Subspaces of $\mathbb{R}^n$

Section 2.8

Math 220
Topics to Cover

1. Definition of a Subspace
2. Examples of Subspaces
3. Column Space of a Matrix
4. Null Space of a Matrix
5. Basis for a Subspace
1. Definition of a Subspace

A subset $S$ of $\mathbb{R}^n$ is a subspace if $S$ satisfies the following three conditions:

(a) $S$ contains $0$, the zero vector.
(b) If $\mathbf{u}$ and $\mathbf{v}$ are in $S$, then $\mathbf{u} + \mathbf{v}$ is also in $S$.
(c) If $r$ is a real number and $\mathbf{u}$ is in $S$, then $r \mathbf{u}$ is also in $S$.

A subset of $\mathbb{R}^n$ that satisfies condition (b) is said to be closed under addition.

A subset of $\mathbb{R}^n$ that satisfies condition (c) is said to be closed under scalar multiplication.

Closed under scalar addition and multiplication ensures that arithmetic performed on vectors in a subspace produce other vectors in the subspace.
2. Examples of Subspaces

Example 1 - Let $\ell_1$ denote a line through the origin in $R^2$. Do the points on $\ell_1$ form a subspace?
Example 2 - Let $\ell_1$ denote a line not through the origin in $\mathbb{R}^2$. Do the points on $\ell_1$ form a subspace?
Example 3 - Let $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$ be vectors in $\mathbb{R}^n$. Is the Span$\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$ a subspace of $\mathbb{R}^n$?
Example 4 - Is the following set a subspace of $\mathbb{R}^3$?

\[
\left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \mid x_1 + x_2 + x_3 = 0 \right\}
\]
2. Examples of Subspaces

Example 5 - Is the following set a subspace of $\mathbb{R}^3$?

\[
\left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \mid x_1 x_2 - x_3^2 = 0 \right\}
\]
Example 6 - Is the following set a subspace of $\mathbb{R}^2$?
2. Examples of Subspaces

Example 7

Let \( \mathbf{v}_1 = \begin{bmatrix} 1 \\ -3 \\ 2 \\ 3 \end{bmatrix} \), \( \mathbf{v}_2 = \begin{bmatrix} 4 \\ -4 \\ 5 \\ 7 \end{bmatrix} \), \( \mathbf{v}_3 = \begin{bmatrix} 5 \\ -3 \\ 6 \\ 5 \end{bmatrix} \), and \( \mathbf{u} = \begin{bmatrix} -1 \\ -7 \\ -1 \\ 2 \end{bmatrix} \).

Determine if \( \mathbf{u} \) is in the subspace of \( \mathbb{R}^4 \) generated by \( \{ \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \} \).
3. Column Space of a Matrix

- The column space of a matrix $A$ is the set of all linear combinations of the columns of matrix $A$.

- The column space can also be thought of as the subspace of $\mathbb{R}^n$ spanned by the column vectors of $A$.

- The column space is denoted by $\text{Col } A$. 

Section 2.8: Subspaces of $\mathbb{R}^n$
3. Column Space of a Matrix Continued

Example

Let \( \mathbf{v}_1 = \begin{bmatrix} -2 \\ 0 \\ 6 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -2 \\ 3 \\ 3 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 0 \\ -5 \\ 5 \end{bmatrix}, \) and \( \mathbf{u} = \begin{bmatrix} -6 \\ 1 \\ 17 \end{bmatrix}. \)

Determine if \( \mathbf{u} \) is in \( \text{Col} \ A \), where \( A = [\mathbf{v}_1 \ \mathbf{v}_2 \ \mathbf{v}_3] \).
4. Null Space of a Matrix

- The null space of a matrix $A$ is the set of all solutions of the homogeneous equation $Ax = 0$.

- The null space is denoted by $\text{Nul } A$.
Example

Let \( \mathbf{v}_1 = \begin{bmatrix} -2 \\ 0 \\ 6 \end{bmatrix} \), \( \mathbf{v}_2 = \begin{bmatrix} -2 \\ 3 \\ 3 \end{bmatrix} \), \( \mathbf{v}_3 = \begin{bmatrix} 0 \\ -5 \\ 5 \end{bmatrix} \), and \( \mathbf{u} = \begin{bmatrix} -5 \\ 5 \\ 3 \end{bmatrix} \).

Determine if \( \mathbf{u} \) is in \( \text{Nul} \ A \), where \( A = [\mathbf{v}_1 \ \mathbf{v}_2 \ \mathbf{v}_3] \).
A basis for a subspace $S$ of $\mathbb{R}^n$ is a linearly independent set of vectors in $S$ that spans $S$.

A basis for a subspace is useful because a subspace usually contains an infinite number of vectors and a smaller set of vectors that span the subspace is easier to work with.

A basis for the column space of matrix $A$ can be constructed from the pivot columns of matrix $A$.

A basis for the null space of matrix $A$ can be constructed from the vectors that make up the solution of $Ax = 0$. 
Example 1 - Given the following matrix $A$ and an echelon form of $A$, find a basis for $\text{Col} \ A$.

$$A = \begin{bmatrix} 3 & -6 & 9 & 0 \\ 2 & -4 & 7 & 2 \\ 3 & -6 & 6 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 5 & 4 \\ 0 & 0 & 3 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$
Example 2 - Given the following matrix $A$ and an echelon form of $A$, find a basis for $\text{Nul } A$.

$$A = \begin{bmatrix} 3 & -6 & 9 & 0 \\ 2 & -4 & 7 & 2 \\ 3 & -6 & 6 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 5 & 4 \\ 0 & 0 & 3 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$