(L)^{-1}$. $\square$

Recall that an element x of an algebraic frame L is called a dense element if $x^{\perp} = 0$. A dense, compact element is called a unit.

Lemma 4.3. *Let L be an algebraic frame satisfying the FIP. For $x, y \in \mathfrak{K}(L)$, $V(x) = U(y)$ (and $U(x) = V(y)$) if and only if $x \wedge y = 0$ and $x \vee y$ is a unit.*

Proof. Suppose that L is an algebraic frame satisfying the FIP. Let x and y be two arbitrary compact elements of L with $V(x) = U(y)$. It follows that $V(x \vee y) = V(x) \cap V(y) = U(y) \cap V(y) = \emptyset$, which means that $x \vee y \nleq p$ for all minimal prime elements $p \in L$. So, $(x \vee y)^{\perp} \leq p$ for all minimal prime elements $p \in L$, and thus $(x \vee y)^{\perp} = 0$, using Theorem 3.6. Hence, $x \vee y$ is dense. Since $x \vee y \in \mathfrak{K}(L)$, it follows that $x \vee y$ is a unit. Finally, we claim that x and y are two disjoint elements of L. We observe that $V(x) = U(y)$ implies that $U(x \wedge y) = U(x) \cap U(y) = U(x) \cap V(x) = \emptyset$. Therefore $x \wedge y \leq p$, for all $p \in Min(L)$ and thus $x \wedge y = 0$.

Conversely, let $x, y \in \mathfrak{K}(L)$ with the condition that $x \wedge y = 0$ and $x \vee y$ is a unit. We then have

$$V(x) \cup V(y) = V(x \wedge y) = V(0) = Min(L)^{-1} \text{ (using Lemma 3.7).}$$

Also, using the fact that $x \vee y$ is dense it follows that $V(x) \cap V(y) = V(x \vee y) = 0$, by Lemma 3.10. Hence, $V(x)$ and $V(y)$ are two disjoint sets whose union is $Min(L)^{-1}$. Since for any $x \in L$, $V(x)$ and $U(x)$ are the set theoretic complement of each other it follows that $V(x) = U(y)$ and $U(x) = V(y)$. $\square$

Since we are interested in determining when $Min(L)^{-1}$ is zero-dimensional, Lemma 4.3 assures us that it is necessary that L possess a unit. Therefore, we shall make that tacit assumption on our statements. Using the above two lemmas we now answer the question about zero-dimensionality of the inverse topology. It will turn out that zero-dimensionality and totally disconnectedness are equivalent for the inverse topology, although these two properties are not equivalent for a general topological space.

Theorem 4.4. *Let L be an algebraic frame satisfying the FIP that also possesses a unit. Then the following statements are equivalent:*

1. *$Min(L)^{-1}$ is zero-dimensional.*

2. *For every $p \in Min(L)^{-1}$ and $x \in \mathfrak{K}(L)$ with $x \leq p$, there exist $y, z \in \mathfrak{K}(L)$ with $x \leq y$ such that $y \leq p$, and furthermore, $y \wedge z = 0$ and $y \vee z$ is dense.*

3. *For each pair of distinct minimal primes p and q there exist compact elements x and z with $x \wedge z = 0$ and $x \vee z$ is dense, such that $x \leq p$ but $x \nleq q$.*

4. *$Min(L)^{-1}$ is totally disconnected.*

5. *For all $a, b \in \mathfrak{K}(L)$ with $a \wedge b = 0$, there exist $x, y \in \mathfrak{K}(L)$ with $x \wedge y = 0$ and $x \vee y$ is a unit such that $a \leq x$ and $b \leq y$.*

Proof. The proof goes like (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (1) and (1) $\Rightarrow$ (5) $\Rightarrow$ (3).

(1) $\Rightarrow$ (2) Suppose that $Min(L)^{-1}$ is zero-dimensional. Let p be in $Min(L)^{-1}$ and $x \in \mathfrak{K}(L)$ with $x \leq p$, then $p \in V(x)$. Since $Min(L)^{-1}$ is zero-dimensional, there exist a clopen set K such that $p \in K \subseteq V(x)$. By Lemma 4.2 we have that $K = V(y_1) = U(z)$, for some $y_1, z \in \mathfrak{K}(L)$. Let $y = y_1 \vee x$, We first notice that since $y_1, x \in \mathfrak{K}(L)$, $y \in \mathfrak{K}(L)$ and $x \leq y$, by definition of y. We also observe that

$$p \in V(x) \cap V(y_1) = V(x \vee y_1) = V(y).$$

Therefore, $y \leq p$. It remains to show that $y \wedge z = 0$ and $y \vee z$ is dense which follows from Lemma 4.3 since, $V(y) = V(y_1 \vee x) = V(y_1) \cap V(x) = V(y_1)$ and so $V(y) = U(z)$.

(2) $\Rightarrow$ (3) Assume that the condition (2) holds, and let $p, q \in Min(L)^{-1}$ with $p \neq q$. Since $Min(L)^{-1}$ is T_1 there exists a compact element x_1 in L such that $p \in V(x_1)$ and $q \notin V(x_1)$. So, $x_1 \leq p$ and $x_1 \nleq q$. Here we can use condition (2) which says that for $p \in Min(L)^{-1}$ and $x_1 \in \mathfrak{K}(L)$ with $x_1 \leq p$, there exist $y, z \in \mathfrak{K}(L)$ with $x_1 \leq y$ such that $y \leq p$, and

furthermore, $y \wedge z = 0$ and $y \vee z$ is dense. If $x = x_1 \vee y$, then $x \in \mathfrak{K}(L)$ as both x_1 and y are in $\mathfrak{K}(L)$. We observe that $p \in V(y) \cap V(x_1) = V(y \vee x_1) = V(x)$, and so $x \leq p$. However, $x \nleq q$ because $x_1 \nleq q$ and $x_1 \leq x$. It remains to prove that $x \wedge z = 0$ and $x \vee z$ is dense. For that we first notice that $y \wedge z = 0$ and $x_1 \leq y$ implies that $x_1 \wedge z = 0$. Thus we have,

$$x \wedge z = (x_1 \vee y) \wedge z = (x_1 \wedge z) \vee (y \wedge z) = 0.$$

Lastly,

$$(x \vee z)^{\perp} = (x_1 \vee y \vee z)^{\perp} = x_1^{\perp} \wedge (y^{\perp} \wedge z^{\perp}) = x_1^{\perp} \wedge 0 = 0.$$

Hence, $x \vee z$ is dense.

(3) $\Rightarrow$ (4) Suppose condition (3) holds, and let $p, q \in Min(L)^{-1}$ with $p \neq q$. To show $Min(L)^{-1}$ is totally disconnected we need to show that $\{p, q\}$ is disconnected. Using (3), there exist $x, y \in \mathfrak{K}(L)$ with $x \leq p$, $x \nleq q$ such that $x \wedge y = 0$ and $x \vee y$ is dense. We thus conclude from Lemma 4.3 that $V(x) = U(y)$ and $U(x) = V(y)$. Now, $x \leq p$ implies that $p \in V(x)$ and $x \nleq q$ says that $q \in U(x) = V(y)$. So finally we have two disjoint open sets $V(x) \cap \{p, q\}$ and $V(y) \cap \{p, q\}$ of the subset $\{p, q\}$ of $Min(L)^{-1}$ which separates $\{p, q\}$. Hence, any subset of $Min(L)^{-1}$ with more than one elements is disconnected; in other words, the only connected subsets of $Min(L)^{-1}$ are the singleton sets, which says that $Min(L)^{-1}$ is totally disconnected.

(4) $\Rightarrow$ (1) Suppose $Min(L)^{-1}$ is totally disconnected. To show that $Min(L)^{-1}$ is zero-dimensional we pick a basic open set $V(x)$ for some compact element x, and let $p \in V(x)$ is a minimal prime element of L. We want to find a clopen set K such that $p \in K \subseteq V(x)$. Since $Min(L)^{-1}$ is totally disconnected it follows that for all $q \in U(x)$ (so that $q \neq p$) there exist a clopen set K_q such that $q \in K_q$ but $p \notin K_q$. Now,

$$Min(L)^{-1} = \left(\bigcup_{q \in U(x)} K_q \right) \bigcup V(x),$$

where $\{K_q : q \in U(x)\} \bigcup V(x)$ is an open cover of $Min(L)^{-1}$ which has a finite subcover since $Min(L)^{-1}$ is compact. So, there exist finitely many elements, $q_1, q_2, \ldots, q_n \in U(x)$, such that

$$Min(L)^{-1} = K_{q_1} \bigcup K_{q_2} \bigcup \ldots \bigcup K_{q_n} \bigcup V(x).$$

We now let $K_1 = K_{q_1} \bigcup K_{q_2} \bigcup \ldots \bigcup K_{q_n}$, then K_1 is clopen and $Min(L)^{-1} = K_1 \bigcup V(x)$. We notice that $p \notin K_{q_i}$ for all $i = 1, \ldots, n$, and so $p \notin K_1$. Let $K = Min(L)^{-1} \backslash K_1$, then K is clopen and $p \in K$. It remains to prove that $K \subseteq V(x)$. We observe that if $q \in K$, then $q \notin K_1$. Hence, $q \notin K_i$, for all $i = 1, \ldots, n$. This in turn implies that $q \in V(x)$. Therefore, $p \in K \subseteq V(x)$.

Thus we have shown the equivalence: (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (1). Finally, we show that (1) $\Rightarrow$ (5) $\Rightarrow$ (3).

(1) $\Rightarrow$ (5) Let $Min(L)^{-1}$ be zero-dimensional, and let a and b be two compact elements of L with $a \wedge b = 0$. We then have $U(a) \bigcap U(b) = U(a \wedge b) = \emptyset$. So, $U(a)$ and $U(b)$ are two disjoint closed sets of a compact zero-dimensional space $Min(L)^{-1}$ and hence can be separated by a clopen set, that is, there exist a clopen set K such that $U(a) \subseteq K$ and $U(b) \bigcap K = \emptyset$. By applying Lemma 4.2, there exist $x_1, y_1 \in \mathfrak{K}(L)$ such that $K = U(x_1) = V(y_1)$. Again, from Lemma 4.3 we have, $x_1 \wedge y_1 = 0$ and $x_1 \vee y_1$ is dense. Let us consider $x = a \vee x_1$ and $y = b \vee y_1$, then $x, y \in \mathfrak{K}(L)$ and clearly $a \leq x$ and $b \leq y$. It just remains to show that $x \wedge y = 0$ and $x \vee y$ is dense. To prove that we first notice, $U(a) \subseteq K = V(y_1)$ implies $U(a) \bigcap U(y_1) = \emptyset$, which says that $a \wedge y_1 = 0$. Similarly, $U(b) \bigcap U(x_1) = \emptyset$ tells us that $b \wedge x_1 = 0$. Thus we have,

$$x \wedge y = (a \vee x_1) \wedge (b \vee y_1) = (a \wedge b) \vee (x_1 \wedge b) \vee (a \wedge y_1) \vee (x_1 \wedge y_1) = 0,$$

since each term is zero. Lastly, since $x_1 \vee y_1$ is dense and $x_1 \leq x$ and $y_1 \leq y$, it follows that $x \vee y$ is also dense.

(5) $\Rightarrow$ (3) Assume condition (5). Let $p, q \in Min(L)^{-1}$ with $p \neq q$. Since p and q are

minimal prime elements of L, using Proposition 3.12 we can find two compact elements a and b in L such that $a \wedge b = 0$ such that $a \leq p$, $a \nleq q$, $b \leq q$, and $b \nleq p$. Applying condition (5), there exist $x, y \in \mathfrak{K}(L)$ with $x \wedge y = 0$ and $x \vee y$ is dense, such that $a \leq x$ and $b \leq y$. To show condition (3), it remains to show that $x \leq p$ and $x \nleq q$. Now, $b \nleq p$ and $b \leq y$ implies that $y \nleq p$. Consequently $x \leq p$, since $x \wedge y = 0$. Following a similar argument we see that $x \nleq q$.

Thus we have proved all five conditions to be equivalent. □

We now have all the tools to prove one of our main theorems of this dissertation. The theorem gives equivalent conditions for the hull-kernel topology on $Min(L)$ to be compact. In addition, we will also have the zero-dimensionality of the inverse topology.

Theorem 4.5. *The following are equivalent for an algebraic frame L satisfying the FIP that also possesses a unit:*

1. *The hull-kernel topology on $Min(L)$ equals the inverse topology $Min(L)^{-1}$.*
2. *The hull-kernel topology on $Min(L)$ is compact and the inverse topology is Hausdorff and zero-dimensional.*
3. *The hull-kernel topology on $Min(L)$ is compact.*
4. *For all $x \in \mathfrak{K}(L)$ and $p \in U(x)$, there exist $y \in \mathfrak{K}(L)$ with $y \leq p$ such that $x \vee y$ is an unit.*
5. *For all $x \in \mathfrak{K}(L)$ there exist $y \in \mathfrak{K}(L)$ with $y \leq x^{\perp}$ such that $y^{\perp} \wedge x^{\perp} = 0$.*

Proof. (1)⇒(2) The implication is clear, since $Min(L)$ is Hausdorff and zero-dimensional and $Min(L)^{-1}$ is compact.

(2)⇒(3) Clear.

(3)⇒(4) Suppose the hull-kernel topology on $Min(L)$ is compact. Let $x \in \mathfrak{K}(L)$ and $p \in U(x)$. We want to find $y \in \mathfrak{K}(L)$, with $y \leq p$, such that $x \vee y$ is a unit. Since $V(x)$ is

open in $Min(L)$, $V(x) = \bigcup_{c_i \in \mathfrak{K}(L)} U(c_i)$. Also, $V(x)$ is a closed subset of the compact space $Min(L)$, and so is compact. Thus, there exist finitely many compact elements, say $c_1, c_2, ...c_n$, so that $V(x) = U(c_1) \cup U(c_2) \cup ... \cup U(c_n) = U(c_1 \vee c_2 \vee .. \vee c_n)$. Let $y = c_1 \vee c_2 \vee .. \vee c_n$, then $y \in \mathfrak{K}(L)$. Therefore, $V(x) = U(y)$ and so $U(x) = V(y)$. So if $p \in U(x)$, then $p \in V(y)$ which implies that $y \leq p$. Lastly we show that $x \vee y$ is a unit.

Let $q \in Min(L)$. If $x^\perp \leq q$, then, $x^\perp \wedge y^\perp \leq q$. On the other hand, if $x^\perp \nleq q$, then $x \leq q$ (since q is prime) which implies that $q \in V(x) = U(y)$. Therefore, $y \nleq q$ and hence $y^\perp \leq q$. So, $x^\perp \wedge y^\perp \leq q$. Thus, $x^\perp \wedge y^\perp \leq q$ for all $q \in Min(L)$ which concludes that $x^\perp \wedge y^\perp = 0$. Hence, $x \vee y$ is dense. Furthermore, since x and y are both compact elements of L, $x \vee y$ is also a compact element of L; thence $x \vee y$ is a unit.

(4)$\Rightarrow$(1) As the hull-kernel topology is finer than the inverse topology on $Min(L)$, to show equality of the two topologies it suffices to show that the inverse topology is finer than the hull-kernel topology. Let $x \in \mathfrak{K}(L)$ and $q \in U(x)$. We want to find a compact element y such that $q \in V(y) \subseteq U(x)$. By (4), there exist $y \in \mathfrak{K}(L)$ with $y \leq q$ such that $x \vee y$ is dense. Hence, $q \in V(y)$ and $x^\perp \wedge y^\perp = 0$. Finally, we want to show that $V(y) \subseteq U(x)$. Let us consider the following:

$$\begin{aligned}
p \in V(y) \quad &\Rightarrow \quad y \leq p, \text{ by the definition of } V(y) \\
&\Rightarrow \quad y^\perp \nleq p, \text{ using Lemma 3.9} \\
&\Rightarrow \quad x^\perp \leq p, \text{ since } x^\perp \wedge y^\perp = 0 \text{ and } p \text{ is prime} \\
&\Rightarrow \quad x \nleq p, \text{ using Lemma 3.9} \\
&\Rightarrow \quad p \in U(x), \text{ by the definition of } U(x).
\end{aligned}$$

Hence, $q \in V(y) \subseteq U(x)$. Therefore, the hull-kernel topology on $Min(L)$ equals the inverse topology.

To complete the proof we show (3) $\Rightarrow$ (5) $\Rightarrow$ (4).

(3)⇒(5) Suppose $Min(L)$ is compact. Let $x \in \mathfrak{K}(L)$, then

$$Min(L) = U(x \vee x^{\perp}) = U(x) \bigcup U(x^{\perp}).$$

Since L is algebraic it follows that, $x^{\perp} = \bigvee_{c \in \mathfrak{K}(L)} c$. Then,

$$Min(L) = U(x) \bigcup U(\bigvee c) = U(x) \bigcup \left(\bigcup_{c} U(c) \right).$$

Since $Min(L)$ is compact there exist finitely many compact elements $c_1, .., c_n$ such that

$$Min(L) = U(x) \bigcup U(c_1) \bigcup ... \bigcup U(c_n).$$

Let $y = c_1 \bigvee ... \bigvee c_n$, then $y \in \mathfrak{K}(L)$. Furthermore, $y \leq \bigvee_{c \in \mathfrak{K}(L)} c = x^{\perp}$. Finally, since $Min(L) = U(x) \bigcup U(y) = U(x \vee y)$ it follows that $x \vee y$ is dense, that is, $x^{\perp} \wedge y^{\perp} = 0$.

(5)⇒(4) Let $x \in \mathfrak{K}(L)$ and $p \in U(x)$. By (5), there exist $y \in \mathfrak{K}(L)$ with $y \leq x^{\perp}$ such that $x^{\perp} \wedge y^{\perp} = 0$, that is, $x \vee y$ is a unit. Finally, we notice the following:

$$\begin{aligned} p \in U(x) \quad &\Rightarrow \quad p \in V(x^{\perp}), \text{ since } U(x) = V(x^{\perp}) \\ &\Rightarrow \quad x^{\perp} \leq p, \text{ by the definition of } V(x^{\perp}) \\ &\Rightarrow \quad y \leq p, \text{ since } y \leq x^{\perp}. \end{aligned}$$

Hence, $y \leq p$. Thus, (4) holds. □

We have been unable to find a better characterization of when $Min(L)$ is a compact extremally disconnected space.

Theorem 4.6. *Suppose that L is an algebraic frame satisfying the FIP. The following conditions are equivalent:*

1. *$Min(L)$ is a compact, extremally disconnected space.*

2. *For each $x \in L$ there exists $y \in \mathfrak{K}(L)$ with $y \leq x^{\perp}$ such that $x \vee y$ is dense.*

3. *Every polar $x \in L$ is of the form $c^{\perp\perp}$, for some $c \in \mathfrak{K}(L)$.*

Proof. (1) $\Leftrightarrow$ (2). Let L be an algebraic frame satisfying the FIP. Let $x \in \mathfrak{K}(L)$, then by the hypothesis it follows that there exist $y \in \mathfrak{K}(L)$ with $y \leq x^{\perp}$ such that $x \vee y$ is dense, that is, $x^{\perp} \wedge y^{\perp} = 0$. Hence by Theorem 4.5, $Min(L)$ is compact. To show extremally disconnected, let us choose $x \in L$. By assumption there exist $y \in \mathfrak{K}(L)$ such that $y \leq x^{\perp}$. Thus, $V(x^{\perp}) \subseteq V(y)$. Also, $y \leq x^{\perp}$ implies that $x \wedge y = 0$, which gives $U(x) \bigcap U(y) = \emptyset$. On the other hand,

$$\begin{aligned} x \vee y \text{ is dense } &\Rightarrow x^{\perp} \wedge y^{\perp} = 0 \\ &\Rightarrow U(x^{\perp}) \bigcap U(y^{\perp}) = \emptyset \\ &\Rightarrow U(x^{\perp}) \bigcap V(y) = \emptyset, \text{ since } y \text{ is a compact element} \\ &\Rightarrow V(y) \subseteq V(x^{\perp}). \end{aligned}$$

Thus $V(x^{\perp}) = V(y)$, which is open. By applying Theorem 3.14 it follows that $Min(L)$ is extremally disconnected.

To prove the converse we notice that given $x \in L$, $V(x^{\perp})$ is clopen by Theorem 3.14 and so $Min(L) \setminus V(x^{\perp})$ is clopen. Therefore, applying Lemma 4.2, $Min(L) \setminus V(x^{\perp}) = U(y)$ for some $y \in \mathfrak{K}(L)$. Hence, $V(x^{\perp}) = V(y)$. Thus we have,

(i) $V(x^{\perp} \wedge y^{\perp}) = V(x^{\perp}) \bigcup V(y^{\perp}) = Min(L)$, concluding that $x^{\perp} \wedge y^{\perp} = 0$. Hence, $x \vee y$ is dense.

(ii) $V(x^{\perp}) \bigcap V(y^{\perp}) = \emptyset$, implying that $x^{\perp} \vee y^{\perp} \nleq p$ for all $p \in Min(L)$. Thus, $(x \wedge y)^{\perp} \nleq p$ for all $p \in Min(L)$. This implies that $x \wedge y = 0$, applying Theorem 3.6. Hence, $y \leq x^{\perp}$.

(2) $\Leftrightarrow$ (3). Suppose that $x \in L$ is a polar, that is, $x = x^{\perp\perp}$. Using the hypothesis in (2), for $x^{\perp} \in L$, we find some $y \in \mathfrak{K}(L)$ with $y \leq (x^{\perp})^{\perp}$ such that $y^{\perp} \wedge x^{\perp\perp} = 0$. So, $y \leq x^{\perp\perp} = x$. This implies that $y^{\perp\perp} \leq x^{\perp\perp} = x$. On the other hand, $y^{\perp} \wedge x^{\perp\perp} = 0$

gives $y^{\perp} \wedge x = 0$, and hence $x \leq y^{\perp\perp}$. It thus follows that $x = y^{\perp\perp}$, with $y \in \mathfrak{K}(L)$. Conversely suppose $x \in L$, then $x^{\perp}$ is a polar. Therefore, there exist $y \in \mathfrak{K}(L)$ with $x^{\perp} = y^{\perp\perp}$. Thus, $y \leq y^{\perp\perp} = x^{\perp}$, and $x^{\perp} \wedge y^{\perp} = y^{\perp\perp} \wedge y^{\perp} = 0$.

$\square$

In conclusion, if L is an algebraic frame satisfying the FIP, then $Min(L)$ endowed with the hull-kernel topology is a zero-dimensional, Hausdorff space, and it is compact precisely when $Min(L) = Min(L)^{-1}$.

CHAPTER 5

The Frame $Rad(L)$

If we start with an algebraic frame L, then we can construct a new frame whose elements are infimum of prime elements of the frame L. The idea behind this construction comes from the well-known commutative ring theory concept of radical ideals of a commutative ring. For a commutative ring A with identity, and any ideal I of A, the radical of I, denoted $\sqrt{I}$, is the intersection of all prime ideals of A containing I. As it turns out, the collection of all radical ideals of A forms a complete lattice which is a frame. This notion of radical ideals from commutative ring theory will be generalized to frame theory to construct a new frame from a given frame.

We start by defining the radical of an algebraic frame. Let L be an algebraic frame. Recall that $Spec(L)$ denotes the collection of all prime elements of L. We define

$$Rad(L) = \left\{ \bigwedge p : p \in S \subseteq Spec(L) \right\}.$$

For any $x \in L$, we will denote $\sqrt{x} = \bigwedge\{p \in Spec(L) : x \leq p\}$. We recall an useful property of the operation meet, $\wedge$, which says that if $S, T \subseteq L$, then $S \subseteq T \Rightarrow \bigwedge T \leq \bigwedge S$.

Lemma 5.1. *Suppose that L is an algebraic frame and $x, y \in L$. The following are true:*

1. $x \leq \sqrt{x}$, *for all* $x \in L$.

2. *$x \leq y \Rightarrow \sqrt{x} \leq \sqrt{y}$. The other implication holds if $x, y \in Rad(L)$.*

3. *$\sqrt{\sqrt{x}} = \sqrt{x}$, for all $x \in L$.*

Proof. Let L be an algebraic frame and $x, y \in L$.

1. It is evident from the definition of $\sqrt{x}$ that $x \leq \sqrt{x}$.

2. Let $x \leq y$. If $y \leq p$, then $x \leq p$. Hence, $\{p \in Spec(L) : y \leq p\} \subseteq \{p \in Spec(L) : x \leq p\}$. We thus have that

$$\sqrt{x} = \bigwedge\{p \in Spec(L) : x \leq p\} \leq \bigwedge\{p \in Spec(L) : y \leq p\} = \sqrt{y}.$$

3. Since $x \leq \sqrt{x}$, from (1), then using (2) it follows that $\sqrt{x} \leq \sqrt{\sqrt{x}}$. On the other hand, $\sqrt{x} = \bigwedge\{p : x \leq p\} \leq p$ for all p. Hence, $\{p : x \leq p\} \subseteq \{p : \sqrt{x} \leq p\}$, concluding that $\sqrt{\sqrt{x}} \leq \sqrt{x}$. Therefore, $\sqrt{\sqrt{x}} = \sqrt{x}$.

□

Given an algebraic frame L we define a function $\gamma : L \to Rad(L)$ by, $\gamma(x) = \sqrt{x}$ for $x \in L$. γ is a closure operator since it satisfies the following three conditions of closure operators, using Lemma 5.1. For $x, y \in L$,

(i) $x \leq \gamma(x)$.

(ii) $x \leq y \Rightarrow \gamma(x) \leq \gamma(y)$.

(iii) $\gamma(\gamma(x)) = \gamma(x)$.

The map $\gamma : L \to Rad(L)$ preserves the partial order of L (using (ii) above)

We observe the following properties of $Rad(L)$.

Proposition 5.2. *Let L be an algebraic frame. $Rad(L)$ is a complete sublattice of L.*

Proof. Suppose that L is an algebraic frame. We claim that $Rad(L)$ is a lattice with meet, $\wedge$, being the same $\wedge$ operation in L and join, $\sqcup$, is given as following; for $x, y \in Rad(L)$,

$$x \sqcup y = \bigwedge\{p \in Spec(L) \mid x, y \leq p\}.$$

The only thing that requires verification is whether $\sqcup$ as defined above satisfies the conditions for supremum. We first observe that $\{p : x, y \leq p\} = \{p : x \vee y \leq p\}$. Hence,

$$x \sqcup y = \bigwedge\{p \in Spec(L) \mid x, y \leq p\} = \bigwedge\{p \in Spec(L) \mid x \vee y \leq p\} = \sqrt{x \vee y}.$$

From Lemma 5.1, it follows that $x \vee y \leq \sqrt{x \vee y} = x \sqcup y$ for all $x, y \in L$. Hence, $x, y \leq x \vee y \leq x \sqcup y$, which implies that $x \sqcup y$ is an upper bound of x and y. Suppose there exist some $z \in Rad(L)$ with $x, y \leq z$. Therefore, $x \vee y \leq z$ and hence from Lemma 5.1 it follows that $\sqrt{x \vee y} \leq \sqrt{z}$. Since $z \in Rad(L)$ it follows that $z = \sqrt{z}$. Hence,

$$x \sqcup y = \sqrt{x \vee y} \leq \sqrt{z} = z.$$

Thus, $x \sqcup y$ is the least upper bound of x and y, which proves that $Rad(L)$ is a lattice with the two operations $\wedge$ and $\sqcup$ as infimum and supremum respectively.

It can be easily verified that $(Rad(L), \wedge, \sqcup)$ is a complete lattice. $\square$

In conclusion we observe the following important properties of $\sqrt{x}$ which will be used several times in this chapter. For any $x \in L$,

$$x \leq \sqrt{x},$$

and

$$x = \sqrt{x} \Leftrightarrow x \in Rad(L).$$

Again, for any $p \in Spec(L)$,

$$x \leq p \Leftrightarrow \sqrt{x} \leq p.$$

It also follows from the definition of $\sqcup$ that for any $x_\alpha \in Rad(L)$,

$$\sqrt{\bigvee_\alpha x_\alpha} = \bigsqcup_\alpha x_\alpha.$$

Therefore,

$$\bigvee_\alpha x_\alpha \leq \bigsqcup_\alpha x_\alpha.$$

Our next goal is to characterize the compact elements of $Rad(L)$ and prove that $Rad(L)$ is an algebraic frame. For that we need next two lemmas.

Lemma 5.3. *Suppose that L is an algebraic frame and $x, y \in L$.*

$$\sqrt{x \vee y} = \sqrt{x} \sqcup \sqrt{y}.$$

Proof. Let L be an algebraic frame and let $x, y \in L$. By definition,

$$\sqrt{x \vee y} = \bigwedge\{p \in Spec(L) : x \vee y \leq p\} = \bigwedge\{p \in Spec(L) : x, y \leq p\},$$

and

$$\sqrt{x} \sqcup \sqrt{y} = \bigwedge\{q \in Spec(L) : \sqrt{x}, \sqrt{y} \leq q\}.$$

Let $p \in Spec(L)$ with $x, y \leq p$. Using definition once again we have that $\sqrt{x} = \bigwedge\{r \in Spec(L) : x \leq r\}$, and hence, $\sqrt{x} \leq p$. Similarly, $\sqrt{y} \leq p$. Therefore, $\sqrt{x} \sqcup \sqrt{y} \leq \sqrt{x \vee y}$. On the other hand, if $q \in Spec(L)$ with $\sqrt{x}, \sqrt{y} \leq q$, then $x \leq \sqrt{x} \leq q$ and $y \leq \sqrt{y} \leq q$. Consequently, $\sqrt{x \vee y} \leq \sqrt{x} \sqcup \sqrt{y}$. Hence, $\sqrt{x} \sqcup \sqrt{y} = \sqrt{x \vee y}$. $\square$

Lemma 5.4. *Suppose that L is an algebraic frame and $x \in \mathfrak{K}(L)$. If $x \nleq t$ for some $t \in L$, then there exist some $v \in L$ with $t \leq v$, which is maximal with respect to $x \nleq v$. Moreover,*

such an element $v \in L$ is prime.

Proof. Let L be an algebraic frame and $x \in \mathfrak{K}(L)$ with $x \nleq t$, for some $t \in L$. Let $S = \{y \in L : x \nleq y\}$. Since $x \nleq t$, therefore $t \in S$, proving that S is a nonempty subset of L. Let $C \subseteq S$ be a chain in S and $c = \bigvee C$. We claim that $c \in S$. If $x \leq c$, then $x \leq \bigvee C$. Since x is a compact element of L, there exist finitely many elements $c_1, c_2, \ldots, c_n \in C$ such that $x \leq c_1 \vee \ldots \vee c_n$. Observe that $c_i \in C$ for all $i = 1, \ldots, n$ implies that $c_1 \vee \ldots \vee c_n$ is an element in C, since C is a chain. Therefore, $c_1 \vee \ldots \vee c_n \in S$. This contradicts the fact that $x \leq c_1 \vee \ldots \vee c_n$. Hence, $x \nleq c$ and so $c \in S$. Thus, every chain in S has a maximal element in S. Using Zorn's Lemma it follows that S has a maximal element, call v. Consequently, there exist some $v \in L$ with $t \leq v$ and maximal with respect to $x \nleq v$.

Finally, we show that $v \in Spec(L)$. Since L is distributive, it is enough to show that $a \wedge b = v \Rightarrow a = v$ or $b = v$. Suppose that $a \wedge b = v$, but both $a > v$ and $b > v$. By maximality of v it follows that $x \leq a$ and $x \leq v$. Hence, $x \leq a \wedge b = v$, a contradiction. Therefore, either $a = v$ or $b = v$; consequently, $v \in Spec(L)$. $\square$

Similar to the case of commutative ring theory, we will show that the compact elements of $Rad(L)$ are precisely the ones generated by the compact elements of L.

Theorem 5.5. *Let L be an algebraic frame. $c \in \mathfrak{K}(Rad(L))$ if and only if $c = \sqrt{k}$ for some $k \in \mathfrak{K}(L)$.*

Proof. Suppose that L is an algebraic frame. We first show that if $k \in \mathfrak{K}(L)$, then $\sqrt{k} \in \mathfrak{K}(Rad(L))$. Let $a_\alpha \in Rad(L)$ for some $\alpha \in I$ with $\sqrt{k} \leq \bigsqcup_{\alpha \in I} a_\alpha$. Since $k \leq \sqrt{k}$, it follows that $k \leq \bigsqcup_{\alpha \in I} a_\alpha$. Recall that $\bigvee_{\alpha \in I} a_\alpha \leq \bigsqcup_{\alpha \in I} a_\alpha$. We claim that $x \leq \bigvee_{\alpha \in I} a_\alpha$. By the way of contradiction, suppose that $x \nleq \bigvee_{\alpha \in I} a_\alpha$. Using Lemma 5.4, there exist $v \in Spec(L)$ with $\bigvee_{\alpha \in I} a_\alpha \leq v$ and $k \nleq v$. It thus follow from the definition of $\sqcup$ that $\bigsqcup_{\alpha \in I} a_\alpha \leq v$. Hence, $\sqrt{k} \leq v$ but $k \nleq v$, which is a contradiction. Thus, $k \leq \bigvee_{\alpha \in I} a_\alpha$. Since k is a compact element of L,

there exist a finite subset $I_0 \subseteq I$ with $k \leq \bigvee_{\alpha \in I_0} a_\alpha$. Therefore, using Lemma 5.1,

$$\sqrt{k} \leq \sqrt{\bigvee_{\alpha \in I_0} a_\alpha} = \bigsqcup_{\alpha \in I_0} a_\alpha.$$

Hence, $\sqrt{k} \in \mathfrak{K}(Rad(L))$.

Conversely, let $c \in \mathfrak{K}(Rad(L))$, then $c = \sqrt{c}$. We claim that there exist some $k \in \mathfrak{K}(L)$ with $c = \sqrt{k}$. Since $c \in L$ and L is an algebraic frame, it follows that $c = \bigvee_\alpha k_\alpha$ for some $k_\alpha \in \mathfrak{K}(L)$. By a known fact we have,

$$c = \bigvee_\alpha k_\alpha \leq \bigsqcup_\alpha k_\alpha \leq \bigsqcup_\alpha \sqrt{k_\alpha}.$$

Since c is a compact element of $Rad(L)$ it follows that $c \leq \sqrt{k_{\alpha_1}} \sqcup \ldots \sqcup \sqrt{k_{\alpha_n}}$, for finitely many $k'_\alpha s$ in $\mathfrak{K}(L)$. Let us denote $k = k_{\alpha_1} \vee \ldots \vee k_{\alpha_n}$, then $k \in \mathfrak{K}(L)$. We will show that $c = \sqrt{k}$. Using Lemma 5.3 we have that

$$\sqrt{k_{\alpha_1}} \sqcup \ldots \sqcup \sqrt{k_{\alpha_n}} = \sqrt{k_{\alpha_1} \vee \ldots \vee k_{\alpha_n}} = \sqrt{k}.$$

Hence, $c \leq \sqrt{k}$. On the other hand, $k \leq \bigvee_\alpha k_\alpha = c$. Consequently, $\sqrt{k} \leq \sqrt{c} = c$. Therefore, $c = \sqrt{k}$. □

Theorem 5.6. *Let L be an algebraic frame. $Rad(L)$ is an algebraic lattice.*

Proof. Let L be an algebraic frame. Let $x \in Rad(L)$, then $x = \sqrt{x}$. Since $x \in L$ and L is algebraic we have $x = \bigvee\{k_\alpha : k_\alpha \in \mathfrak{K}(L)\}$. For each α we have from Theorem 5.5 that $\sqrt{k_\alpha} \in \mathfrak{K}(Rad(L))$. We claim that $x = \bigsqcup_\alpha \sqrt{k_\alpha}$. We first notice that

$$\sqrt{x} = \sqrt{\bigvee_\alpha k_\alpha} = \bigsqcup_\alpha k_\alpha \leq \bigsqcup_\alpha \sqrt{k_\alpha},$$

using Lemma 5.1 and the definition of $\sqcup$. Therefore, $x \leq \bigsqcup_\alpha \sqrt{k_\alpha}$.

On the other hand,

$$\begin{aligned} k_\alpha \leq \bigvee_\alpha k_\alpha, \text{ for all } \alpha \quad &\Rightarrow \quad \sqrt{k_\alpha} \leq \sqrt{\bigvee_\alpha k_\alpha}, \text{ for all } \alpha \\ &\Rightarrow \quad \bigsqcup_\alpha \sqrt{k_\alpha} \leq \sqrt{\bigvee_\alpha k_\alpha} \\ &\Rightarrow \quad \bigsqcup_\alpha \sqrt{k_\alpha} \leq \sqrt{x} \\ &\Rightarrow \quad \bigsqcup_\alpha \sqrt{k_\alpha} \leq x, \text{ since } x = \sqrt{x}. \end{aligned}$$

Therefore we have that $x = \bigsqcup_\alpha \sqrt{k_\alpha}$; thence, $Rad(L)$ is algebraic. □

We have proved that for an algebraic frame $(L, \wedge, \vee)$, $(Rad(L), \wedge, \sqcup)$ is a complete algebraic lattice. As it turns out, $Rad(L)$ is an algebraic frame. To prove that we use Proposition 2.5 from Chapter 2 and show that $Rad(L)$ is a distributive lattice.

Theorem 5.7. *Let L be an algebraic frame, then $Rad(L)$ is distributive.*

Proof. Let L be an algebraic frame and let $a, b, c \in Rad(L)$. We want to show that $a \wedge (b \sqcup c) = (a \wedge b) \sqcup (a \wedge c)$. Notice that $a \wedge b \leq a \wedge (b \sqcup c)$ and $a \wedge c \leq a \wedge (b \sqcup c)$. Hence it is always true that

$$(a \wedge b) \sqcup (a \wedge c) \leq a \wedge (b \sqcup c).$$

To show the other inequality, we observe that,

$$(a \wedge b) \sqcup (a \wedge c) = \sqrt{(a \wedge b) \vee (a \wedge c)} = \sqrt{a \wedge (b \vee c)},$$

since L is distributive and $a, b, c \in L$. If possible, let $a \wedge (b \sqcup c) \nleq (a \wedge b) \sqcup (a \wedge c)$. This means that $\{p \in Spec(L) : a \wedge (b \vee c) \leq p\} \nsubseteq \{p \in Spec(L) : a \wedge (b \sqcup c) \leq p\}$. So, there exist some $p \in Spec(L)$ such that $a \wedge (b \vee c) \leq p$, but $a \wedge (b \sqcup c) \nleq p$. Since p is a prime element, $a \wedge (b \vee c) \leq p$ implies that either $a \leq p$ or $b \vee c \leq p$. If $a \leq p$, then

$a \wedge (b \sqcup c) \leq p$, which is a contradiction. If, instead, $b \vee c \leq p$, then $b \leq p$ and $c \leq p$. This implies that $b \sqcup c \leq p$; whence, $a \wedge (b \sqcup c) \leq p$, which is again a contradiction. Therefore, $\{p \in Spec(L) : a \wedge (b \vee c) \leq p\} \subseteq \{p \in Spec(L) : a \wedge (b \sqcup c) \leq p\}$. Consequently,

$$\begin{aligned} a \wedge (b \sqcup c) &= \bigwedge \{p \in Spec(L) : a \wedge (b \sqcup c) \leq p\} \\ &\leq \bigwedge \{p \in Spec(L) : a \wedge (b \vee c) \leq p\} \\ &= \sqrt{a \wedge (b \vee c)} \\ &= (a \wedge b) \sqcup (a \wedge c) \end{aligned}$$

Thus, $Rad(L)$ is a distributive lattice, and hence is an algebraic frame. □

An obvious question at this point is the following: Is the property of FIP inherited by $Rad(L)$ from L?

Theorem 5.8. *Let L be an algebraic frame satisfying the FIP. The algebraic frame $Rad(L)$ also satisfies the FIP.*

Proof. Suppose that L is an algebraic frame satisfying the FIP and let $\sqrt{k_1}, \sqrt{k_2} \in \mathfrak{K}(Rad(L))$, for some $k_1, k_2 \in \mathfrak{K}(L)$ (Theorem 5.5). So, using the property of FIP in L it follows that $k_1 \wedge k_2 \in \mathfrak{K}(L)$. Therefore, using Theorem 5.5 again, we conclude that $\sqrt{k_1 \wedge k_2} \in \mathfrak{K}(Rad(L))$. We want to show that $\sqrt{k_1} \wedge \sqrt{k_2} \in \mathfrak{K}(Rad(L))$. Let $x_\alpha \in Rad(L)$, for some $\alpha \in I$, with $\sqrt{k_1} \wedge \sqrt{k_2} \leq \bigsqcup_{\alpha \in I} x_\alpha$. Since $\sqrt{k_1 \wedge k_2} \leq \sqrt{k_1} \wedge \sqrt{k_2}$, we have that $\sqrt{k_1 \wedge k_2} \leq \bigsqcup_{\alpha} x_\alpha$. Hence, $\sqrt{k_1 \wedge k_2} \leq \bigsqcup_{\alpha \in I_0} x_\alpha$, for some finite subset $I_0 \subseteq I$, since $\sqrt{k_1 \wedge k_2}$ is a compact element of $Rad(L)$. By definition, $\bigsqcup_{\alpha \in I_0} x_\alpha = \bigwedge \{p \in Spec(L) : \bigvee_{\alpha \in I_0} x_\alpha \leq p\}$. For all such $p \in Spec(L)$

with $\bigvee_{\alpha \in I_0} x_\alpha \leq p$, we have the following:

$$\begin{aligned}
\sqrt{k_1 \wedge k_2} \leq p \quad &\Rightarrow \quad k_1 \wedge k_2 \leq p, \text{ since } k_1 \wedge k_2 \leq \sqrt{k_1 \wedge k_2} \\
&\Rightarrow \quad \text{either } k_1 \leq p \text{ or } k_2 \leq p, \text{ since } p \in Spec(L) \\
&\Rightarrow \quad \text{either } \sqrt{k_1} \leq p \text{ or } \sqrt{k_2} \leq p \\
&\Rightarrow \quad \sqrt{k_1} \wedge \sqrt{k_2} \leq p, \text{ for all } p \\
&\Rightarrow \quad \sqrt{k_1} \wedge \sqrt{k_2} \leq \bigwedge p = \bigsqcup_{\alpha \in I_0} x_\alpha.
\end{aligned}$$

Hence, if $\sqrt{k_1}, \sqrt{k_2} \in \mathfrak{K}(Rad(L))$, then $\sqrt{k_1} \wedge \sqrt{k_2} \in \mathfrak{K}(Rad(L))$. Therefore, $Rad(L)$ satisfies the FIP. □

Lemma 5.9. *Let L be an algebraic frame. The following are true:*

1. $0_{Rad(L)} = 0_L$.
2. $1_{Rad(L)} = 1_L$.
3. $\sqrt{x} \wedge \sqrt{y} = \sqrt{x \wedge y}$, *for all* $x, y \in Rad(L)$.
4. $\sqrt{\bigsqcup_\alpha x_\alpha} = \bigsqcup_\alpha \sqrt{x_\alpha}$, *for all* $x_\alpha \in Rad(L)$ *and for any* α.

Proof. Suppose that L is an algebraic frame.

1. Since L is an algebraic frame, using Theorem 3.6, we have $0_L = \bigwedge\{p : p \in Spec(L)\}$. Hence, $0_L \in Rad(L)$ and $0_L = 0_{Rad(L)}$.
2. Notice that 1_L is the empty intersection of prime elements of L. Therefore, $1_L \in Rad(L)$ and $1_L = 1_{Rad(L)}$.
3. Since $x, y \in Rad(L)$, it follows that $x = \sqrt{(}x)$ and $y = \sqrt{y}$. Also, $x \wedge y \in Rad(L)$ and hence $\sqrt{x \wedge y} = x \wedge y$. Therefore,

$$\sqrt{x} \wedge \sqrt{y} = x \wedge y = \sqrt{x \wedge y}.$$

4. Since $x_\alpha \in Rad(L)$ for all α, it follows that $\bigsqcup_\alpha x_\alpha \in Rad(L)$. Also, $\sqrt{x_\alpha} = x_\alpha$ for all α. Therefore,

$$\sqrt{\bigsqcup_\alpha x_\alpha} = \bigsqcup_\alpha x_\alpha = \bigsqcup_\alpha \sqrt{x_\alpha}.$$

□

It thus follows that the closure operator, γ, restricted to $Rad(L)$, $Rad(L) \to Rad(L)$ is a frame homomorphism.

Given a function f, we define $Fix(f)$ to be the set of all the elements of the domain that is fixed by f. So for the map $\gamma : L \to Rad(L)$, $Fix(\gamma) = \{x \in L : \gamma(x) = x\}$.

Proposition 5.10. *Suppose L is an algebraic frame and $\gamma : L \to Rad(L)$ is a closure operator mapping x into $\sqrt{x}$. $Rad(L) = Fix(\gamma)$.*

Proof. Suppose that L is an algebraic frame and $\gamma : x \mapsto \sqrt{x}$ is a closure operator between L and $Rad(L)$. We observe that if $x \in Rad(L)$, then $x = \sqrt{x} = \gamma(x)$. Hence, $Rad(L) \subseteq Fix(\gamma)$.

On the other hand, it is clear that if $x \in L$ with $\gamma(x) = x$, then $x \in Rad(L)$.

Hence, $Rad(L) = Fix(\gamma)$. □

It thus follows that if we start with an algebraic frame L then the closure operator $\gamma : L \to Rad(L)$, restricted to $Rad(L)$, is a frame homomorphism. In fact, $\gamma|_{Rad(L)} : Rad(L) \to Rad(L)$ is an identity map, and hence is a bijection. We will now discuss the minimal prime element space of the frame $Rad(L)$ compared to the minimal prime element space of the algebraic frame L. As it turns out that the space $Min(L)$ is in bijective correspondence with the space $Min(Rad(L))$. We will use closure operator $\gamma : x \mapsto \sqrt{x}$ to prove this fact.

Theorem 5.11. *Let L be an algebraic frame and $\gamma : L \to Rad(L)$ mapping $x \mapsto \sqrt{x}$ is a closure operator between. γ restricted to $Spec(L)$ is a bijection from $Spec(L)$ onto $Spec(Rad(L))$. In fact, considering $Rad(L)$ as a subset of L it follows that $Spec(L) = Spec(Rad(L))$.*

Proof. Suppose L is an algebraic frame and $\gamma : L \to Rad(L)$ mapping $x \mapsto \sqrt{x}$ is a closure operator. We first show that $Spec(L) \subseteq Spec(Rad(L))$. Let $p \in Spec(L)$ and $x, y \in Rad(L)$ with $x \wedge y \leq p$. Since $x, y \in L$, we have that $x \leq p$ or $y \leq p$. Hence $p \in Spec(Rad(L))$.

Conversely, suppose that $p \in Spec(Rad(L))$. To show that $p \in Spec(L)$, let $x, y \in L$ with $x \wedge y \leq p$. Recall once again that by definition, for any $x \in L$ and $q \in Spec(L)$,

$$x \leq q \Leftrightarrow \sqrt{x} \leq q.$$

Since $p \in Rad(L)$, it follows that $p = \bigwedge\{q : q \in S \subseteq Spec(L)\}$. We thus have,

$$\begin{aligned} x \wedge y \leq p &\Rightarrow x \wedge y \leq q, \text{ for all } q \in S \\ &\Rightarrow x \leq q \text{ or } y \leq q, \text{ for all } q \in S(\text{ since } q \in Spec(L)) \\ &\Rightarrow \sqrt{x} \leq q \text{ or } \sqrt{y} \leq q, \text{ for all } q \in S \\ &\Rightarrow \sqrt{x} \wedge \sqrt{y} \leq q, \text{ for all } q \in S \\ &\Rightarrow \sqrt{x} \wedge \sqrt{y} \leq p. \end{aligned}$$

Since $p \in Spec(Rad(L))$ and $\sqrt{x}, \sqrt{y} \in Rad(L)$, it follows that either $\sqrt{x} \leq p$ or $\sqrt{y} \leq p$. Hence, $x \leq p$ or $y \leq p$, proving that $p \in Spec(L)$. Also, notice that if $p, q \in Spec(L)$ and $\sqrt{p} = \sqrt{q}$, then $p = q$, since $p, q \in Rad(L)$.

Therefore, $Spec(L) = Spec(Rad(L))$. Moreover, $\gamma : p \mapsto \sqrt{p}$ is an identity map between $Spec(L)$ and $Spec(Rad(L))$, and hence is a bijection. □

Corollary 5.12. *For an algebraic frame L, $Min(L) = Min(Rad(L))$.*

Proof. Suppose L is an algebraic frame. Theorem 5.11 says that the closure operator $\gamma : p \mapsto \sqrt{p}$, restricted to $Spec(L)$, is an identity map between $Spec(L)$ and $Spec(Rad(L))$. Let $p \in Min(Rad(L))$, then $p \in Spec(Rad(L))$ and hence $p \in Spec(L)$. By Zorn's Lemma, there exists $q \in Min(L)$ with $q \leq p$. Therefore, $q \in Spec(Rad(L))$ with $q \leq p$. Hence, $p = q \in Min(L)$. So, $Min(Rad(L)) \subseteq Min(L)$.

Conversely, suppose that $p \in Min(L)$, then $p \in Spec(Rad(L))$. Again by Zorn's Lemma there exist some $q \in Min(Rad(L))$ with $q \leq p$. But then $q \in Min(L)$ It thus follows that $p = q$ and so $p \in Min(Rad(L))$.

Therefore, $Min(L) = Min(Rad(L))$. □

In conclusion, we have that under the map $\gamma : x \mapsto \sqrt{x}$, restricted to $Min(L)$, is an identity map between $Min(L)$ and $Min(Rad(L))$. We thus have the following theorem:

Theorem 5.13. *Suppose that L is an algebraic frame satisfying the FIP. Let $\gamma : L \to Rad(L)$ defined by $\gamma(x) = \sqrt{x}$ is a closure operator; then,*

1. *$Min(L)$ is homeomorphic to $Min(Rad(L))$, with respect to the hull-kernel topology.*

2. *$Min(L)^{-1}$ is homeomorphic to $Min(Rad(L))^{-1}$, with respect to the inverse topology.*

Proof. Suppose that L is an algebraic frame satisfying the FIP, then $Rad(L)$ is an algebraic frame satisfying the FIP by Theorem 5.8. Recall that the basic open sets of the hull-kernel topology on $Min(L)$ are $\{U(k) : k \in \mathfrak{K}(L)\}$, where $U(k) = \{p \in Min(L) : k \nleq p\}$. Similarly, the basic open sets of the inverse topology on $Min(L)$ are $\{V(k) : k \in \mathfrak{K}(L)\}$. Therefore, the basic open sets for the hull-kernel topology and the inverse topology of $Min(Rad(L))$ are of the form $\{U(\sqrt{k}) : k \in \mathfrak{K}(L)\}$ and $\{V(\sqrt{k}) : k \in \mathfrak{K}(L)\}$, respectively, using Theorem 5.5. We observe again that for any $k \in L$ and $p \in Min(L)$,

$$k \leq p \Leftrightarrow \sqrt{k} \leq p.$$

Therefore we have,

$$\begin{aligned} U(\sqrt{k}) &= \{p \in Min(Rad(L)) : \sqrt{k} \nleq p\} \\ &= \{p \in Min(L) : \sqrt{k} \nleq p\}, \text{ by Corollary 5.12} \\ &= \{p \in Min(L) : k \nleq p\} \\ &= U(k). \end{aligned}$$

Hence, the basic open sets of $Min(L)$ with respect to the hull-kernel topology are precisely the basic open sets of $Min(Rad(L))$ with respect to the hull-kernel topology. Consequently, $Min(L)$ is homeomorphic to $Min(Rad(L))$ with respect to the hull-kernel topology.

Similarly, we can prove that $V(k) = V(\sqrt{k})$, and hence $Min(L)^{-1}$ is homeomorphic to $Min(Rad(L))^{-1}$. $\square$

Remark 5.14. It is clear from Theorem 4.5 that if $Min(L)$ is not compact then $Min(L)$ with the hull-kernel topology is NOT homeomorphic to $Min(Rad(L))^{-1}$ with the inverse topology and similarly, $Min(L)^{-1}$ with the inverse topology is NOT homeomorphic to $Min(Rad(L))$ with the hull-kernel topology.

CHAPTER 6

Rigidity Of Algebraic Frames

In this chapter, as the name suggest, we will introduce different rigid extensions for algebraic frames. Recall that if G and H are two lattice-ordered groups with G an ℓ-subgroup of H, then G is said to be rigid in H, or in other words, H is a rigid extension of G if for each $h \in H$ there is a $g \in G$ such that $g'' = h''$. For a reference see [4] and [17]. In these articles the authors discussed some other types of extensions for ℓ-groups, which are in general weaker versions of the rigid extension, viz, the r-extension and the r^*-extension. It has been shown that a rigid subgroup of an ℓ-group is both an r-subgroup and an r^*-subgroup. In our dissertation we will define these notions of different rigid extensions for algebraic frames, and prove some results that are analogous to the theory of lattice-ordered groups.

The idea behind the consideration of these rigid extensions came from the following question: What type of extensions between $G \leq H$ will ensure that $Min(G) = Min(H)$? The answer lead to the construction of rigid extensions. For our convenience, a fourth type of extension, called $r^\flat$-extension, were introduced for ℓ-groups. The definition goes by: An extension $G \leq H$ is an r^b-extension if the contraction map $P \to P \cap G$ takes a minimal prime subgroup of H to a minimal prime subgroup of G in a bijective manner. It turned out that both an r-extension and an r^*-extension implies r^b-extension and hence r^b-extension is the weakest of all four rigid extensions for ℓ-groups. As an example of rigid extension we

suppose that A is an f-ring (an f-ring is a lattice-ordered ring in which $a \wedge b = 0$ implies that $a \wedge bc = 0$ for all $c \geq 0$) and qA denotes the classical ring of quotients of A. It follows that A is an f-subring of qA and furthermore, A is rigid in qA.

Before defining the notions of rigid extensions for algebraic frames, we recall some concepts on frame homomorphisms. Let L and M be two algebraic frames and $f : L \to M$ is a frame homomorphism. If f is an *injective* map, then by identifying $f(x)$ with x, L can be viewed as a *subframe* of M. In this case we will write $L \leq M$. f is called *coherent* if it maps compact elements of L into compact elements of M, that is, $f(\mathfrak{K}(L)) \subseteq \mathfrak{K}(M)$. We call $L \leq M$ a *coherent extension* when $\mathfrak{K}(L) \subseteq \mathfrak{K}(M)$. Although the definitions of rigid extensions can be stated for two arbitrary algebraic frames L and M without the condition that L is a subframe of M, we will consider $L \leq M$ a coherent extension, in the definition, for simplicity. Also, for $L \leq M$ and $x \in M$, we will denote by $x^{\perp}$ the polar of x in M, and by x' the polar of x in L. We observe that when $L \leq M$, for any $x \in M$, $x' \leq x^{\perp}$.

Suppose that L and M are two algebraic frames satisfying the FIP, and $L \leq M$ is a coherent frame homomorphism, that is, $\mathfrak{K}(L) \subseteq \mathfrak{K}(M)$.

(i) *Rigid extension:* L is said to be rigid in M, or M is a rigid extension of L if for each $k \in \mathfrak{K}(M)$ there exists $c \in \mathfrak{K}(L)$ with the property $k^{\perp\perp} = c^{\perp\perp}$ (equivalently, $k^{\perp} = c^{\perp}$).

(ii) *r-extension:* M is an r-extension of L if for each $k \in \mathfrak{K}(M)$ and $p \in Min(M)$ with $k \nleq p$, there exists $c \in \mathfrak{K}(L)$ with $c \nleq p$ such that $c^{\perp\perp} \leq k^{\perp\perp}$.

(iii) *r^*-extension:* M is said to be an r^*-extension of L if for each $k \in \mathfrak{K}(M)$ and $p \in Min(M)$ with $k \leq p$, there exists $c \in \mathfrak{K}(L)$ with $c \leq p$ such that $k^{\perp\perp} \leq c^{\perp\perp}$.

(iv) *$r^\flat$-extension* M is an $r^\flat$-extension of L if the contraction map $p \to p^{\mathfrak{c}} = \bigvee\{c \in \mathfrak{K}(L) : f(c) \leq p\}$ is a bijective correspondence between $Min(M)$ and $Min(L)$.

Our next aim is to find relations between these different extensions of frames. Similar to the theory of lattice-ordered groups, we will show that rigid-extension will imply both the r-extension and the r^*-extension for algebraic frames.

Theorem 6.1. *Let L and M be two algebraic frames satisfying the FIP and let $L \leq M$ be a coherent frame homomorphism. If M is a rigid extension of L, then M is both an r-extension and an r^*-extension of L.*

Proof. Suppose that $L \leq M$ is a coherent extension of algebraic frames satisfying the FIP. We assume that M is a rigid extension of L.

To show that M is an r-extension of L, let $k \in \mathfrak{K}(M)$ and $p \in Min(M)$ with $k \nleq p$. Since L is rigid in M there exist some $c \in \mathfrak{K}(L)$ with the property that $k^\perp = c^\perp$. Since p is prime, $k^\perp \leq p$ and so $c^\perp \leq p$. This in turn implies, applying Lemma 3.9, that $c \nleq p$. Also it is clear that $c^{\perp\perp} \leq k^{\perp\perp}$. Hence, M is an r-extension of L.

Next we pick $k \in \mathfrak{K}(M)$ and $p \in Min(M)$ with $k \leq p$. Using rigid extension, there exist $c \in \mathfrak{K}(L)$ with $k^{\perp\perp} = c^{\perp\perp}$ and equivalently, $k^\perp = c^\perp$. We apply Lemma 3.9 on p to conclude that $k^\perp \nleq p$ and so $c^\perp \nleq p$ as well. Since p is a prime element of M, it follows that $c \leq p$. It is also evident that $k^{\perp\perp} \leq c^{\perp\perp}$, proving that M is an r^*-extension of L. □

Lemma 6.2. *Suppose $L \leq M$ is a coherent extension of algebraic frames satisfying the FIP. If M is a rigid extension of L, then the dense elements of L are dense in M.*

Proof. Suppose that $L \leq M$ is a coherent, rigid extension of algebraic frames satisfying the FIP. We recall that an element $x \in L$ is dense if $x' = 0$. We will show that for an element $x \in L$, $x' = 0$ implies that $x^\perp = 0$, that is, x is dense in M.

Suppose that $k \in \mathfrak{K}(M)$ with $k \leq x^\perp$. Using rigid extension, there exist $c \in \mathfrak{K}(L)$ such that $k^{\perp\perp} = c^{\perp\perp}$. We have the following:

$$\begin{aligned} k \leq x^\perp &\Rightarrow k^{\perp\perp} \leq x^\perp \\ &\Rightarrow c \leq c^{\perp\perp} = k^{\perp\perp} \leq x^\perp \\ &\Rightarrow c \wedge x = 0 \\ &\Rightarrow c \leq x' \end{aligned}$$

Since $x' = 0$, this implies that $c = 0$. Therefore,

$$k \leq k^{\perp\perp} = c^{\perp\perp} = 0^{\perp\perp} = 0.$$

Hence for all $k \in \mathfrak{K}(M)$, $k \leq x^{\perp}$ implies that $k = 0$. Since M is algebraic we conclude that $x^{\perp} = 0$. □

As a consequence of the above lemma we have the following proposition.

Proposition 6.3. *Suppose $L \leq M$ is a coherent extension of algebraic frames satisfying the FIP. If M is a rigid extension of L, then $x'' \leq x^{\perp\perp}$, for any $x \in L$.*

Proof. Let $L \leq M$ be a coherent frame extension and L is rigid in M. Using Lemma 6.2 it follows that a dense element of L is dense in M. We observe that for any $x \in L$, $x' \leq x^{\perp}$, since $x' \wedge x = 0$. Therefore for any $x \in L$,

$$x'' \wedge x^{\perp} \leq (x')^{\perp} \wedge x^{\perp} = (x' \vee x)^{\perp} = 0,$$

since $x' \vee x$ is dense in L, and hence $x' \vee x$ is dense in M. Therefore, $x'' \wedge x^{\perp} = 0$, proving that $x'' \leq x^{\perp\perp}$. □

We can characterize rigid extensions between algebraic frames using the d-elements of a frame. Let L be an algebraic frame. An element x in L is called a d-element if for any $c \in \mathfrak{K}(L)$, $c \leq x$ implies that $c^{\perp\perp} \leq x$. We denote the collection of d-elements of L by dL. Our standard reference is [16]. Notice that if $x \in dL$, then

$$x = \bigvee \{c^{\perp\perp} : c \leq x, c \in \mathfrak{K}(L)\}.$$

As it turns out, for a frame L, the collection of d-elements is a complete lattice and in fact, a frame. As for examples of d-elements it can be verified easily that the minimal prime

elements of an algebraic frame L are d-elements. Also, for any $x \in L$, the element $x^{\perp} \in L$ is an d-element.

Suppose $L \leq M$ is a coherent, frame homomorphism between two algebraic frames L and M satisfying the FIP. Let us define a map $F : dM \to dL$ as follows: if $x \in dM$, then $x = \bigvee\{k^{\perp\perp} : k \in \mathfrak{K}(M), k \leq x\}$; we define $F(x) = \bigvee\{c'' : c \in \mathfrak{K}(L), c \leq x\}$. We first show that the map F is well-defined. If, furthermore, M is a rigid extension of L, then F is an isomorphism.

We recall here that for any $x, y \in L$,

$$x^{\perp\perp} \vee y^{\perp\perp} \leq (x^{\perp} \wedge y^{\perp})^{\perp} = (x \vee y)^{\perp\perp}.$$

Lemma 6.4. *Suppose that $L \leq M$ is a coherent frame homomorphism between two algebraic frames L and M satisfying the FIP. The map F between dM and dL defined above is a well-defined map.*

Proof. Let $L \leq M$ be a coherent frame homomorphism, and let $F : dM \to dL$ is defined by $F(x) = \bigvee\{c'' : c \in \mathfrak{K}(L), c \leq x\}$. It follows that $F(x) \in L$. We claim that $F(x) \in dL$. Let $c \in \mathfrak{K}(L)$ with $c \leq F(x)$. So, $c \leq c_1'' \vee \ldots \vee c_n''$, for some $c_1, \ldots, c_n \in \mathfrak{K}(L)$ with $c_1, \ldots, c_n \leq x$. Let $k = c_1 \vee \ldots \vee c_n$, then $k \in \mathfrak{K}(L)$ By the definition of F, it follows that $k'' \leq F(x)$. Finally we have,

$$c'' \leq (c_1'' \vee \ldots \vee c_n'')'' \leq (c_1 \vee \ldots \vee c_n)'''' = k'''' = k'' \leq F(x).$$

Hence, $F(x) \in dL$. □

We notice from the above lemma that if $L \leq M$ is a coherent frame extension of algebraic frames, then following hold: if $c \in \mathfrak{K}(L)$ and $x \in dM$ with $c \leq F(x)$, then $c \leq c'' \leq k'' \leq F(x)$ for some $k \in \mathfrak{K}(L)$ with $k \leq x$. S Using this fact we prove another property of d-elements.

Lemma 6.5. *Suppose $L \leq M$ is a coherent frame homomorphism between two algebraic frame L and M with M a rigid extension of L. Suppose $F : dM \to dL$ is defined as*

$F(x) = \bigvee\{c'' : c \in \mathfrak{K}(L), c \leq x\}$. *For any* $c \in \mathfrak{K}(L)$, $c \leq F(x) \Leftrightarrow c \leq x$.

Proof. Let $L \leq M$ is a coherent, rigid extension of algebraic frames satisfying the FIP. Suppose that $c \in \mathfrak{K}(L)$ and $x \in dM$. We first notice that if $c \leq x$, then by the definition of the map F it follows that $c'' \leq F(x)$. Hence, $c \leq F(x)$.

Conversely if $c \leq F(x)$, then $c \leq c'' \leq k'' \leq F(x)$, for some $k \in \mathfrak{K}(L)$ with $k \leq x$. So $k^{\perp\perp} \leq x$, since $x \in dM$ and $\mathfrak{K}(L) \subseteq \mathfrak{K}(M)$. From Proposition 6.3 it follows that $k'' \leq k^{\perp\perp}$, since $L \leq M$ is a rigid extension. Hence, $c \leq k'' \leq k^{\perp\perp} \leq x$. □

It thus follows from the above lemma that for any $c \in \mathfrak{K}(L)$ and $x \in dM$,

$$c \nleq x \Leftrightarrow c'' \nleq F(x).$$

Theorem 6.6. *Suppose* $L \leq M$ *is a coherent extension of algebraic frames* L *and* M *satisfying the FIP.* M *is a rigid extension of* L *precisely when the map* $F : dM \to dL$ *is an isomorphism between* dM *and* dL.

Proof. Let $L \leq M$ be a coherent extension of algebraic frames L and M satisfying the FIP. Let $F : dM \to dL$ be defined by $F(x) = \bigvee\{c'' : c \in \mathfrak{K}(L), c \leq x\}$.

($\Rightarrow$) Suppose that M is a rigid extension of L. Let $x, y \in dM$ with $x \neq y$. Since M is an algebraic frames, there exist some $k \in \mathfrak{K}(M)$ such that $k \leq x$ but $k \nleq y$. So $k^{\perp\perp} \leq x$, but $k^{\perp\perp} \nleq y$. Using the definition of rigid extension, there exists $c \in \mathfrak{K}(L)$ with $k^{\perp\perp} = c^{\perp\perp}$. Therefore, $c \leq c^{\perp\perp} = k^{\perp\perp} \leq x$ but $c^{\perp\perp} \nleq y$. This means that $c \nleq y$, since y is a d-element. It thus follows from Lemma 6.5 that $c'' \leq d(x)$, but $c'' \nleq d(y)$. Hence $c \leq F(x)$ but $c \nleq F(y)$, since $F(y) \in dL$. These proves that $F(x) \neq F(y)$, concluding that F is injective.

Let $x \in dL$ and define $x_1 = \bigvee\{c^{\perp\perp} : c \in \mathfrak{K}(L), c \leq x\}$. We notice that $x_1 \in dM$ since, if $k \in \mathfrak{K}(M)$ with $k \leq x_1$, then by a similar argument as before

$$k \leq k^{\perp\perp} \leq c^{\perp\perp} \leq x_1,$$

for some $c \in \mathfrak{K}(L)$ with $c \leq x$. We claim that $F(x_1) = x$, proving that F is a surjective map and hence a bijection. We recall that $F(x_1) = \bigvee\{k'' : k \in \mathfrak{K}(L), k \leq x_1\}$. If $c \in \mathfrak{K}(L)$ with $c \leq x$, then $c \leq c^{\perp\perp} \leq x_1$. These implies that $c \leq c'' \leq F(x_1)$. Hence, $x \leq F(x_1)$. To show the reverse inequality let $k \in \mathfrak{K}(L)$ with $k \leq x_1$, that is, $k'' \leq F(x_1)$. Therefore $k \leq k^{\perp\perp} \leq c^{\perp\perp} \leq x_1$, for some $c \in \mathfrak{K}(L)$ and $c \leq x$. We have the following:

$$\begin{aligned} k \leq c^{\perp\perp} &\Rightarrow k \wedge c^{\perp} = 0 \\ &\Rightarrow k \wedge c' = 0, \text{ since } c' \leq c^{\perp} \\ &\Rightarrow k \leq c'' \\ &\Rightarrow k'' \leq c'' \end{aligned}$$

Since $x \in dL$ and $c \leq x$, it follows that $c'' \leq x$. Therefore, $k'' \leq x$. Consequently, $F(x_1) = \bigvee\{k'' : k \in \mathfrak{K}(L), k \leq x_1\} \leq x$, thence $F(x_1) = x$.

Finally we claim that F is a frame homomorphism. Let $x, y \in dM$ with $x \leq y$. If $c \in \mathfrak{K}(L)$ with $c \leq x$, then $c \leq y$. Therefore, $c'' \leq F(x)$ implies that $c'' \leq F(y)$. Thus, $F(x) = \bigvee\{c'' : c \in \mathfrak{K}(L), c \leq x\} \leq F(y)$. Hence, $F : dM \to dL$ is a frame homomorphism and therefore a frame isomorphism.

($\Leftarrow$) Conversely, suppose that the map $F : dM \to dL$, defined by $F(x) = \bigvee\{c'' : c \in \mathfrak{K}(L), c \leq x\}$, is an isomorphism between dM and dL. We claim that for any $x \in dM$, $x = \bigvee\{c^{\perp\perp} : c \in \mathfrak{K}(L), c \leq x\}$. Let us denote $y = \bigvee\{c^{\perp\perp} : c \in \mathfrak{K}(L), c \leq x\}$, then $y \in dM$ and $y \leq x$. Therefore $F(y) \leq F(x)$. On the other hand, if $k \leq F(x)$, then $k \leq c'' \leq F(x)$, for some $c \in \mathfrak{K}(L)$ with $c \leq x$. Therefore, $c \leq c^{\perp\perp} \leq y$ and so $c'' \leq F(y)$. Thus $k \leq F(y)$, proving that $F(y) = F(x)$. Since F is an injective map we have that $x = y = \bigvee\{c^{\perp\perp} : c \in \mathfrak{K}(L), c \leq x\}$. To show that M is a rigid extension of L, we pick an arbitrary $k \in \mathfrak{K}(M)$. $k^{\perp\perp} \in dM$ and so $k^{\perp\perp} = \bigvee\{c^{\perp\perp} : c \in \mathfrak{K}(L), c \leq k^{\perp\perp}\}$. Since $k \leq k^{\perp\perp}$, there exists $c \in \mathfrak{K}(L)$ with $c \leq k^{\perp\perp}$ such that $k \leq c^{\perp\perp} \leq k^{\perp\perp}$. Hence, $k^{\perp\perp} = c^{\perp\perp}$, thence M

is a rigid extension of L. $\square$

We observe an immediate corollary to the preceding theorem, which uses the fact that the composition of maps $dM \to dL$ and $dN \to dM$, as in the theorem, is again the map $dN \to dL$ with the same properties.

Corollary 6.7. *Suppose that L, M and N are algebraic frames satisfying the FIP. If M is a rigid extension of L and N is a rigid extension of M, then N is a rigid extension of L; that is, rigidity is a transitive property for algebraic frames.*

To have a better understanding of rigid extensions we consider a nice class of algebraic frames and specify their rigid subframes. We begin with some definitions. Recall that an element x in a frame L is *complemented* if $x \vee x^{\perp} = 1$. A frame is *zero-dimensional* if every element is the supremum of complemented elements. It turns out that an algebraic frame is zero-dimensional precisely when every compact element is complemented. Furthermore, such a frame satisfies the FIP. A frame is called a *boolean* frame if every element of the frame is complemented. Hence, a boolean algebraic frame is zero-dimensional.

Lemma 6.8. *Let L be an algebraic frame and $x \in L$. If x is complemented, then $x = x^{\perp\perp}$.*

Proof. Suppose that L is an algebraic frame and $x \in L$ with $x \vee x^{\perp} = 1$. We know that $x \leq x^{\perp\perp}$, and therefore $x^{\perp\perp} \wedge x = x$. For the other inequality we notice that, $x^{\perp\perp} \wedge (x \vee x^{\perp}) = x^{\perp\perp} \wedge 1 = x^{\perp\perp}$. On the other hand, using distributivity we have,

$$\begin{aligned} x^{\perp\perp} \wedge (x \vee x^{\perp}) &= (x^{\perp\perp} \wedge x) \vee (x^{\perp\perp} \wedge x^{\perp}) \\ &= (x^{\perp\perp} \wedge x) \vee 0 \\ &= (x^{\perp\perp} \wedge x) \end{aligned}$$

Hence, $x^{\perp\perp} = x^{\perp\perp} \wedge x = x$. $\square$

Proposition 6.9. *Let $L \leq M$ be a coherent extension of algebraic frames L and M satisfying the FIP and let M be a zero-dimensional frame. The extension is rigid precisely when $L = M$. In particular, the only rigid subframe of an algebraic boolean frame is itself.*

Proof. Suppose that $L \leq M$ is a coherent extension of algebraic frames satisfying the FIP and M is zero-dimensional. It follows immediately that if $L = M$, then $L \leq M$ is a rigid extension. On the other hand, suppose that the extension is rigid. For each $k \in \mathfrak{K}(M)$ there exists $c \in \mathfrak{K}(L)$ such that $k^{\perp\perp} = c^{\perp\perp}$. Since M is zero-dimensional, $k^{\perp\perp} = k$ for all $k \in \mathfrak{K}(M)$. We notice that in this case both k and c are compact elements of M, the extension being coherent. Hence we have the following:

$$c = c^{\perp\perp} = k^{\perp\perp} = k.$$

Therefore, $\mathfrak{K}(M) = \mathfrak{K}(L)$. Since both L and M are algebraic, it follows that $L = M$. $\square$

We recall that if $(X, \mathcal{O})$ is a topological space then the topology $\mathcal{O}$ forms a frame under inclusion. A frame is *spacial* if it is isomorphic to some topology on a set. Suppose that $(X, \mathcal{O})$ is a Hausdorff space (that is, any two distinct points of X can be separated by disjoint open sets). Also assume that the frame $\mathcal{O}$ be algebraic. Since the topology is Hausdorff, every compact subset is closed in the topology. Likewise, if $c \in \mathcal{O}$ is a compact element, then c is a compact subset of X and so is closed in the topology. Also, c is open and hence is clopen. Therefore, every compact element of the algebraic (Hausdorff) frame $\mathcal{O}$ is clopen in the topology. Since $\mathcal{O}$ is algebraic, any $U \in \mathcal{O}$ can be expressed as $U = \bigvee\{c : c \in \mathfrak{K}(\mathcal{O})\}$. In other words, $U = \bigcup\{c : c \text{ clopen}\}$. So every open set in the topology can be written as an union of clopen sets, which in turn says that the topology $\mathcal{O}$ is zero-dimensional (A topological space (X, τ) is *zero-dimensional* if it has a base of clopen sets). Hence, a Hausdorff topology $\mathcal{O}$ is an algebraic frame only if it is zero-dimensional. We further notice that if $c \in \mathfrak{K}(\mathcal{O})$, then $c^{\perp} = \bigcup\{U \in \mathcal{O} : U \cap c = \emptyset\}$, which in turn says that $c^{\perp} = X \setminus \mathrm{cl}_X(c)$. So, if $\mathcal{O}$ is a Hausdorff topology, then for every compact element c in the algebraic frame $\mathcal{O}$,

$c^{\perp\perp} = c$. Based on these facts we have a corollary to Proposition 6.9.

Corollary 6.10. *Suppose $\mathcal{O}_1$ and $\mathcal{O}_2$ are two Hausdorff topologies. If $\mathcal{O}_1 \leq \mathcal{O}_2$ is a coherent extension of algebraic frames satisfying the FIP, then the extension is rigid if and only if $\mathcal{O}_1 = \mathcal{O}_2$.*

As an example we can consider discrete topologies, $\mathcal{D}_1$ and $\mathcal{D}_2$, on two spaces X_1 and X_2. Since a discrete topology is algebraic and Hausdorff, it is a zero-dimensional frame. $\mathcal{D}_1 \leq \mathcal{D}_2$ is a rigid extension if and only if $X_1 = X_2$. The question then is, what happens if the topologies are not Hausdorff to begin with?

Example 6.11. Let us consider the set of all integers, $\mathbb{Z}$. Let $\mathcal{O}_1$ be the indiscrete topology and $\mathcal{O}_2$ be the cofinite topology on $\mathbb{Z}$. We observe that $\mathcal{O}_2$ is a non-Hausdorff, T_1 topology. In fact, every pair of distinct open sets in $\mathcal{O}_2$ intersects non-trivially and so every open set is dense. This means that $x^\perp = 0$, for every $x \in \mathcal{O}_2$. Moreover, every $x \in \mathcal{O}_2$ is compact, whence $\mathcal{O}_2$ is an algebraic frame satisfying the FIP. Hence it follows that for every $k \in \mathfrak{K}(\mathcal{O}_2)$, $k^{\perp\perp} = 1 = 1^{\perp\perp}$. Also we notice that $\mathcal{O}_1 = \{0, 1\}$, where $0 = \emptyset$ and $1 = \mathbb{Z}$. Therefore, $1 \in \mathfrak{K}(\mathcal{O}_1)$, and $k^{\perp\perp} = 1^{\perp\perp}$ for all compact element k of $\mathcal{O}_2$. Thus, $\mathcal{O}_1 \leq \mathcal{O}_2$ is a coherent rigid extension but $\mathcal{O}_1 \neq \mathcal{O}_2$.

We observe that in general if we have a coherent extension of algebraic frames satisfying the FIP, say $L \leq M$, and M is a coherent frame (that is, $1 \in \mathfrak{K}(M)$) has the property that every nonzero compact element is dense, then the extension is rigid.

We recall from the theory of ℓ-groups the notion of the contraction map: For ℓ-groups $G \leq H$, the contraction map $(C)(H) \to (C)(G)$ is defined by $K \mapsto K \cap G$. For algebraic frames the definition goes like this: Suppose $L \leq M$ is a coherent extension of algebraic frames. For any $x \in M$ we define a map $M \to L$ by $x \mapsto x^{\mathfrak{c}}$ where, $x^{\mathfrak{c}} = \bigvee\{c \in \mathfrak{K}(L) : c \leq x\}$. $x^{\mathfrak{c}}$ is the *contraction* of x to L and the map is called the *contraction map* from M into L. We notice the following properties of the contraction map $x \mapsto x^{\mathfrak{c}}$ between M and L:

1. The set $\{c \in \mathfrak{K}(L) : c \leq x\}$ is nonempty since $0 \in \mathfrak{K}(L)$ belongs to the set and therefore $x^{\mathfrak{c}} \in L$.

2. The contraction map is always surjective: Let $x \in L$, then $x \in M$ and $x^{\mathfrak{c}} = \bigvee\{c \in \mathfrak{K}(L) : c \leq x\} = x$.

3. The contraction map preserves order: Let $x, y \in M$ with $x \leq y$. If $c \in \mathfrak{K}(L)$ such that $c \leq x$, then $c \leq y$. Therefore,

$$x^{\mathfrak{c}} = \bigvee\{c \in \mathfrak{K}(L) : c \leq x\} \leq \bigvee\{k \in \mathfrak{K}(L) : k \leq y\} = y^{\mathfrak{c}}.$$

4. Since $\{c \in \mathfrak{K}(L) : c \leq x\} \subseteq \{k \in \mathfrak{K}(M) : k \leq x\}$, it follows that

$$x^{\mathfrak{c}} \leq x, \text{ for every } x \in M.$$

5. Let $c \in \mathfrak{K}(L)$. We notice from (4) that $c \leq x^{\mathfrak{c}}$ implies that $c \leq x$. On the other hand, if $c \leq x$, then by definition of $x^{\mathfrak{c}}$ it follows that $c \leq x^{\mathfrak{c}}$. Therefore for any $c \in \mathfrak{K}(L)$,

$$c \leq x^{\mathfrak{c}} \Leftrightarrow c \leq x.$$

Let L be a frame and $\mathfrak{F} \subseteq L$. $\mathfrak{F}$ is a *filter base* of L if

(a) $0 \notin \mathfrak{F}$, and

(b) $x, y \in \mathfrak{F}$ will imply that $x \wedge y \in \mathfrak{F}$.

Given a filter base $\mathfrak{F}$ of a frame L we define $\overline{\mathfrak{F}} = \{x \in L : x \geq f, \text{ for some } f \in \mathfrak{F}\}$. It can be verified that $\overline{\mathfrak{F}}$ is a filter containing $\mathfrak{F}$. So, every filter base can be extended to a filter.

Lemma 6.12. *Let $L \leq M$ be a coherent extension of algebraic frames satisfying the FIP. The contraction map $x \mapsto x^{\mathfrak{c}}$ restricted to $Spec(M)$ maps $p \in Spec(M)$ to $p^{\mathfrak{c}} \in Spec(L)$. Furthermore if $q \in Min(L)$, then there exists a $p \in Min(M)$ such that $p^{\mathfrak{c}} = q$.*

Proof. Suppose that $L \leq M$ is a coherent extension of algebraic frames L and M satisfying the FIP. Let $p \in Spec(M)$ and $x, y \in \mathfrak{K}(L)$ with $x \wedge y \leq p^{\mathfrak{c}}$. Since $p^{\mathfrak{c}} \leq p$, we have $x \wedge y \leq p$. Since p is a prime element of M it follows that either $x \leq p$ or $y \leq p$. By definition, this implies that either $x \leq p^{\mathfrak{c}}$ or $y \leq p^{\mathfrak{c}}$. Hence, $p^{\mathfrak{c}} \in Spec(L)$.

Finally, let $q \in Min(L)$. Using Lemma 3.3, the collection $\mathfrak{U} = \{c : c \in \mathfrak{K}(L), c \nleq q\}$ is an ultrafilter of compact elements of L and $q = \bigvee\{c^{\perp} : c \in \mathfrak{U}\}$. We notice that $\mathfrak{U}$ is a filter base of compact elements of M since,

1. $0 \leq q$ implies that $0 \notin \mathfrak{U}$.

2. Let $c_1, c_2 \in \mathfrak{U}$. Since $c_1^{\perp} \leq q$ and $c_2^{\perp} \leq q$, it follows that both $c_1, c_2 \nleq q$, by Lemma 3.9. Hence, $c_1 \wedge c_2 \nleq q$, since q is a prime element of L. Consequently, $c_1 \wedge c_2 \in \mathfrak{U}$.

$\mathfrak{U}$ can be extended to a filter and hence by using Zorn's Lemma the filter can be further extended to an ultrafilter of compact elements, namely $\overline{\mathfrak{U}}$, of M. Let us consider $p = \bigvee\{k^{\perp} : k \in \mathfrak{U}\}$. Using Lemma 3.4 we conclude that $p \in Min(M)$. To finish the proof we claim that $p^{\mathfrak{c}} = q$. Let $c \in \mathfrak{K}(L)$ with $c \nleq q$, then $c \in \mathfrak{U} \subseteq \overline{\mathfrak{U}}$. By the definition of p we then have that $c^{\perp} \leq p$; whence $c \nleq p$, by Lemma 3.9. Therefore, for any $c \in \mathfrak{K}(L)$, $c \nleq q \Rightarrow c \nleq p$. In other words, $c \leq p \Rightarrow c \leq q$, for any $c \in \mathfrak{K}(L)$. Hence, $p^{\mathfrak{c}} = \bigvee\{d \in \mathfrak{K}(L) : d \leq p\} \leq q$. Since $q \in Min(L)$ and we have proved that $p^{\mathfrak{c}} \in Spec(L)$, it follows that $p^{\mathfrak{c}} = q$. $\square$

Observe that for any $c \in \mathfrak{K}(L)$ and $p \in Min(M)$, $c \nleq p \Leftrightarrow c \nleq p^{\mathfrak{c}}$.

Let $L \leq M$ be a coherent extension of algebraic frames satisfying the FIP. Recall that $L \leq M$ is an $r^{\flat}$-extension if the contraction map $p \mapsto p^{\mathfrak{c}}$ is a bijective correspondence between $Min(M)$ and $Min(L)$. In general, a contraction map between M and L need not map a minimal prime element of M into a minimal prime element of L. Also, such a map may not be a bijection between $Min(M)$ and $Min(L)$. The next two examples will demonstrate these facts.

Example 6.13. Let L and M be as follows:

$$L = \begin{tikzcd} 1 \arrow[d, dash] \\ a \arrow[d, dash] \\ 0 \end{tikzcd} \qquad \text{and,} \qquad M = \begin{tikzcd} & 1 \arrow[dl, dash] \arrow[dr, dash] & \\ a \arrow[dr, dash] & & b \arrow[dl, dash] \\ & 0 & \end{tikzcd}$$

We notice that $L \leq M$ is a coherent subframe, $Min(M) = \{a, b\}$ and $Min(L) = \{0\}$. Under the contraction mapping, $a^{\mathfrak{c}} = a \notin Min(L)$. Hence, the contraction map restricted to $Min(M)$ does not map into $Min(L)$.

Example 6.14. Let $L \leq M$ be as in the diagram below.

$$\begin{tikzcd} 1 \arrow[d, dash] \\ \\ 0 \end{tikzcd} \qquad\qquad \begin{tikzcd} & 1 \arrow[dl, dash] \arrow[dr, dash] & \\ a \arrow[dr, dash] & & b \arrow[dl, dash] \\ & 0 & \end{tikzcd}$$

$$L \qquad\qquad\qquad\qquad M$$

In this case, $Min(M) = \{a, b\}$ and $Min(L) = \{0\}$. Furthermore, $a^{\mathfrak{c}} = b^{\mathfrak{c}} = 0$, concluding that the contraction map restricted to $Min(M)$ is not a bijection.

We have proved that rigid extension is stronger than both the r-extension and the r^*-extension for algebraic frames. We now proceed to prove that $r^\flat$-extension is, in fact, weaker than both r-extension and r^*-extension for frames, which will be our next two theorems.

Theorem 6.15. *Let L and M be two algebraic frames satisfying the FIP and $L \leq M$ is a coherent extension. If M is an r-extension of L then M is an $r^\flat$-extension of L, and furthermore the contraction map is a homeomorphism between $Min(L)$ and $Min(M)$ with respect to the hull-kernel topology on both the minimal prime element spaces.*

Proof. Suppose that $L \leq M$ is a coherent extension of algebraic frames satisfying the FIP. Assume that $L \leq M$ is an r-extension. We define the contraction map restricted to $Min(M)$, $Min(M) \to Min(L)$ by $p \mapsto p^{\mathfrak{c}}$, where $p^{\mathfrak{c}} = \bigvee\{c \in \mathfrak{K}(L) : c \leq p\}$. We will first prove that the contraction map restricted to $Min(M)$ is well-defined. We have already proved in Lemma 6.12 that for any $p \in Min(M)$, $p^{\mathfrak{c}} \in Spec(L)$. It remains to show that $p^{\mathfrak{c}}$ is a minimal prime element of L. By Zorn's Lemma, there exists $q \in Min(L)$ with $q \leq p^{\mathfrak{c}}$. Using Lemma 6.12 again it follows that there exists $r \in Min(M)$ with $r^{\mathfrak{c}} = q \leq p^{\mathfrak{c}}$. We know from the Lemma on Ultrafilter 3.3 that $p = \bigvee\{k^{\perp} : k \in \mathfrak{K}(M), k \nleq p\}$. Let us choose such a $k \in \mathfrak{K}(M)$ with $k \nleq p$. Using r-extension, there exist some $c \in \mathfrak{K}(L)$ with $c \nleq p$ and $c^{\perp\perp} \leq k^{\perp\perp}$. Therefore, $c \nleq p^{\mathfrak{c}}$ and hence $c \nleq r^{\mathfrak{c}}$. Consequently, $c \nleq r$, which says that $c^{\perp\perp} \nleq r$, since $r \in Min(M)$. Hence, $k^{\perp\perp} \nleq r$ and so $k \nleq r$. As a conclusion we observe that $p = \bigvee\{k^{\perp} : k \in \mathfrak{K}(M), k \nleq p\} \leq r$. Since both p and r are minimal prime elements of M, it follows that $p = r$, thence $p^{\mathfrak{c}} = r^{\mathfrak{c}} = q \in Min(L)$. So, the contraction map restricted to $Min(M)$ is a well-defined map.

We next show that the contraction map restricted to $Min(M)$ is injective. Let $p, q \in Min(M)$ with $p \neq q$, then there exists $k \in \mathfrak{K}(M)$ with $k \leq p$ but $k \nleq q$. Using r-extension, there exists $c \in \mathfrak{K}(L)$ with $c \nleq q$ such that $c^{\perp\perp} \leq k^{\perp\perp}$. Since $c \nleq q$, it follows that $c \nleq q^{\mathfrak{c}}$. On the other hand, $k \leq p$ implies that $k^{\perp\perp} \leq p$, since p is a minimal prime element. We thus have,

$$c \leq c^{\perp\perp} \leq k^{\perp\perp} \leq p.$$

Therefore, $c \leq p^{\mathfrak{c}}$ but $c \nleq q^{\mathfrak{c}}$. Since $c \in \mathfrak{K}(L)$ and L is algebraic, it concludes that $p^{\mathfrak{c}} \neq q^{\mathfrak{c}}$.

The surjectivity of the contraction map restricted to $Min(M)$ follows from Lemma 6.12. Hence, $L \leq M$ is an $r^{\flat}$-extension.

To prove homeomorphism we need to show that the contraction map $p \mapsto p^{\mathfrak{c}}$ is a continuous, open map between $Min(M)$ and $Min(L)$ with respect to the hull-kernel topology. We will use $U_M(x)$ ($V_M(x)$) and $U_L(x)$ ($V_L(x)$) to denote the open (closed) sets of $Min(M)$ and $Min(L)$ respectively. Also, we will use the notation $U^{\mathfrak{c}}$ to mean the image of U_M under the

contraction map. Recall once again that for $c \in \mathfrak{K}(L)$ and $p \in Min(M)$,

$$c \leq p \Leftrightarrow c \leq p^{\mathfrak{c}} \text{ and } c \nleq p \Leftrightarrow c \nleq p^{\mathfrak{c}}.$$

To show continuity, let $c \in \mathfrak{K}(L)$ and consider the basic open set $U_L(c)$ of $Min(L)$. We claim that $U_L(c)^{-1}$ is open in $Min(M)$. Since the extension is coherent, $c \in \mathfrak{K}(M)$ and $U_M(c) = \{p \in Min(M) : c \nleq p\}$ is an open set in $Min(M)$. Also, $U_L(c)^{-1} = \{q \in Min(M) : q^{\mathfrak{c}} \in U_L(c)\}$ Let us consider the following:

$$\begin{aligned} p \in U_M(c) &\Rightarrow c \nleq p \\ &\Rightarrow c \nleq p^{\mathfrak{c}} \\ &\Rightarrow p^{\mathfrak{c}} \in U_L(c) \\ &\Rightarrow p \in U_L(c)^{-1}. \end{aligned}$$

Therefore, $U_M(c) \subseteq U_L(c)^{-1}$. To show the converse we consider the following string of implications:

$$\begin{aligned} q \in U_L(c)^{-1} &\Rightarrow q^{\mathfrak{c}} \in U_L(c) \\ &\Rightarrow c \nleq q^{\mathfrak{c}} \\ &\Rightarrow c \nleq q \\ &\Rightarrow q \in U_M(c). \end{aligned}$$

Hence, $U_L(c)^{-1} \subseteq U_M(c)$. Therefore, $U_L(c)^{-1} = U_M(c)$ is open in $Min(M)$. So the contraction map restricted to $Min(M)$ maps basic open sets of $Min(L)$ to open sets of $Min(M)$ and hence is continuous.

To show that the contraction map is open, let us pick $k \in \mathfrak{K}(M)$ and consider the basic open set $U_M(k) = \{p \in Min(M) : k \nleq p\}$ of $Min(M)$. We want to show that $U(k)^{\mathfrak{c}} =$

$\{p^{\mathfrak{c}} \in Min(L) : p \in U_M(k)\}$ is open in $Min(L)$. For each $p \in U_M(k)$, by the definition of r-extension, there exist $c_p \in \mathfrak{K}(L)$ with $c_p) \nleq p$ and $c^{\perp\perp} \leq k^{\perp\perp}$. So, $k^{\perp} = k^{\perp\perp\perp} \leq c_p^{\perp\perp\perp} = c_p^{\perp}$, for all $p \in U_M(k)$. Let us denote $c = \bigvee\{c_p : p \in U_M(k))\}$, then $c \in L$. We claim that $U(k)^{\mathfrak{c}} = U_L(c)$:

$$\begin{aligned} p^{\mathfrak{c}} \in U(k)^{\mathfrak{c}} &\Rightarrow p \in U_M(k) \\ &\Rightarrow c_p \nleq p, \text{ by construction} \\ &\Rightarrow c \nleq p, \text{ since } c_p \leq c \\ &\Rightarrow p \in U_L(c) \end{aligned}$$

Hence, $U(k)^{\mathfrak{c}} \subseteq U_L(c)$.

To show the other inclusion we first notice that $k^{\perp} \leq c^{\perp}$, because we have $k^{\perp} \leq c_p^{\perp}$ for all $p \in U_M(k)$ and hence,

$$k^{\perp} \leq \bigwedge\{c_p^{\perp} : p \in U_M(k)\} = \left(\bigvee c_p\right)^{\perp} = c^{\perp}.$$

Let $q \in U_L(c)$, then there exist $p \in Min(M)$ such that $q = p^{\mathfrak{c}}$.

$$\begin{aligned} p^{\mathfrak{c}} \in U_L(c) &\Rightarrow c \nleq p^{\mathfrak{c}} \\ &\Rightarrow c \nleq p \\ &\Rightarrow c^{\perp} \leq p, \text{ since } p \text{ is prime} \\ &\Rightarrow k^{\perp} \leq p, \text{ since } k^{\perp} \leq c^{\perp} \\ &\Rightarrow k \nleq p, \text{ by Lemma 3.9} \\ &\Rightarrow p \in U_M(k) \\ &\Rightarrow p^{\mathfrak{c}} \in U(k)^{\mathfrak{c}} \end{aligned}$$

Thus, $U_L(c) \subseteq U(k)^{\mathfrak{c}}$.

It thus follows that the contraction map restricted to $Min(M)$ maps open sets of $Min(M)$ into open sets of $Min(L)$ and hence is a open map.

Therefore, the contraction map $p \mapsto p^{\mathfrak{c}}$ is a bijective, continuous, and open map between $Min(M)$ and $Min(L)$ with respect to the hull-kernel topology, and hence is a homeomorphism. $\square$

Theorem 6.16. *Suppose that L and M are algebraic frames satisfying the FIP and $L \leq M$ is a coherent extension. If M is an r^*-extension of L then M is an $r^\flat$-extension of L, and furthermore the contraction map is a homeomorphism between $Min(M)^{-1}$ and $Min(L)^{-1}$.*

Proof. Suppose that $L \leq M$ is a coherent extension of algebraic frames satisfying the FIP. Assume that $L \leq M$ is an r^*-extension. We define the contraction map restricted to $Min(M)^{-1}$, $Min(M)^{-1} \to Min(L)^{-1}$ by $p \mapsto p^{\mathfrak{c}}$, where $p^{\mathfrak{c}} = \bigvee\{c \in \mathfrak{K}(L) : c \leq p\}$. First of all, we prove that the contraction map restricted to $Min(M)^{-1}$ is well-defined. We have already proved in Lemma 6.12 that for any $p \in Min(M)^{-1}$, $p^{\mathfrak{c}} \in Spec(L)$. As before, It remains to show that $p^{\mathfrak{c}}$ is a minimal prime element of L. By Zorn's Lemma, there exists $q \in Min(L)^{-1}$ with $q \leq p^{\mathfrak{c}}$. Using Lemma 6.12 again it follows that there exists $r \in Min(M)^{-1}$ with $r^{\mathfrak{c}} = q \leq p^{\mathfrak{c}}$. Let us choose a $k \in \mathfrak{K}(M)$ with $k \leq r$. Using r^*-extension, there exist some $c \in \mathfrak{K}(L)$ with $c \leq r$ and $k^{\perp\perp} \leq c^{\perp\perp}$. Therefore, $c \leq r^{\mathfrak{c}}$ and hence $c \leq p^{\mathfrak{c}}$. Consequently, $c \leq p$, which says that $c^{\perp\perp} \leq p$, since $p \in Min(M)$. Hence, $k^{\perp\perp} \leq p$ and so $k \leq p$. Since M is algebraic it follows that $r \leq p$. Since both p and r are minimal prime elements of M, it follows that $p = r$, thence $p^{\mathfrak{c}} = r^{\mathfrak{c}} = q \in Min(L)$. So, the contraction map restricted to $Min(M)^{-1}$ is a well-defined map.

We next show that the contraction map restricted to $Min(M)^{-1}$ is injective. Let $p, q \in Min(M)^{-1}$ with $p \neq q$, then there exists $k \in \mathfrak{K}(M)$ with $k \leq q$ but $k \nleq p$. Using r^*-extension, there exists $c \in \mathfrak{K}(L)$ with $c \leq q$ such that $k^{\perp\perp} \leq c^{\perp\perp}$. Since $c \leq q$, it follows that $c \leq q^{\mathfrak{c}}$. On the other hand, $k \nleq p$ implies that $k^{\perp\perp} \nleq p$, since p is a minimal prime element. Hence, $c^{\perp\perp} \nleq p$. Therefore, $c \nleq p$ and hence, $c \nleq p^{\mathfrak{c}}$. Therefore, $c \nleq p^{\mathfrak{c}}$ but $c \leq q^{\mathfrak{c}}$. Since $c \in \mathfrak{K}(L)$ and L is algebraic, it concludes that $p^{\mathfrak{c}} \neq q^{\mathfrak{c}}$.

The surjectivity of the contraction map restricted to $Min(M)^{-1}$ follows from Lemma 6.12. Hence, $L \leq M$ is an $r^\flat$-extension.

We next prove that the contraction map restricted to $Min(M)^{-1}$ is a homeomorphism with respect to the inverse topology. To prove homeomorphism we need to show that the contraction map $p \mapsto p^\mathfrak{c}$ is a continuous, open map between $Min(M)^{-1}$ and $Min(L)^{-1}$. As before, We will use $V_M(x)$ ($U_M(x)$) and $V_L(x)$ ($U_L(x)$) to denote the open (closed) sets of $Min(M)^{-1}$ and $Min(L)^{-1}$, respectively. Also, we will use the notation $V^\mathfrak{c}$ to mean the image of V_M under the contraction map. Recall once again that for $c \in \mathfrak{K}(L)$ and $p \in Min(M)$,

$$c \leq p \Leftrightarrow c \leq p^\mathfrak{c}.$$

To show continuity, let $c \in \mathfrak{K}(L)$ and consider the basic open set $V_L(c)$ of $Min(L)^{-1}$. We claim that $V_L(c)^{-1}$ is open in $Min(M)^{-1}$. Since the extension is coherent, $c \in \mathfrak{K}(M)$ and $V_M(c) = \{p \in Min(M)^{-1} : c \leq p\}$ is an open set in $Min(M)^{-1}$. Also, $V_L(c)^{-1} = \{q \in Min(M)^{-1} : q^\mathfrak{c} \in V_L(c)\}$. Let us consider the following:

$$\begin{aligned} p \in V_M(c) &\Rightarrow c \leq p \\ &\Rightarrow c \leq p^\mathfrak{c} \\ &\Rightarrow p^\mathfrak{c} \in V_L(c) \\ &\Rightarrow p \in V_L(c)^{-1}. \end{aligned}$$

Therefore, $V_M(c) \subseteq V_L(c)^{-1}$. To show the converse we consider the following string of

implications:

$$\begin{aligned} q \in V_L(c)^{-1} &\Rightarrow q^{\mathfrak{c}} \in V_L(c) \\ &\Rightarrow c \leq q^{\mathfrak{c}} \\ &\Rightarrow c \leq q \\ &\Rightarrow q \in V_M(c). \end{aligned}$$

Hence, $V_L(c)^{-1} \subseteq V_M(c)$. Therefore, $V_L(c)^{-1} = V_M(c)$ is open in $Min(M)^{-1}$. So the contraction map restricted to $Min(M)^{-1}$ maps basic open sets of $Min(L)^{-1}$ to open sets of $Min(M)^{-1}$ and hence is continuous.

To show that the contraction map restricted to $Min(M)^{-1}$ is open, let us pick $k \in \mathfrak{K}(M)$ and consider the basic open set $V_M(k) = \{p \in Min(M)^{-1} : k \leq p\}$ of $Min(M)^{-1}$. We want to show that $V(k)^{\mathfrak{c}} = \{p^{\mathfrak{c}} \in Min(L)^{-1} : p \in V_M(k)\}$ is open in $Min(L)^{-1}$. Let $p^{\mathfrak{c}} \in V(k)^{\mathfrak{c}}$, then $p \in V_M(k)$. By the definition of r^*-extension, there exists $c \in \mathfrak{K}(L)$ with $c \leq p$ and $k^{\perp\perp} \leq c^{\perp\perp}$. Therefore, $c \leq p^{\mathfrak{c}}$ and so $p^{\mathfrak{c}} \in V(c)$. We claim that $V(c) \subseteq V(k)^{\mathfrak{c}}$. Let $q^{\mathfrak{c}} \in V(c)$, then $c \leq q$. Since $q \in Min(L)^1$ it follows that $c^{\perp\perp} \leq q$, and hence $k^{\perp\perp} \leq q$. Thus $k \leq q$, which means that $q^{\mathfrak{c}} \in V(k)^{\mathfrak{c}}$. Hence, $p \in V(c) \subseteq V(k)^{\mathfrak{c}}$. Consequently, $V(k)^{\mathfrak{c}}$ is open in $Min(L)^{-1}$. It thus follows that the contraction map restricted to $Min(M)$ maps open sets of $Min(M)$ into open sets of $Min(L)$ and hence is an open map.

Therefore, the contraction map $p \mapsto p^{\mathfrak{c}}$ is a bijective, continuous, and open map between $Min(M)$ and $Min(L)$ with respect to the hull-kernel topology, and hence is a homeomorphism. □

Now the next obvious question is whether the converse of the preceding theorems hold or not. Although the answer is not obvious, the answer is in the affirmative. Our next two theorems will prove that the converse of both the Theorems 6.15 and 6.16 are true.

Theorem 6.17. *Suppose L and M are two algebraic frames satisfying the FIP with $L \leq M$ a coherent extension. Suppose also that $L \leq M$ is an $r^\flat$-extension. If the contraction map*

$x \mapsto x^{\mathfrak{c}}$ is a homeomorphism between $Min(M)$ and $Min(L)$ with respect to the hull-kernel topology, then $L \leq M$ is an r-extension.

Proof. Suppose $L \leq M$ is a coherent extension of algebraic frames satisfying the FIP. We also assume that the contraction map $x \mapsto x^{\mathfrak{c}}$ is a homeomorphism with respect to the hull-kernel topology on the minimal prime element spaces.

Let $k \in \mathfrak{K}(M)$ and $p \in Min(M)$ with $k \nleq p$. Since $k \nleq p$, it follows that $p \in U(k)$, this in turn says that $p^{\mathfrak{c}} \in U(k)^{\mathfrak{c}}$. Since the contraction map is an open map, $U(k)^{\mathfrak{c}}$ is open in $Min(L)$. Therefore, there exist a basic open set $U(c) \in Min(L)$ for some $c \in \mathfrak{K}(L)$ such that

$$p^{\mathfrak{c}} \in U(c) \subseteq U(k)^{\mathfrak{c}}.$$

Clearly, $p^{\mathfrak{c}} \in U(c) \Rightarrow c \nleq p^{\mathfrak{c}} \Rightarrow c \nleq p$, where the last inequality follows from the definition of $p^{\mathfrak{c}}$. Finally, to prove that $c^{\perp\perp} \leq k^{\perp\perp}$ we show that $k^{\perp} \wedge c = 0$, because, $k^{\perp} \wedge c = 0 \Rightarrow k^{\perp} \leq c^{\perp} \Rightarrow c^{\perp\perp} \leq k^{\perp\perp}$. We notice the following cases.

(a) If $q \in U(k)$, then $k \nleq q$. Since q is a prime element, it follows that $k^{\perp} \leq q$ and so $k^{\perp} \wedge c \leq q$.

(b) On the other hand if $q \in V(k)$, then $k \leq q$. If $c \nleq q$ then $c \nleq q^{\mathfrak{c}}$, by the definition of $q^{\mathfrak{c}}$. So $q^{\mathfrak{c}} \in U(c)$ which implies that $q^{\mathfrak{c}} \in U(k)^{\mathfrak{c}}$. This in turn implies that $q \in U(k)$ (since the contraction map is an injection), which is a contradiction. So, $c \leq q$ and thus $k^{\perp} \wedge c \leq q$.

Thus we proved that $k^{\perp} \wedge c \leq q$, for all $q \in Min(M)$ and so $k^{\perp} \wedge c = 0$ (by Lemma 3.6).

Hence, $L \leq M$ is an r-extension. □

A similar theorem holds true if we replace the hull-kernel topology on $Min(L)$ by the inverse topology and the r-extension by r^*-extension.

Theorem 6.18. *Suppose that $L \leq M$ is a coherent extension of algebraic frames satisfying*

the FIP which is also an $r^\flat$-extension. If the contraction map $x \mapsto x^\mathfrak{c}$ is a homeomorphism between $Min(M)^{-1}$ and $Min(L)^{-1}$, then $L \leq M$ is an r^*-extension.

Proof. Let us assume that L and M are two algebraic frames with $L \leq M$ is an $r^\flat$-extension. Also let the contraction map be a homeomorphism between the minimal prime element spaces with respect to the inverse topology.

We want to show that M is an r^*-extension of L: Let $k \in \mathfrak{K}(M)$ and $p \in Min(M)$ with $k \leq p$. So, $p \in V(k)$, which implies that $p^\mathfrak{c} \in V(k)^\mathfrak{c}$ Since the contraction map is an open map, $V(k)^\mathfrak{c}$ is open in $Min(L)^{-1}$ Therefore, there exist some $c \in \mathfrak{K}(L)$ such that $p^\mathfrak{c} \in V(c) \subseteq V(k)^\mathfrak{c}$ (since $\{V(c) : c \in \mathfrak{K}(L)\}$ forms a base for $Min(L)^{-1}$). We now claim that $c \leq p$ and $c^\perp \leq k^\perp$.

1. $p^\mathfrak{c} \in V(c)$ implies that $c \leq p^\mathfrak{c}$. So, $c \leq p$.

2. Pick $q \in Min(M)^{-1}$.

   If $q \in V(k)$, then $k \leq q$. This implies that $c^\perp \wedge k \leq q$.

   If $q \in U(k)$, then $k \nleq q$. If possible, let $c \leq q$, then $c \leq q^\mathfrak{c}$. So, $q^\mathfrak{c} \in V(c) \subseteq V(k)^\mathfrak{c}$, concluding that $k \leq q$. This is a contradiction to the assumption that $k \nleq q$. Hence it follows that $c \nleq q$. Since q is a prime element, $c^\perp \leq q$ and so, $c^\perp \wedge k \leq q$.

   Thus we proved that for all $q \in Min(M)^{-1}$, $c^\perp \wedge k \leq q$. It thus follows that $c^\perp \wedge k = 0$, by Theorem 3.6. Hence, $c^\perp \leq k^\perp$.

Thus, $L \leq M$ is an r^*-extension. □

We thus have the following diagram about rigidity of frames.

$$\begin{tikzcd}
 & r^*\text{-extension} \arrow[dr] & \\
\text{rigid extension} \arrow[ur] \arrow[dr] & & r^\flat\text{-extension} \\
 & r\text{-extension} \arrow[ur] &
\end{tikzcd}$$

It is natural to ask the question, at this point, about the relationship between frame rigidity and ℓ-group rigidity. As it turns out, the rigid extensions for ℓ-groups are not quite equivalent to the rigid extensions for algebraic frames satisfying the FIP. For the equivalence of these two different concepts, we need to assume that the extensions between ℓ-groups is a major extension. Let $G \leq H$ be two *ell*-groups. We say that the extension of ℓ-groups is a *major* extension, or, H *majorizes* G, if $H(G) = H$, that is, the convex ℓ-subgroup generated by G in H is H.

Theorem 6.19. *Suppose G and H are two ℓ-groups and $G \leq H$ is a major extension of ell-groups.*

1. *G is a rigid subgroup of H if and only if $\mathcal{C}(H)$ is a rigid extension of $\mathcal{C}(G)$.*

2. *G is an r-subgroup of H if and only if $\mathcal{C}(H)$ is an r-extension of $\mathcal{C}(G)$.*

3. *G is an r^*-subgroup of H if and only if $\mathcal{C}(H)$ is an r^*-extension of $\mathcal{C}(G)$.*

Proof. Suppose G and H are two ℓ-groups with $G \leq H$, and let $\mathcal{C}(G)$ and $\mathcal{C}(H)$ be their respective convex ℓ-subgroups. Let us define a map $\phi : C(G) \to C(H)$ by, $\phi(K) = H(K) =$ convex ℓ-subgroup of H generated by K. We recall that for an element $g \in G$, we denote by $G(g)$ the smallest convex ℓ-subgroup in G containing g. Furthermore, $\mathfrak{K}(\mathcal{C}(G)) = \{G(g) : g \in G\}$. We notice the following:

(a) $H(G(g)) = H(g)$, for all $g \in G$.

Since $g \in G(g) \subseteq H(G(g))$ and $H(G(g))$ is convex in H, it follows that $H(g) \subseteq H(G(g))$. On the other hand, $H(g) \cap G = G(g)$ and so we have that $G(g) \subseteq H(g)$. Hence, $H(G(g)) \subseteq H(g)$.

(b) If $K_1, K_2 \in \mathcal{C}(G)$ with $H(K_1) = H(K_2)$, then $K_1 = K_2$.

$$\begin{aligned} H(K_1) = H(K_2) \quad &\Rightarrow \quad K_1 \subseteq H(K_2) \text{ and } K_2 \subseteq H(K_1) \\ &\Rightarrow \quad K_1 \subseteq K_2 \text{ and } K_2 \subseteq K_1, \text{since } K_1 \text{ and } K_2 \text{ are convex in } G \\ &\Rightarrow \quad K_1 = K_2. \end{aligned}$$

(c) $H\left(\bigvee h_\alpha\right) = \bigvee H(h_\alpha)$ and $H(h_1 \wedge h_2) = H(h_1) \wedge H(h_2)$.

We have the following properties of ϕ from above; From (a) it follows that for any $g \in G$, $\phi(G(g)) = H(g)$, that is, ϕ takes the compact elements of $\mathcal{C}(G)$ to compact elements of $\mathcal{C}(H)$. (b) implies that ϕ is injective, and finally, ϕ is a frame homomorphism, which follows from (c). ϕ will be the required rigid map between the frames. Let us now prove the results:

1. ($\Rightarrow$) Suppose G is a rigid subgroup of H. Let $h \in H$ with $H(h) \in \mathcal{C}(H)$. Since G is rigid in H, there exist $g \in G$ such that $g'' = h''$ (in H), that is, $H(g)^{\perp\perp} = H(h)^{\perp\perp}$. So, $\phi(G(g))^{\perp\perp} = H(g)^{\perp\perp} = H(h)^{\perp\perp}$. Hence, $\mathcal{C}(G)$ is rigid in $\mathcal{C}(H)$ as frames.

   ($\Leftarrow$) Suppose $\mathcal{C}(G)$ is rigid in $\mathcal{C}(H)$ as frames, and let $h \in H$. $H(h) \in \mathcal{C}(H)$, so by rigidity, there exist $g \in G$ such that $\phi(G(g))^{\perp\perp} = H(h)^{\perp\perp}$. Thus, $H(g)^{\perp\perp} = H(h)^{\perp\perp}$, or in other words, $g'' = h''$. Hence, G is rigid in H.

2. ($\Rightarrow$) Suppose G is an r-subgroup of H. Let $h \in H$ with $H(h) \in \mathcal{C}(H)$ and $P \in Min(H)$ such that $H(h) \nleq P$ (We recall that by $Min(H)$ we actually mean $Min(\mathcal{C}(H))$). Thus P does not contain h. Since H is an r-extension of G, there exist some $g \in G \setminus P$ such that $g'' \subseteq h''$ (in H). So, $H(g) \nleq P$ and $H(g)^{\perp\perp} \leq H(h)^{\perp\perp}$; which implies that $\phi(G(g)) = H(g) \nleq P$ and $\phi(G(g))^{\perp\perp} \leq H(g)^{\perp\perp}$, satisfying the conditions for $\mathcal{C}(H)$ to be an r-extension of $\mathcal{C}(G)$.

   ($\Leftarrow$) To show the other direction we assume that $\mathcal{C}(H)$ is an r-extension of $\mathcal{C}(G)$. Let $h \in H$ and $P \in Min(H)$ that does not contain h. Therefore, $H(h) \in \mathcal{C}(H)$ and $H(h) \nleq P$. Using the assumption of r-extension, there exist some $G(g) \in \mathcal{C}(G)$ (for

some $g \in G$) with $\phi(G(g)) \nleq P$ such that $\phi(G(g))^{\perp\perp} \leq H(h)^{\perp\perp}$. In other words these implies that $H(g) \nleq P$ and $H(g)^{\perp\perp} \leq H(h)^{\perp\perp}$, which essentially concludes that $g \in G \setminus P$ and $g'' \subseteq h''$. Hence, G is an r-subgroup of H.

3. Following a similar string of arguments as in (2) above, the result holds true for r^*-extension also.

□

The next obvious question is, like in the case of ℓ-groups, whether rigid extension is different than r-extension or r^*-extension for algebraic frames? These questions are answered positively in the articles [4] and [17] for ℓ-groups. There are examples of ℓ-groups $G \leq H$ which demonstrate that the arrows are not reversible in the context of ℓ-groups. Unfortunately, none of these examples is useful in our context since the extensions are not majorizing and therefore $\mathcal{C}(G)$ is not a subframe of $\mathcal{C}(H)$. We provide an example of a majorizing extension of ℓ-groups $G \leq H$ which is an r^*-extension but not a rigid extension. Theorem 6.19 can then be applied to conclude that there exists an extension of algebraic frames which is an r^*-extension but not a rigid extension. Moreover, in this example r-extension implies rigid extension. Therefore it is also an example which proves that r-extension and r^*-extension are two different concepts for algebraic frames satisfying the FIP.

We first give some preliminaries required for the following example. A topological space X is *zero-dimensional* if the topology has a base of clopen sets. We will assume that all of our spaces X are Tychonoff, that is, Hausdorff and completely regular. For a space X, $C(X)$ and $C(X, \mathbb{Z})$ denote respectively the rings of all real-valued and integer-valued continuous functions defined on X. For $p \in X$, O_p is the set of all functions in $C(X)$ which vanishes in a neighborhood of p. It turns out that O_p is an ideal of $C(X)$. For an $f \in C(X)$, the subset $f^{-1}(0) \in X$ is called a *zero* set. Likewise, a *cozero* set of X is a set of the form $\{x \in X : f(x) \neq 0\}$, for some $f \in C(X)$. We denote the zero set of f by $Z(f)$ and the cozero set of f by $coz(f)$. A space X is an *F-space* if every finitely generated ideal in $C(X)$

is principal. The assumption of X being an F-space guarantees that O_p is a prime ideal of $C(X)$. A space X is *basically disconnected* if the closure of every cozero set of X is open.

Lemma 6.20. *Let $f \in C(X)$ and S be a collection of all functions h in $C(X)$ such that coz$(f) \subseteq Z(h)$, then cl(coz$(f)) = \bigcap\{Z(h) : h \in S\}$.*

Proof. Suppose $f \in C(X)$ and $\mathrm{coz}(f) \subseteq Z(h)$ for all $h \in S$. It is clear then that $cl\mathrm{coz}(f) \subseteq \bigcap\{Z(h) : h \in S\}$. To show the other inclusion, let $x \notin cl\mathrm{coz}(f)$. Since X is Tychonoff, there exist some $h' \in C(X)$ such that $h'(x) = 1$ and $h'(cl\mathrm{coz}(f)) = \{0\}$. So, $cl\mathrm{coz}(f) \subseteq Z(h')$ but $x \notin Z(h')$. Since $h' \in S$, it proves that $cl\mathrm{coz}(f) = \bigcap\{Z(h) : h \in S\}$. □

Lemma 6.21. *Suppose $f, g \in C(X)$, then $g^{\perp\perp} \subseteq f^{\perp\perp}$ if and only if coz$(g) \subseteq cl$coz(f).*

Proof. Let $f, g \in C(X)$ and $\mathrm{coz}(g) \subseteq cl\mathrm{coz}(f)$. It suffices to show that $f^{\perp} \subseteq g^{\perp}$. Suppose $h \in f^{\perp}$, then $|h| \wedge |f| = 0$. In order to show that $h \in g^{\perp}$ we need to show that $|h| \wedge |g| = 0$, that is, to show that $\mathrm{coz}(h) \cap \mathrm{coz}(g) = \emptyset$. If $h(x) \neq 0$, then $f(x) = 0$. Suppose that $g(x) \neq 0$, that is, $x \in \mathrm{coz}(g)$. By assumption, $x \in cl\mathrm{coz}(f) \backslash \mathrm{coz}(f)$. Since $x \in \mathrm{coz}(h)$ and $\mathrm{coz}(h)$ is an open neighborhood of x, so $\mathrm{coz}(h) \cap \mathrm{coz}(f) \neq \emptyset$. This implies that $|h| \wedge |f| \neq 0$, which is a contradiction. Hence, $g(x) = 0$ and so $\mathrm{coz}(h) \cap \mathrm{coz}(g) = \emptyset$.

For the other direction we first compute $f^{\perp\perp}$, for some $f \in C(X)$. By definition, $f^{\perp} = \{h \in C(X) : |h| \wedge |f| = 0\} = \{h \in C(X) : \mathrm{coz}(f) \subseteq Z(h)\} = \{h \in C(X) : cl\mathrm{coz}(f) \subseteq Z(h)\}$. Thus, $f^{\perp\perp} = \{h' \in C(X) : \mathrm{coz}(h') \subseteq \cap Z(h), \text{for all} h \in f^{\perp}\} = \{h' \in C(X) : \mathrm{coz}(h') \subseteq cl\mathrm{coz}(f)\}$, using the above Lemma. Hence, $g^{\perp\perp} \subseteq f^{\perp\perp}$ implies that $g \in f^{\perp\perp}$ and so $\mathrm{coz}(g) \subseteq cl\mathrm{coz}(f)$. □

We are now ready to provide an example of ℓ-groups G and H where G majorizes H such that G is an r^*-subgroup of H but not a rigid subgroup. So, the corresponding convex ℓ-subgroups of these ℓ-groups will be the required algebraic frames with the FIP that has the property that one is an r^*-extension of another without being a rigid extension.

Example 6.22. Let X be a compact, zero-dimensional F-space which is not basically disconnected, e.g. $X = \beta\mathbb{N} \setminus \mathbb{N}$. Now consider the ℓ-group $C(X)$ and its ℓ-subgroup $C(X, \mathbb{Z})$.

Clearly $C(X,\mathbb{Z})$ majorizes $C(X)$. Also, $C(X,\mathbb{Z})$ is not a rigid extension since X is not basically disconnected (see [14]). We notice that for a compact F-space X, the minimal primes of $C(X)$ are precisely the ideals O_p for $p \in X$. To show that $C(X,\mathbb{Z})$ is an r^*-subgroup, we consider an $f \in C(X)$ and $p \in X$ with $f \in O_p$. We have to show that there exist some $g \in C(X,\mathbb{Z})$ such that $g \in O_p$ and $f^{\perp\perp} \subseteq g^{\perp\perp}$. In other words, this means that for all $p \in int(Z(f))$, there exist some $g \in C(X,\mathbb{Z})$ with $p \in int(Z(g))$ such that $\mathrm{coz}(f) \subseteq cl(\mathrm{coz}(g))$ (using above lemma). Since $g \in C(X,\mathbb{Z})$, $cl(\mathrm{coz}(g))$ and $int(Z(g))$ are clopen sets in X. Therefore, for every $p \in int(Z(f))$ we have to find a clopen set K of X satisfying the property that $p \in K \subseteq Z(f)$. Since $p \in int(Z(f))$ and X is zero dimensional, there exist some basic clopen set K such that $p \in K \subseteq int(Z(f))$ and hence, $p \in K \subseteq Z(f)$. Thus, $C(X,\mathbb{Z}) \subseteq C(X)$ is an r^*-subgroup which is not a rigid subgroup.

Furthermore, we notice in this example that $C(X,\mathbb{Z}) \subseteq C(X)$ is an r-subgroup if and only if X is basically disconnected and hence the extension is rigid.

Proposition 6.23. *Suppose X is a compact, zero-dimensional F-space. The following statements are equivalent.*

1. *X is basically disconnected.*
2. *$C(X,\mathbb{Z}) \leq C(X)$ is a rigid extension.*
3. *$C(X,\mathbb{Z}) \leq C(X)$ is an r-extension.*

Proof. 1) $\Rightarrow$ 2) Proposition 3.5 in [14].

2) $\Rightarrow$ 3) Clear.

3) $\Rightarrow$ 1) Suppose that X is not basically disconnected. Therefore, there exists a $f \in C(X)$ such that $cl_X(\mathrm{coz}(f))$ is not open. Let $p \in cl_X(\mathrm{coz}(f))$ be arbitrary, then $p \notin int_X(Z(f))$, and therefore $f \notin O_p$. If $C(X,\mathbb{Z}) \leq C(X)$ is an r-extension, then there will exist some $g_p \in C(X,\mathbb{Z})$ with $g_p \notin O_p$ and $g_p^{\perp\perp} \subseteq f^{\perp\perp}$. These reduce to saying that there will exist some $g_p \in C(X,\mathbb{Z})$ with $p \notin int_X(Z(g_p)) = Z(g_p)$ and $\mathrm{coz}(g_p) \subseteq cl_X(\mathrm{coz}(f))$. In other

words these would imply that $p \in \mathrm{coz}(g_p) \subseteq cl_X(\mathrm{coz}(f))$. Since p is chosen arbitrarily, the preceding statement implies that $cl_X(\mathrm{coz}(f))$ is open, which is a contradiction. Hence, $C(X, \mathbb{Z}) \leq C(X)$ is not an r-extension. $\square$

Therefore, Example 6.22 provides us with an example of algebraic frames $L \leq M$ satisfying the FIP in which an r-extension and an r^*-extension are not equivalent.

A different question is whether an r-extension and an r^*-extension together gives a rigid extension? We give a partial answer to this question. Notice that if $L \leq M$ is a coherent extension, then it is both an r-extension and an r^*-extension if and only if $Min(L)$ is homeomorphic to $Min(M)$ with respect to both the hull-kernel topology and the inverse topology. Moreover, if the basic open sets of $Min(M)$ are mapped to basic open sets of $Min(L)$ with respect to both topologies, then $L \leq M$ will be a rigid extension. We denote a basic open set of $Min(M)$ by $U_M(k)$ and a basic open set of $Min(M)^{-1}$ by $V_M(k)$, for some $k \in \mathfrak{K}(M)$, and denote the set $\{p^{\mathfrak{c}} : p \in U_M(k)\}$ by $U_M(k)^{\mathfrak{c}}$.

Lemma 6.24. *Let $L \leq M$ be an $r^\flat$-extension. If $U_M(k)^{\mathfrak{c}} = U_L(c)$, for some $k \in \mathfrak{K}(M)$ and $c \in \mathfrak{K}(L)$, then $V_M(k)^{\mathfrak{c}} = V_L(c)$.*

Proof. Observe that for $k \in \mathfrak{K}(M)$ and $p \in Min(M)$, $p \in U_M(k) \Leftrightarrow p^{\mathfrak{c}} \in U_M(k)^{\mathfrak{c}}$. $\square$

Theorem 6.25. *Let $L \leq M$ be a coherent extension of algebraic frames. Suppose that the contraction map $Min(M) \to Min(L)$ is a homeomorphism with respect to both the hull-kernel topology and the inverse topology. $L \leq M$ is a rigid extension if and only if the contraction map takes basic open sets of $Min(M)$ ($Min(M)^{-1}$) into basic open sets of $Min(L)$ ($Min(L)^{-1}$).*

Proof. ($\Rightarrow$) Let $L \leq M$ be a rigid extension. Let $k \in \mathfrak{K}(M)$ and consider $U_M(k) = \{p \in Min(M) : k \nleq p\}$. By rigidity, there exists $c \in \mathfrak{K}(L)$ with $k^\perp = c^\perp$. Notice that for any $q \in U_L(c)$, $q = p^{\mathfrak{c}}$ for some $p \in U_M(k)$. Hence, $U_M(k)^{\mathfrak{c}} = U_L(c)$. Using Lemma 6.24 it follows that $V_M(k)^{\mathfrak{c}} = V_L(c)$.

($\Leftarrow$) Let $k \in \mathfrak{K}(M)$ and consider $U_M(k) = \{p \in Min(M) : k \nleq p\}$. By the hypothesis, there exists some $c \in \mathfrak{K}(L)$ such that $U(k)^{\mathfrak{c}} = U_L(c)$. We show that $k^{\perp} \wedge c = 0 = k \wedge c^{\perp}$, proving that $k^{\perp} = c^{\perp}$. Suppose $p \in Min(M)$ is arbitrary. If $p \in U_M(k)$, then $k^{\perp}, c^{\perp} \leq p$, which implies that $k^{\perp} \wedge c \leq p$ and $k \wedge c^{\perp} \leq p$. If $p \in V_M(k)$, then $k \leq p$ and so $c \leq p$ thereby proving that $k^{\perp} \wedge c \leq p$ and $k \wedge c^{\perp} \leq p$. Hence, $k^{\perp} \wedge c$ and $k \wedge c^{\perp}$ are below every minimal prime of M and so $k^{\perp} \wedge c = 0 = k \wedge c^{\perp}$. Thus, the extension is rigid. $\square$

We proved that an extension which is both an r-extension and an r^*-extension can be characterized as an extension for which the contraction map between $Min(M)$ and $Min(L)$ is a homeomorphism with respect to both the hull-kernel topology and the inverse topology. So together, an extension which is both an r-extension and an r^*-extension is "close" to being a rigid extension. We are unable to determine whether this extension is in fact rigid.

CHAPTER 7

Basis

Similar to the concept of basis in an ℓ-group, we will define basis for an algebraic frame. For a detailed reference on the basis of ℓ-group, we should look at [5] and [4].

Definition 7.1. Suppose L is an algebraic frame. A nonzero element $b \in \mathfrak{K}(L)$ is called *basic* if $\{x \in L : x \leq b\}$ forms a chain.

L is said to have a *basis* if it has a maximal set B of pairwise disjoint elements, where each $b \in B$ is basic.

We immediately notice the fact that to check basic ness of an element b we only need to show that the set of all compact elements below b forms a chain.

Proposition 7.2. *Suppose L is an algebraic frame An element $b \in \mathfrak{K}(L)$ is basic if and only if $\{c \in \mathfrak{K}(L) : c \leq b\}$ is a chain.*

Proof. Let L be an algebraic frame. Suppose $b \in \mathfrak{K}(L)$ is basic, then by the definition it follows that $\{c \in \mathfrak{K}(L) : c \leq b\}$ forms a chain. On the other hand, let $x, y \in L$ with $x, y \leq b$. Since L is algebraic, $x = \vee c_\alpha$ and $y = \vee d_\beta$, where $c_\alpha, d_\beta \in \mathfrak{K}(L)$. We notice that for all α and β, $c_\alpha, d_\beta \leq b$. Thus by hypothesis, $\{c_\alpha, d_\beta\}$ is a chain. Two cases may arise:

1. Each c_α is below some d_β; so that $x = \vee c_\alpha \leq \vee d_\beta = y$.

2. There exist some c_α which is above all the d_βs; that is, $y = \vee d_\beta \leq c_\alpha \leq x$.

Hence, any two elements below b are comparable, proving that b is basic. $\square$

Proposition 7.3. *Let L be an algebraic frame and $b \in L$ is basic. If $x \leq b$, then either $x = 0$ or $x^{\perp} = b^{\perp}$.*

Proof. Suppose L is an algebraic frame and $b \in L$ is basic. Let $x \leq b$. It follows immediately that $b^{\perp} \leq x^{\perp}$. To show the other inclusion we let $x^{\perp} = \vee\{c \in \mathfrak{K}(L) : c \leq x^{\perp}\}$. Thus, $c \wedge x = 0$ for all $c \leq x^{\perp}$. For each $c \leq x^{\perp}$, $c \wedge b \leq b$. So by the definition of basic element, x and $c \wedge b$ are comparable, for every c. If $x \leq c \wedge b$ for some c, then $x \leq c$ and so $0 = c \wedge x = x$. Thus, $x = 0$. Suppose $x \neq 0$, then $c \wedge b \leq x$ for all c. Now if $c \wedge b \leq x$, then $0 = (c \wedge x) \wedge b = (c \wedge b) \wedge x = c \wedge b$. Hence, $c \leq b^{\perp}$ for all $c \leq x^{\perp}$ and so $x^{\perp} \leq b^{\perp}$. Thus, either $x = 0$ or $x^{\perp} = b^{\perp}$. $\square$

Corollary 7.4. *Suppose L is an algebraic frame. For any polar $x \in L$ and basic element $b \in L$ either $x \wedge b = 0$ or $b \leq x$.*

Proof. Let L be an algebraic frame and b is a basic element in L. Suppose $x \in L$ is a polar, that is, $x^{\perp\perp} = x$. If $x \wedge b \neq 0$, then by Proposition 7.3 $(x \wedge b)^{\perp} = b^{\perp}$. Thus it follows that $b \leq b^{\perp\perp} = (x \wedge b)^{\perp\perp} \leq x^{\perp\perp} = x$. So, $b \leq x$. $\square$

Introduction of basic elements and basis for an algebraic frame will lead us to a result regarding rigidity of frames. We will answer the following question: Which extensions between algebraic frames $L \leq M$ will ensure that L has a basis if and only if M has a basis? To answer the question we will need to have an extra condition on the frame, the condition of disjointification.

Definition 7.5. Suppose L is an algebraic frame. L is said to be a frame with *disjointification* if for each pair of compact elements x and y in L there exist a disjoint pair of compacts c and d in L with $c \leq x$ and $d \leq y$ such that $c \vee y = d \vee x = x \vee y$. These frames are also known as relatively normal.

From now on we will assume our frame satisfies the condition of disjointification, unless otherwise stated.

Theorem 7.6. *Suppose L is an algebraic frame and $b \in \mathfrak{K}(L)$, then the following statements are equivalent:*

1. *b is basic.*

2. *$b^{\perp} \in Min(L)$.*

3. *$b^{\perp} \in Spec(L)$.*

4. *$\downarrow b^{\perp\perp}$ is a chain and $b^{\perp\perp}$ is maximal with respect to this property.*

Proof. Suppose L is an algebraic frame and let $b \in \mathfrak{K}(L)$.

(1) $\Rightarrow$ (2). Let b be a basic element in L and let $x, y \in L$ with $x, y \neq 0$ such that $x \wedge y \leq b^{\perp}$. So, $x \wedge y \wedge b = 0$. Consider $x \wedge b$ and $y \wedge b$. Since b is basic, $x \wedge b$ and $y \wedge b$ are comparable. Without loss of generality we consider $x \wedge b \leq y \wedge b$. Then it follows that $x \wedge b = (x \wedge b) \wedge (y \wedge b) = x \wedge y \wedge b = 0$. Hence $x \leq b^{\perp}$, proving that b is prime. Suppose $p \in Min(L)$ with $p \leq b^{\perp}$. If $b^{\perp} \nleq p$, then $b \leq p$ since $b \in \mathfrak{K}(L)$. This will imply that $b \leq b^{\perp}$, which is a contradiction. Hence, $b^{\perp} = p$ and so $b^{\perp} \in Min(L)$.

(2) $\Rightarrow$ (3). Clear.

(3) $\Rightarrow$ (4). Let $x, y \in \mathfrak{K}(L)$ with $x, y \leq b^{\perp\perp}$. Without loss of generality we assume that $x \neq 0$ and $y \neq 0$. So, $x \nleq b^{\perp}$ and $y \nleq b^{\perp}$, which implies that $x \wedge y \nleq b^{\perp}$ since $b^{\perp}$ is prime. Thus, $x \wedge y \neq 0$. By the condition of disjointification we have $c, d \in \mathfrak{K}(L)$ satisfying $c \leq x$, $d \leq y$, $c \wedge d = 0$ and $c \vee y = d \vee x = x \vee y$. We first notice that $b^{\perp} \leq x^{\perp}$ for, since $x \leq b^{\perp\perp}$, $x \wedge b^{\perp} = 0$ which means $b^{\perp} \leq x^{\perp}$. Similarly, $b^{\perp} \leq y^{\perp}$. Now, $c \wedge d = 0$ implies that $c \leq b^{\perp}$ or $d \leq b^{\perp}$. If $c \leq b^{\perp}$, then $c \leq x \wedge x^{\perp} = 0$ and hence $c = 0$. Thus, $x \vee y = c \vee y = y$ which says that $x \leq y$. On the other hand if $d \leq b^{\perp}$, then following the same argument as above we have $y \leq x$. Hence, $\downarrow b^{\perp\perp}$ is a chain.

To show that $b^{\perp\perp}$ is maximal we choose some $x \in L$ such that $b^{\perp\perp} \leq x$. Since $b \leq x$ it follows that $x^{\perp} \leq b^{\perp}$. Again, x being strictly above $b^{\perp\perp}$, $x \wedge b^{\perp} \neq 0$. So $b^{\perp} \nleq x^{\perp}$. Hence, $x^{\perp}$ is strictly below $b^{\perp}$ which is a minimal prime. So, $x^{\perp}$ is not prime and consequently x is not a basic element.

(4) $\Rightarrow$ (1). Since $\downarrow b^{\perp\perp}$ is a chain and $b \leq b^{\perp\perp}$, it follows that $\downarrow b$ is a chain. Thus, b is basic.

□

Corollary 7.7. *Let L be an algebraic frame and $c, d \in L$. If c and d are basic elements in L, then either $c \wedge d = 0$ or c and d are comparable.*

Proof. Suppose L is an algebraic frame and c and d are two basic elements of L. Using Corollary 7.4 we can conclude that either $c \leq d^{\perp}$ or $c \wedge d^{\perp} = 0$. In the first case if $c \leq d^{\perp}$, then $c \wedge d = 0$. Otherwise $c \wedge d^{\perp} = 0$ which implies $c \leq d^{\perp\perp}$. Since $c, d \leq d^{\perp\perp}$, Theorem 7.6 tells us that c and d are comparable (since $\downarrow b^{\perp\perp}$ is a chain). □

Let us consider frames with basis. The first result will tell us that every compact elements in a frame exceeds a basic element; and consequently every element in the algebraic frame exceeds a supremum of basic elements.

Proposition 7.8. *Let L be an algebraic frame with a basis B. For every nonzero $c \in \mathfrak{K}(L)$ there exist some basic element $b \in L$ such that $b \leq c$.*

Proof. Let B be a basis of the algebraic frame L. So, B is a maximal set of pairwise disjoint elements and each element in B is basic. Let $c \in \mathfrak{K}(L)$ and suppose $c \notin B$. By maximality of B there exist some $b \in B$ such that $c \wedge b \neq 0$. So $c \wedge b$ is compact and $c \wedge b \leq b$, which implies that $c \wedge b$ is basic. Thus, c lies above a basic element $c \wedge b$. □

Let us recall once again about the minimal prime element space of an algebraic frame with respect to the hull-kernel topology. We will show in the following theorem that in order for $Min(L)$ to have a dense discrete subspace it is necessary and sufficient that L has a basis. Recall Theorem 3.16 from Chapter (3). For our convenience we restate the theorem.

Proposition 7.9. *Let $p \in Min(L)$. The following statements are equivalent.*

i) p is isolated in $Min(L)$ with respect to hull-kernel topoology.

ii) For some $c \in \mathfrak{K}(L)$, $p = c^{\perp}$.

iii) p is a polar.

Theorem 7.10. *Let L be an algebraic frame with the FIP. L has a basis if and only if $Min(L)$ has a dense discrete subspace with respect to the hull-kernel topology.*

Proof. Suppose L is an algebraic frame with the FIP.

($\Rightarrow$) Suppose L has a basis B. Let us consider the set $S = \{c^{\perp} : c \in B\}$. Using Theorem 7.6 it follows that $S \subseteq Min(L)$. Next, applying Proposition 7.9 we conclude that each $c^{\perp}$ is isolated in the hull-kernel topology and consequently S is a discrete subspace of $Min(L)$. We claim that S is dense in $Min(L)$. To prove our claim we choose $p \in Min(L) \setminus S$ and $d \in \mathfrak{K}(L)$ with $p \in U(d)$. So, $d \nleq p$. Consequently, $d^{\perp} \leq p$. Thus $d^{\perp}$ is not prime and so it follows that $d \notin B$. By maximality of B we have some $c \in B$ such that $c \wedge d \neq 0$. Hence $d \nleq c^{\perp}$, which implies that $c^{\perp} \in U(d)$. Thus $U(d) \bigcap S \neq \emptyset$, concluding that S is dense in $Min(L)$.

($\Leftarrow$) Conversely suppose that $Min(L)$ has a dense discrete subspace D, with respect to the hull-kernel topology. For each $p \in D$, p is isolated and so by Proposition 7.9 it follows that $p = c^{\perp}$, for some $c \in \mathfrak{K}(L)$. Let us define $B = \{c \in \mathfrak{K}(L) : c^{\perp} \in D\}$. We will show that B is a basis in L. We first notice that each $c \in B$ is basic, using Theorem 7.6. Also for any two distinct elements c and d in B, $c \wedge d = 0$, using Corollary 7.7. Finally we need to show that B is a maximal set of pairwise disjoint elements. Let $k \in L$ such that $k \notin B$. If $k \wedge c = 0$ for all $c \in B$, then $k \leq c^{\perp}$ for all $c^{\perp} \in D$. Consequently, $D \subseteq V(k)$. Since D is dense in $Min(L)$ it follows that $Min(L) = cl(D) \subseteq cl(V(k)) = V(k)$. So, $k = 0$. Therefore, for every nonzero element k in L there exist some $c \in B$ such that $k \wedge c \neq 0$, or in other words B is a maximal set of pairwise disjoint elements. Hence, L has a basis B. $\square$

It will now follow immediately from the preceding theorem that r-extension will ensure that an algebraic frame has a basis if and only if its extension has a basis.

Corollary 7.11. *Suppose L and M are two algebraic frames and M is an r-extension of L. L has a basis if and only if M has a basis.*

Proof. Suppose L and M are two algebraic frames. Since M is an r-extension of L it follows that $Min(L)$ is homeomorphic to $Min(M)$ with respect to the hull-kernel topology. Thus,

$$\begin{aligned} L \text{ has a basis} &\Leftrightarrow Min(L) \text{ has a dense discrete subspace} \\ &\Leftrightarrow Min(M) \text{ has a dense discrete subspace} \\ &\Leftrightarrow M \text{ has a basis.} \end{aligned}$$

□

CHAPTER 8

Open Questions

We finish the manuscript with several open questions which will lead to further studies.

1. What conditions on L will make $Min(L)^{-1}$ extremally disconnected or even discrete?

2. When is $Min(L)$ basically disconnected?

3. When is $Min(L)^{-1}$ basically disconnected?

4. If $L \leq M$ is a coherent extension of algebraic frames, is there a natural extension between $Rad(L)$ and $Rad(M)$? In this case, will the rigid extensions between the original frames $L \leq M$ be inherited by the extension between $Rad(L)$ and $Rad(M)$?

5. What are the properties of $Rad(L)$ if L is an algebraic non-distributive lattice? If L is a complete algebraic lattice which is not a frame, is $Rad(L)$ still a frame?

6. Let $L \leq M$. Is rigid extension and r-extension equivalent for a coherent extension of algebraic frames? If not, provide a counter-example.

7. Does r- and r^*-extension together imply rigid extension, for a coherent extension of algebraic frames?

8. Let us define a weaker version of rigid extension for algebraic frames $L \leq M$, where we do not necessarily assume that $1_L = 1_M$, and the definition is the same as the

original rigid extension. Is the weak rigid extension equivalent to rigid extension? If not, provided a counter-example. Is the weak rigid extension for algebraic frames equivalent to the rigid extension for ℓ-groups without the assumption of majorizing condition?

9. Can the other types of extensions for ℓ-groups, such as a-extension, *dense* extension, or *essential* extension be generalized for algebraic frames? For the definitions we refer to [4] and [14].

10. Considering the category of algebraic frames (which does not necessarily satisfy the finite intersection property), can the results shown here be generalized?

Finally, we make a conjecture:

Conjecture 8.1. Let $L \leq M$ be an extension of finite frames L and M. $L \leq M$ is a rigid extension if and only if $L \leq M$ is an $r^\flat$-extension.

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