Basics of Fourier Analysis

Fourier’s Theorem:

Let \( y(t) \) be any function that is periodic in time with period \( T \), i.e. \( y(t + T) = y(t) \), then:

\[
y(t) = \frac{1}{2} B_0 + \sum_{n=1}^{\infty} \left( A_n \sin(2\pi f_n t) + B_n \sin(2\pi f_n t) \right)
\]

where

\( f_1 = \frac{1}{T}, \quad f_2 = 2f_1, \quad f_3 = 3f_1, \ldots, \quad f_n = nf_1, \ldots, \)

\( (n = 1, 2, 3 \ldots) \) with

\[
A_n = \frac{2}{T} \int_{-T/2}^{T/2} y(t) \sin(2\pi f_n t) dt ,
\]

\( (n = 1, 2, 3 \ldots) \) and

\[
B_n = \frac{2}{T} \int_{-T/2}^{T/2} y(t) \cos(2\pi f_n t) dt .
\]

\( (n = 0, 1, 2, 3 \ldots) \)

For the requirements on the function \( y(t) \) in order to make the above theorem mathematically rigorous, see: Robert S. Strichartz, A Guide to Distribution Theory and Fourier Transforms (Academic Press, New York, 2006).
Some Examples:

1. The Fourier series for a certain periodic function is

\[ f(t) = \frac{4}{\pi} \left( \frac{\sin \pi t}{1^2} - \frac{\sin 3\pi t}{3^2} + \frac{\sin 5\pi t}{5^2} - \frac{\sin 7\pi t}{7^2} + \ldots \right). \]

Describe the function.

2. Find the fourier coefficients of the sine function i.e. find \( A_n \) and \( B_n \) for \( y(t) = \sin(t) \).

3. Find the fourier coefficients of the square wave, and verify that (1) with these values of \( A_n \) and \( B_n \) gives the square wave.