NOTES ON A SIMPLE GRAVITY PENDULUM

A simple pendulum is an idealization involving these two assumptions:

- The rod/string/cable on which the bob is swinging is massless and always remains taut (rigid);
- Motion occurs in a 2-dimensional plane, i.e. the bob does not trace an ellipse.

Figure 1. Trigonometry of a simple gravity pendulum.

\[ y_0 \]
\[ y_1 \]
\[ \theta \]
\[ \theta_0 \]
\[ h \]

\[ ^1 \text{Adapted from Wikipedia.org} \]
Consider Figure 2. The vertically downward arrow is the gravitational force acting on the bob, the other 2 arrows are that same force resolved into components parallel and perpendicular to the bob's instantaneous motion, which is always perpendicular to the cable/rod. Newton's second law

\[ F = ma \]

where \( F \) is the force acting on mass \( m \), causing it to accelerate at \( a \) meters per second\(^2\). Because the bob is constrained to move on the circular arc, there is no need to consider any force other than the one responsible for instantaneous acceleration parallel to the circular path. Thus

\[ F = -mg \sin \theta = ma \]
\[ a = -g \sin \theta \]

where
$g$ is the acceleration due to gravity near the surface of the earth. It is negative because it is pointing downward.

This linear acceleration $a$ can be related to the change in angle $\theta$ by the arc length formulas:

$$s = \ell \theta$$
$$v = \frac{ds}{dt} = \ell \frac{d\theta}{dt}$$
$$a = \frac{d^2s}{dt^2} = \ell \frac{d^2\theta}{dt^2}$$

Thus:

$$\ell \frac{d^2\theta}{dt^2} = -g \sin \theta$$

$$\frac{d^2\theta}{dt^2} + \frac{g}{\ell} \sin \theta = 0 \quad (1)$$

This is the differential equation which, when solved for $\theta(t)$, will yield the motion of the pendulum. It can also be obtained via the conservation of mechanical energy principle: any given object which fell a vertical distance $h$ would have acquired kinetic energy equal to that which it lost to the fall. In other words, gravitational potential energy is converted into kinetic energy. Change in potential energy is given by

$$\Delta U = mgh$$

change in kinetic energy (body started from rest) is given by

$$\Delta K = \frac{1}{2}mv^2$$

Since no energy is lost, those two must be equal

$$\frac{1}{2}mv^2 = mgh$$

$$v = \sqrt{2gh}$$

Using the arc length formula above, this equation can be re-written as
\[
\frac{d\theta}{dt} = \frac{1}{l} \sqrt{2gh}
\]

What is \( h \)? It is the vertical distance the pendulum fell. If the pendulum starts its swing from some initial angle \( \theta_0 \), then \( y_0 \), the vertical distance from the screw, is given by

\[
y_0 = l \cos \theta_0
\]

(See Figure 1). Similarly, for \( y_1 \), we have

\[
y_1 = l \cos \theta
\]

then \( h \) is the difference of the two

\[
h = l (\cos \theta - \cos \theta_0)
\]

substituting this into the equation for \( \frac{d\theta}{dt} \) gives

\[
\frac{d\theta}{dt} = \sqrt{\frac{2g}{l} (\cos \theta - \cos \theta_0)} \quad (2)
\]

This equation is known as the first integral of motion, it gives the velocity in terms of the location and includes an integration constant related to the initial displacement (\( \theta_0 \)). We can differentiate, by applying the chain rule, with respect to time to get the acceleration

\[
\frac{d}{dt} \frac{d\theta}{dt} = \frac{d}{dt} \sqrt{\frac{2g}{l} (\cos \theta - \cos \theta_0)}
\]

Thus, we obtain:

\[
\frac{d^2\theta}{dt^2} = -\frac{g}{l} \sin \theta \quad (3)
\]

which is the same as obtained through force analysis.

EXERCISE: Show that equation (3) follows from eqn. (2).
The differential equation (equation 1) which represents the approximate motion of the pendulum is

\[ \frac{d^2 \theta}{dt^2} + \frac{g}{\ell} \sin \theta = 0. \]

As stated above the first integral of motion is

\[ \frac{d\theta}{dt} = \sqrt{\frac{2g}{\ell}} (\cos \theta - \cos \theta_0). \]  \hspace{1cm} (2)

It gives the velocity in terms of the location and includes an integration constant related to the initial displacement ($\theta_0$). One can compute the exact period by inverting equation (2)

\[ \frac{dt}{d\theta} = \frac{1}{\sqrt{2}} \frac{\ell}{g} \frac{1}{\sqrt{\cos \theta - \cos \theta_0}} \]

and integrating over one complete cycle,

\[ T = \theta_0 \rightarrow 0 \rightarrow -\theta_0 \rightarrow 0 \rightarrow \theta_0, \]

or twice the half-cycle

\[ T = 2 (\theta_0 \rightarrow 0 \rightarrow -\theta_0), \]

or 4 times the quarter-cycle

\[ T = 4 (\theta_0 \rightarrow 0), \]

which leads to

\[ T = 4 \frac{1}{\sqrt{2}} \sqrt{\frac{\ell}{g}} \int_{\theta_0}^{0} \frac{1}{\sqrt{\cos \theta - \cos \theta_0}} d\theta. \]

This integral cannot be evaluated in terms of elementary functions. It can be re-written in the form of the elliptic function of the first kind (see Jacobi's elliptic functions in Wikepedia),
where $F(k, \phi)$ is Legendre's elliptic function of the first kind

$$F(k, \phi) = \int_0^\phi \frac{1}{\sqrt{1 - k^2 \sin^2 \theta}} d\theta.$$ 

Elliptic functions are generalizations of sine and cosine functions, and they occur naturally in solutions of many equations with spherical symmetry.

NOTE: For a swing of $180^\circ$ the bob is balanced over its pivot point and so $T = \infty$ (keep in mind the pendulum is made of a rigid rod). Thus $F(k, 180^\circ)$ is infinity.

**Small-angle approximation**

The problem with the equations developed in the previous section is that they are very complicated. To shed some simpler light on the behavior of the pendulum we shall make an approximation. Namely, we restrict the motion of the pendulum to a relatively small amplitude, that is, relatively small $\theta$. How small? Small enough that the following approximation is true within some desirable tolerance

$$\sin \theta \approx \theta$$

if and only if

$$|\theta| \ll 1.$$ 

Substituting this approximation into (1) yields

$$\frac{d^2 \theta}{dt^2} + \frac{g}{\ell} \theta = 0.$$
Under the initial conditions $\theta(0) = \theta_0$ and $\frac{d\theta}{dt}(0) = 0$, the solution to this equation is a well-known harmonic function

$$\theta(t) = \theta_0 \cos \left( \sqrt{\frac{g}{\ell}} t \right) \quad |\theta_0| \ll 1.$$ 

where $\theta_0$ is the semi-amplitude of the oscillation, that is the maximum angle between the rod of the pendulum and the vertical.

Since

$$\omega = \sqrt{\frac{g}{\ell}} = \frac{2\pi}{T_0},$$

the period of a complete oscillation can be easily found, and we have obtained Huygens’s law:

$$T_0 = 2\pi \sqrt{\frac{\ell}{g}}$$

$$T_0 = 2\pi \sqrt{\frac{\ell}{g}} \quad |\theta_0| \ll 1.$$