LAGRANGIAN DYNAMICS AND THE PARCEL METHOD IN ATMOSPHERIC MODELS

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ABSTRACT

The Lagrangian "parcel method" of stability analysis is systematically presented and rendered rigorous in its application to nondivergent flow on a plane with constant Coriolis parameter. The method is applied in detail to two familiar models. (1) Zonal geostrophic current with linear horizontal shear. Here the parcel method is valid only for uniform initial disturbance ("infinite wave-length"), in which case the usual stability criterion applies, \(-\tau_0 < f\). Another particular solution has been given by Lord Kelvin. (2) Anticyclonic vortex with concentric circular contours. Here, the parcel-method stability criterion is \(-\tau_0 < f/2\) (as suggested by the balance equation) and is valid for any initial disturbance with balanced velocity field. So far, no non-symmetric solution to the balance equation for this pressure field has been discovered.

1. Introduction

Any meteorologist familiar with the theoretical literature cannot fail to be impressed by the discrepancy between the inadequacies of the "parcel method" analysis of hydrodynamic stability and the universal acceptance of the results obtained thereby. This method, which consists in the introduction of accelerations into an equilibrium while constraining the force field to remain constant, seems to have only the slightest theoretical justification. Yet, for such models of equilibria as hydrostatic balance, the circular vortex or the parallel zonal current, stability criteria derived by the parcel method have long been part of the conceptual equipment of all practicing meteorologists.

A definite advance on this problem was made in 1950 R. Fjørtoft in his dissertation [1]. Fjørtoft introduced a variational principle according to which the condition for stability of an equilibrium is that the total "energy store" available to increase the kinetic energy of the system be a minimum, subject to the constraint that the accelerations satisfy the equations of motion. Principles of this type have, of course, a firm foundation and a long history of application in classical dynamics, but in the more complex systems of continuous media their use has been much less systematic. Nevertheless, Fjørtoft was able to substantiate stability criteria derived previously by the conventional Eulerian wave analysis. This remarkable achievement, elegantly reformulated by Eliassen and Kleinschmidt [2], still leaves the theoretical puzzle of an unfounded technique yielding correct results. These authors [2, p. 70] give a sentence or two of interpretation of this anomalous situation but cannot be said to go to the heart of the matter.

I propose to discuss this problem, first by some general remarks on Lagrangian dynamics, and secondly by application of these remarks to some exact solutions, Lagrangian and Eulerian, of models sufficiently simple that interpretation of the results is not open to doubt. From these results, I conclude that in any given stability analysis, if not in general, it is possible to assess the validity of the parcel method and that, far from being a superceded technique, this method has wide applicability and opens up new lines of theoretical research.

2. Lagrangian dynamics

Meteorologists are not well served by their literature on this topic. The most complete reference is Lichtenstein [3], texts of dynamic meteorology presenting only the basic equations. As Lagrangian coordinates, we shall use the Cartesian \(x\) and \(y\), with initial values (at time \(t = 0\)) \(a\) and \(b\). Thus,

\[
\begin{align*}
x &= x(a, b, t), & y &= y(a, b, t) \\
a &= x(a, b, 0), & b &= y(a, b, 0).
\end{align*}
\]

The motion is two-dimensional non-divergent, and the system rotates with Coriolis parameter \(f\) about the local vertical. For purposes of the present paper, the Coriolis parameter is assumed constant. This means, of course, that attention is focused upon motions covering an area of the earth's surface not so large that the sine of the latitude varies significantly from its northernmost to its southernmost boundaries. The linear part of the variation of this quantity (that is, the
beta effect) renders even the simplest geostrophic wind profiles non-linear, and, although the problem may be treated successfully by the techniques of reference [4], the emphasis of the present paper is upon the methodology of stability analysis rather than upon meteorological generality. With the Coriolis parameter constant, a notational convenience is gained in the equations without loss of generality by expressing time non-dimensionally in units of \( f^{-1} \). The equations of motion are then

\[
\begin{align*}
\frac{\partial x}{\partial t} + \frac{\partial y}{\partial a} - \frac{\partial y}{\partial a} - \frac{\partial x}{\partial a} = -\frac{\partial \phi}{\partial a} \\
\frac{\partial x}{\partial b} + \frac{\partial y}{\partial b} - \frac{\partial y}{\partial b} - \frac{\partial x}{\partial b} = -\frac{\partial \phi}{\partial b}.
\end{align*}
\]

(1)

Here, \( \phi(a, b, t) \) is the geopotential or the barotropic pressure function, depending on whether the motion is in a pressure surface or a level surface, respectively. The notation

\[ x = \left( \frac{\partial x}{\partial t} \right)_{a, b \text{ constant}} \]

is used for the substantial derivative as the least ambiguous and most suggestive.

If the flow is to be non-divergent, the Jacobian of the transformation \( (a, b) \rightarrow (x, y) \) which expresses the ratio of the area occupied by an infinitesimal fluid mass at time \( t \) to its initial area must be independent of the time. Thus, the continuity equation in Lagrangian coordinates is

\[
\frac{\partial}{\partial t} J \left( \frac{x, y}{a, b} \right) = J \left( \frac{\dot{x}, \dot{y}}{a, b} \right) + J \left( \frac{x, y}{a, b} \right) = 0 \]

(2)

where

\[ J \left( \frac{x, y}{a, b} \right) = \frac{\partial (x, y)}{\partial (a, b)} = \frac{\partial x}{\partial a} \frac{\partial y}{\partial b} - \frac{\partial y}{\partial a} \frac{\partial x}{\partial b}. \]

In terms of these quantities, the two-dimensional divergence \( \delta \) is

\[ \delta = -\frac{\dot{\rho}}{\rho} = -\left[ J \left( \frac{x, y}{a, b} \right) \right] \frac{\partial}{\partial t} J \left( \frac{x, y}{a, b} \right) = 0. \]

(3)

Since the equation of continuity is concerned primarily with changes on a fluid parcel, it is in some sense more naturally expressed in Lagrangian variables, but it will be noted that whatever has been gained is at the expense of the equation's becoming non-linear (more precisely, quasi-non-linear). Therefore, the solution of eq (1) and (2), both now non-linear, is at least as difficult as the solution of the corresponding Eulerian equations for two-dimensional barotropic flow. The conventional wave-perturbation technique attempts to gain certain information about the flow by linearization of the Eulerian acceleration terms. If the imposed perturbation is expressed as a sine wave or a member of some other set of orthogonal functions appropriate to the geometry of the system, the development of an arbitrary perturbation can be obtained by superposition.

The parcel method proceeds quite differently. The force field in (1) is first expressed in mixed Eulerian-Lagrangian terms: if

\[ \phi(a, b, t) = \Phi(x, y, t), \]

it follows that

\[
\begin{align*}
\frac{\partial \phi}{\partial a} &= \frac{\partial \Phi}{\partial x} \frac{\partial x}{\partial a} + \frac{\partial \Phi}{\partial y} \frac{\partial y}{\partial a}, \\
\frac{\partial \phi}{\partial b} &= \frac{\partial \Phi}{\partial x} \frac{\partial x}{\partial b} + \frac{\partial \Phi}{\partial y} \frac{\partial y}{\partial b}.
\end{align*}
\]

(4)

It is impossible to evaluate these expressions without having first solved the problem since \( x \) and \( y \) must be known as functions of \( a \) and \( b \) at all times. However, the parcel method suggests that a stationary field \( \Phi \) be sought such that

\[ \Phi(x, y, t) = \Phi(x, y, 0). \]

If such a field exists, we may represent its space derivatives by the geostrophic wind field:

\[ \frac{\partial \Phi}{\partial x} = v_x(x, y), \quad \frac{\partial \Phi}{\partial y} = -u_y(x, y). \]

The functions \( u_x \) and \( v_y \) are therefore arbitrary since they are known initially. If these expressions are substituted in (3), which in turn are substituted into (1), the resulting equations are

\[
\begin{align*}
\frac{\partial x}{\partial a} \left[ \ddot{x} - \dot{y} + v_x(x, y) \right] + \frac{\partial y}{\partial a} \left[ \ddot{y} + \dot{x} - u_y(x, y) \right] &= 0, \\
\frac{\partial x}{\partial b} \left[ \ddot{x} - \dot{y} + v_x(x, y) \right] + \frac{\partial y}{\partial b} \left[ \ddot{y} + \dot{x} - u_y(x, y) \right] &= 0.
\end{align*}
\]

(5)

Either the bracketed terms must vanish independently, or the determinant of the coefficients, \( J(x, y/a, b) \), is identically zero. The only case of interest is the former, whence

\[
\begin{align*}
\ddot{x} - \dot{y} &= -v_x(x, y), \\
\ddot{y} + \dot{x} &= u_y(x, y).
\end{align*}
\]

These are the conventional parcel-method equations. From the preceding analysis, it is evident that they determine a solution of the Lagrangian eq (1) under the given assumption. That is, the force field represented by \( u_x, v_y \) will in fact remain constant in time with the parcels moving in accordance with (5). To determine whether this motion is non-divergent eq (2) must be applied. This requires some discussion.
Eq (5) are of the second-order. Therefore, initial velocities

\[ u_0(a, b) = x(a, b, 0), \quad v_0(a, b) = y(a, b, 0) \]

must be assigned to every point in the field. To emphasize the functional dependence of the solutions \( x \) and \( y \) on these quantities, the following explicit notation is adopted:

\[ x(a, b, t) = X[a, b, u_0(a, b), v_0(a, b), t] \]
\[ y(a, b, t) = Y[a, b, u_0(a, b), v_0(a, b), t] \]

Identities follow of the form

\[ \frac{\partial x}{\partial a} = \frac{\partial X}{\partial a} + \frac{\partial X}{\partial u_0} \frac{\partial u_0}{\partial a} + \frac{\partial X}{\partial v_0} \frac{\partial v_0}{\partial a}, \quad \text{etc.} \]

Now say that the solutions \( x \) and \( y \) have been determined from equations (5). If these solutions are substituted into the Jacobian \( J(x, y/a, b) \), the result is

\[ J\left(\frac{x}{a}, b\right) = \left[ \frac{\partial X}{\partial a} + \frac{\partial X}{\partial u_0} \frac{\partial u_0}{\partial a} + \frac{\partial X}{\partial v_0} \frac{\partial v_0}{\partial a} \right] \]
\[ \cdot \left[ \frac{\partial Y}{\partial b} + \frac{\partial Y}{\partial u_0} \frac{\partial u_0}{\partial b} + \frac{\partial Y}{\partial v_0} \frac{\partial v_0}{\partial b} \right] \]
\[ - \left[ \frac{\partial X}{\partial b} + \frac{\partial X}{\partial u_0} \frac{\partial u_0}{\partial b} + \frac{\partial X}{\partial v_0} \frac{\partial v_0}{\partial b} \right] \]
\[ \cdot \left[ \frac{\partial Y}{\partial a} + \frac{\partial Y}{\partial u_0} \frac{\partial u_0}{\partial a} + \frac{\partial Y}{\partial v_0} \frac{\partial v_0}{\partial a} \right] \]

\[ = J\left(\frac{X, Y}{a, b}\right) + \frac{\partial u_0}{\partial a} J\left(\frac{X, Y}{u_0, b}\right) \]
\[ + \frac{\partial v_0}{\partial b} J\left(\frac{X, Y}{a, v_0}\right) + \frac{\partial v_0}{\partial a} J\left(\frac{X, Y}{v_0, b}\right) \]
\[ + \frac{\partial u_0}{\partial b} J\left(\frac{X, Y}{a, u_0}\right) + J\left(\frac{X, Y}{u_0, v_0}\right) J\left(\frac{u_0, v_0}{a, b}\right) . \]

The six Jacobians involving \( X \) and \( Y \) will in general be functions of time. If the flow \( x(a, b, t), y(a, b, t) \) is to be non-divergent, the Jacobian on the left-hand side must be independent of time, in accordance with eq (2). This condition can be satisfied only by properly specifying the initial velocity field \( u_0(a, b), v_0(a, b) \).

Thus, the continuity equation restricts the solutions of (5) by imposing conditions on the initial flow. Only solutions of (5) which also satisfy (2) represent valid two-dimensional non-divergent flows. If the initial velocity field is restricted to the point of triviality, the conclusion is that no motion worthy of interest is one that will maintain a stationary pressure pattern. On the other hand, it may be that the conditions imposed by (2) make possible a wide variety of initial flows. In this case, the solution is not trivial and none of the usual criticisms applied to the parcel method are operative; instead, they reduce to the statement that the initial guess as to the form of the stationary pressure field \( (u_0, v_0) \) was a lucky one. No linearization is employed, nor is a wave-form assumed for the perturbation; the solutions are exact.

3. Stability of a zonal current

The foregoing remarks will gain concreteness by being applied to some familiar models, and we may begin with that of a zonal geostrophic wind field with linear profile:

\[ u_0 = Sy, \quad S = \text{constant.} \]
\[ v_0 = 0 \]

Under the assumptions already formulated, parcel motion in this field is represented by the equations

\[ \ddot{x} - \dot{y} = 0 \]
\[ \ddot{y} + \dot{x} = Sy. \]

The solution is easily determined as

\[ x = a - \mu(b - u_0)t + \lambda^{-1}(u_0\lambda^{-2} + \mu b) \sin \lambda t \]
\[ + v_0\lambda^{-2}(1 - \cos \lambda t) \]
\[ y = \lambda^{-2}(b - u_0) + (\lambda^{-2}u_0 + \mu b) \cos \lambda t \]
\[ + \lambda^{-1}v_0 \sin \lambda t \]

where \( \lambda^2 = 1 - S \) and \( \mu = 1 - \lambda^2 \). It is customarily observed that the condition for stability is that \( \lambda^2 \) be positive — that is, in dimensional terms, that \( 1 - f^{-1}S \) be positive or

\[ f - S > 0 \]

and analogy is made to the comparable condition for the circular vortex.

However, at this point the problem solved by (7) is one of classical particle mechanics rather than of hydrodynamics. The motions of the different parcels are quite unrelated because the initial velocities have not been prescribed for the entire field. If it is now assumed that

\[ u_0 = u_0(a, b) \]
\[ v_0 = v_0(a, b), \]

the continuity eq (2) may be applied to the solution (7). The first step is the computation of the Jacobian:
\[ J(x, y) = \frac{1}{\lambda^2} \left[ 1 - \frac{\partial u_0}{\partial b} - \left( 1 - \frac{2}{\lambda^2} \right) \frac{\partial v_0}{\partial a} + \frac{2}{\lambda^2} J \left( \frac{u_0, v_0}{a, b} \right) \right] \]

\[ + \cos \lambda \left[ 1 - \frac{1}{\lambda^2} \frac{\partial u_0}{\partial b} + \frac{1}{\lambda^2} \frac{\partial v_0}{\partial a} - \frac{2}{\lambda^2} J \left( \frac{u_0, v_0}{a, b} \right) \right] \]

\[ + \sin \lambda \left[ \frac{1}{\lambda^2} \frac{\partial u_0}{\partial a} - \frac{1}{\lambda} \frac{\partial v_0}{\partial b} \right] \]

\[ + \mu \cos \lambda \frac{\partial u_0}{\partial a} + \mu \frac{\partial v_0}{\partial b} \]

The derivative \( \partial J/\partial t \) set equal to zero yields

\[ 0 = \frac{1}{\lambda} \sin \lambda \left[ S - \frac{\partial u_0}{\partial b} + \frac{1}{\lambda^2} \frac{\partial v_0}{\partial a} + \left( 1 + \frac{1}{\lambda^2} \right) J \left( \frac{u_0, v_0}{a, b} \right) \right] \]

\[ + \cos \lambda \left[ \frac{\partial u_0}{\partial a} + \frac{\partial v_0}{\partial b} \right] - \lambda \mu \sin \lambda \frac{\partial u_0}{\partial a} \]

\[ + \mu \cos \lambda \left[ \frac{\partial v_0}{\partial b} + J \left( \frac{u_0, v_0}{a, b} \right) \right] \]

In order that this equation holds for all times, the coefficients of the functions of time must vanish separately:

\[ \frac{\partial u_0}{\partial a} = 0, \]

\[ \delta_0 = \frac{\partial u_0}{\partial a} + \frac{\partial v_0}{\partial b} = 0, \]

\[ \frac{\partial v_0}{\partial a} + J \left( \frac{u_0, v_0}{a, b} \right) = 0, \]

\[ S - \frac{\partial u_0}{\partial b} + \frac{1}{\lambda^2} \frac{\partial v_0}{\partial a} + \left( 1 + \frac{1}{\lambda^2} \right) J \left( \frac{u_0, v_0}{a, b} \right) = 0. \]

The first two equations applied to the third yield

\[ \frac{\partial v_0}{\partial a} \left( 1 - \frac{\partial u_0}{\partial b} \right) = 0, \]

whence from the last equation

\[ \frac{\partial u_0}{\partial b} \left( 1 + \frac{\partial v_0}{\partial a} \right) = S. \]

These two results together give two possible distributions of initial velocities:

(i) \( \frac{\partial v_0}{\partial a} = 0, \frac{\partial u_0}{\partial b} = S; \)

(ii) \( \frac{\partial v_0}{\partial a} = S - 1, \frac{\partial u_0}{\partial b} = 1. \)

The second of these distributions, with a linear variation of the north-south component, is of no meteorological interest. The first states that the initial condition

\[ u_0 = Sb, \]

\[ v_0 = \text{constant} \]

results in a flow which is non-divergent at all times.

The pressure field exactly balances the \( u_0 \)-component, which thus represents no disturbance from the geostrophic equilibrium. The uniform \( v_0 \)-component has no pressure gradient to support it and is completely ageostrophic.

For the initial conditions (8), the solution (7) reduces to

\[ x = a + Sbt + \lambda^{-1} v_0 (1 - \cos \lambda t), \]

\[ y = b + \lambda^{-1} v_0 \sin \lambda t. \]

This highly restricted solution shows that the parcel method has been applied to parallel flow with the tacit assumption of much greater generality than is warranted by the dynamic theory. It is tempting to identify the uniform \( v_0 \)-perturbation with a sinusoidal perturbation of "infinite wave length," or a symmetric (zero-wave-number) perturbation of a vortex, and to state that for these perturbations the parcel stability criterion is valid. However, it is known from small-perturbation theory that the variation of the Coriolis parameter, here neglected, is the dominating effect for long waves; therefore, the first interpretation is subject to question. So far as the symmetric "ring" perturbation of the vortex is concerned, the subsequent section treats this problem in detail.

The following question naturally arises: what motion results when we allow, for example, an east-west variation in \( v_0 \)? The most that the parcel method can state is that under this condition the pressure field does not remain stationary and the solution (7) is no longer valid. However, here we may apply a result recorded by Lord Kelvin [5] and greatly generalized by Kao [6]. Kelvin noted that, for the initial conditions

\[ u_0 = Sb, \]

\[ v_0 = V(x), \]

the non-linear Eulerian equation for the conservation
of vorticity \( \xi \)

\[
\frac{\partial \xi}{\partial t} + u \frac{\partial \xi}{\partial x} + v \frac{\partial \xi}{\partial y} = 0
\]

is satisfied by the vorticity field

\[
\xi = V'(x - Syt) - S,
\]

where \( V' \) is the derivative of \( V \) with respect to its single argument, with the velocity components

\[
u = \frac{V(x - Syt)}{1 + S^2 t^2},
\]

\[
u = \frac{1 + S^2 t^2}{1 + S^2 t^2}.
\]

This is a motion damped by the shear, and, if \( V(x) \) has a wave form, the crests and troughs tilt in such a way as to transfer \( x \)-momentum from the perturbation to the zonal current. Since the Coriolis parameter is assumed constant, the above equation and solution are equally valid in the system discussed in this paper. The trajectories may be computed by using the result,

\[
\frac{d}{dt}(x - Syt) = u - Sy - St = 0.
\]

They are

\[
x = a + \int_0^t u dt = a + Sbt + V(a) \tan^{-1} (St)
\]

\[
y = b + \int_0^t v dt = b + S^{-1} V(a) \tan^{-1} (St)
\]

which resemble the trajectories (7') not at all. There is no oscillatory motion whatever, and no instability for any values of the parameters. Note also that the solutions are dimensional as they stand and do not depend on the Coriolis parameter. The parcels move as though they were unaware of the rotation of the earth. As time increases, the \( x \)-coordinate is more and more a simple translation

\[
x \rightarrow a + [Sb + \pi V(a)/2] t
\]

while the \( y \)-coordinate approaches asymptotically the value

\[
y \rightarrow b + \pi V(a)/2S.
\]

The pressure field can be computed from the accelerations. Only in the motion of the pressure field does the system depend upon the earth's rotation, and this will be made explicit by writing the equations in dimensional form:

\[
- \frac{\partial \phi}{\partial x} = -fu_x = \frac{V(x - Syt)}{(1 + S^2 t^2)^2}[2S - f(1 + S^2 t^2)]
\]

\[
- \frac{\partial \phi}{\partial y} = fu_y = fSy - \frac{S^2 V(x - Syt)}{(1 + S^2 t^2)^2} \times [2S - f(1 + S^2 t^2)].
\]

As was expected, the pressure field readjusts itself continuously so as to maintain a divergence-free flow.

It is natural to inquire how this flow, represented by (9), can reduce to the completely different oscillatory flow (7') in the special case constant, and the surprising answer is that it does not. To resolve this paradox, the two solutions may be compared in Eulerian coordinates. The stream function

\[
\psi = -\frac{1}{2} S^2 y^2 + F(t)x - G(t)y
\]

with \( F(0) \neq 0, G(0) = 0 \) satisfies automatically all of the following conditions: zero-divergence, conservation of vorticity, and the initial velocity field (8). These conditions do not determine a unique flow field. The Kelvin solution has

\[
F(t) = \frac{v_0}{1 + S^2 t^2}, \quad G(t) = \frac{Sv_0}{1 + S^2 t^2}
\]

whereas the equally valid parcel solution has

\[
F(t) = v_0 \cos \lambda t, \quad G(t) = \lambda^{-1} v_0 \sin \lambda t.
\]

The resolution lies in the fact that the parcel solution has implicitly added another initial condition, which is the specification of the initial pressure field in the form of the geostrophic wind, but, because this field does not change with time, it does not appear formally as an initial condition. Thus, at the time \( t = 0 \), the parcel solution has the geostrophic wind

\[
u_a = Sb
\]

\[
v_a = 0
\]

whereas the Kelvin solution has by (9)

\[
u_a = Sy
\]

\[
v_a = (1 - 2f^{-1}S)v_0.
\]

The Kelvin solution has partial geostrophic balance in the north-south flow. The balance is exact in the trivial case \( S = 0 \), and the only motion is the uniform north-south flow conserving the momentum of the initial impulsive motion.

Any initial force field, represented by the geopotential \( \psi \), and any initial flow field, represented by the stream function \( \psi \), may be imposed as long as these are in balance (i.e., do not generate divergence). There is a variety of these possible balanced initial accelerations, and we see that very different flow patterns can result. Kelvin's is one valid particular solution (with much the wider range of application); the parcel equations are another. Both solutions have the property that they satisfy the non-linear equations as well as the small perturbation equations.

The problem of uniqueness has been greatly advanced in a recent study by O. A. Ladyzhenskaia [8] of solutions of the equations of homogeneous, incom-
pressible, viscous flow in two-dimensions (brought to my attention by E. A. Coddington). Without stating Ladyzhenskai's very powerful theorem in detail, we may note that, of her sufficient conditions for existence and uniqueness of solutions, the only one not satisfied by the two solutions of this section is that the motion vanish at some boundary or at infinity. This clearly answers the natural question as to what feature of the model permits non-unique solutions. Since conditions at infinity are rarely applied in problems of horizontal atmospheric motion, it would seem that the uniqueness of the solutions offered should be more often a consideration in stability analyses.

4. The anticyclonic vortex

The constant-shear zonal wind field of the previous section is an example of the general linear geostrophic wind field,

\[ u_0 = \alpha_0 x + \alpha_2 y, \quad v_0 = \beta_1 x + \beta_2 y. \]  

(10)

The parcel motions arising from this force field have been exhaustively analyzed over a period of four decades, variation of the four parameters providing a wide variety of flow patterns. Van Mieghem [7] gives a review and references to the literature. A simple case will be selected, that of the vortex with circular contours

\[ u_0 = Sy, \quad v_0 = - Sx, \]

so that the geostrophic vorticity is now \( \zeta_0 = -2S. \) The equations of parcel motion are then

\[ \ddot{x} - \dot{y} = Sx, \quad \ddot{y} + \dot{x} = Sy \]

and the solution is greatly simplified by the introduction of a complex variable

\[ z = x + iy, \quad z_0 = a + ib, \]

whence the equations may be formulated as

\[ \dot{z} + iz = -Sz = 0. \]

The substitution

\[ z = Z(z_0, \dot{z}_0, t)e^{-i\omega t/2} \]

yields the equation

\[ \dot{Z} + \lambda^2 Z = 0, \]

where \( \lambda^2 = \frac{1}{4} - S, \) with the initial conditions

\[ Z_0 = z_0, \quad \dot{Z}_0 = \dot{z}_0 + i\omega_0/2. \]

Solutions will be of the form \( \cos (\lambda t). \) Thus, the condition for parcel stability is \( 4S < 1 \) or in dimensional terms,

\[ \zeta_0 > -\frac{1}{2}f. \]  

(11)

This transition from oscillatory to exponential solutions is contained in the more complicated frequency formulas in the literature [6, pp. 440-441] but has not, apparently, been associated with the transition in the so-called “balance equation” from elliptic form to hyperbolic. That this association is justified should become clear from the subsequent paragraphs.

If real and imaginary parts are separated, the solution becomes

\[ x = \cos (t/2) \left[ a \cos \lambda t + \lambda^{-1}(u_0 - b/2) \sin \lambda t\right] + \sin (t/2) \left[b \cos \lambda t + \lambda^{-1}(v_0 + a/2) \sin \lambda t\right], \]

\[ y = \cos (t/2) \left[ b \cos \lambda t + \lambda^{-1}(v_0 + a/2) \sin \lambda t\right] + \sin (t/2) \left[-a \cos \lambda t + \lambda^{-1}(-u_0 + b/2) \sin \lambda t\right]. \]

These may be verified by direct substitution. Their form is chosen to facilitate computation of the Jacobian, a tedious but straightforward operation, the details of which are outlined in Appendix I. The resulting continuity equation is

\[ 0 = \delta_0 \cos (2\lambda t) \]

\[ + (2\lambda)^{-1} \left[ \dot{\zeta}_0 + 2J \left( \frac{u_0, v_0}{a, b} \right) + 2fS \right] \sin (2\lambda t) \]

where \( \delta_0 \) and \( \zeta_0 \) are the initial divergence and vorticity, respectively. Thus, in order that the parcel solution represent a barotropic motion, the coefficients of both trigonometric terms must vanish. With dimension restored, these two conditions are

\[ \delta_0 = \frac{\partial u_0}{\partial a} + \frac{\partial v_0}{\partial b} = 0 \]

\[ f\delta_0 + 2J \left( \frac{u_0, v_0}{a, b} \right) + 2fS = 0. \]

The first condition implies that a stream function \( \psi_0 \) exists for the initial flow, and, if this expression replaces the velocities and if account is taken of the geostrophic vorticity \( \zeta_0 = -2S, \) the two equations may be formulated as one:

\[ f\psi_0 + 2 \left[ \frac{\partial^2 \psi_0}{\partial \alpha^2} \frac{\partial^2 \psi_0}{\partial \beta^2} - \left( \frac{\partial^2 \psi_0}{\partial \alpha \beta} \right)^2 \right] = f\dot{\zeta}_0. \]

This is the balance equation for the initial flow, the variation in \( f \) neglected. Thus, the conclusion may be drawn that, if the initial wind field is a balanced one, the parcel solution represents a true barotropic motion, and the condition (11) for stability is valid. The difficulties encountered in integrating the balance equation in regions where \( \zeta_0 < -f/2 \) have, therefore, a dynamical foundation.
It is not surprising to find the balance equation in the context of non-divergence. Of course, the initial flow must be balanced or the flow field would exhibit divergence at the next moment. But it is equally obvious that an initially-balanced flow has no guarantee of remaining balanced. The equation of continuity (2) is needed here, so that the conditions it imposes on the initial flow will in general be much stronger than balance. The preceding section affords an example of this. The field (7') is non-divergent, and the balance equation in this case takes the (dimensional) form

\[ f(V' - S) - 2SV' = -fS \]

which implies

\[ V'(a)(f - 2S) = 0. \]

Thus, the imposed condition \( V' = 0 \) is sufficient but not necessary. The particular initial shear \( S = f/2 \) satisfies the balance equation but not the equation of continuity.

The following question naturally arises at this point: is the strong anticyclonic vortex unstable for non-radially-symmetric perturbations? This now resolves itself into the problem of whether or not a non-symmetric flow field can satisfy the balance equation, given a uniform geostrophic vorticity. Symmetry is more easily discussed in plane polar coordinates. The only non-invariant term in the balance equation is the Jacobian, and here we may write

\[ J\left(\frac{u, v}{x, y}\right) = J\left(\frac{u, v}{r, \theta}\right) J\left(\frac{r, \theta}{x, y}\right) = \frac{1}{r} J\left(\frac{u, v}{r, \theta}\right). \]

If we denote the polar velocity components by

\[ U = r\dot{\theta}, \quad V = r\dot{r}, \]

the transformation with the Cartesian components is

\[ u = U \cos \theta - V \sin \theta \]
\[ v = U \sin \theta + V \cos \theta \]

so that

\[ J\left(\frac{u, v}{x, y}\right) = \frac{1}{r} \left(\frac{U \cos \theta - V \sin \theta, U \sin \theta + V \cos \theta}{r, \theta}\right) \]
\[ = \frac{1}{r} J\left(\frac{U, V}{r, \theta}\right) \frac{1}{r} \frac{\partial}{\partial r}(U^2 + V^2). \]

In polar coordinates, then, the initial conditions to insure divergence-free flow take the form

\[ \delta_0 = \frac{\partial U_0}{\partial r_0} + \frac{U_0}{r_0} + \frac{1}{r_0} \frac{\partial V_0}{\partial \theta_0} = 0 \]

\[ f\dot{\delta}_0 = f\dot{\delta}_0 + \frac{2}{r_0} J\left(\frac{U_0, V_0}{r_0, \theta_0}\right) + \frac{1}{r_0} \frac{\partial}{\partial r_0}(U_0 \dot{\theta} + V_0 \dot{r}). \]

As in Cartesian coordinates, the balance equation is the result of setting \( \delta \) and \( \dot{\delta} \) equal to zero in the divergence equation. If a stream-function \( \psi \) be introduced, and subscripts used to denote partial differentiation, we have

\[ f\dot{\delta}_0 = f\dot{\delta}_0 + \frac{1}{r} \frac{\partial}{\partial r}(\psi_{rr} + \frac{1}{r^2} \psi_{\theta \theta}) + \frac{2}{r^2} (\psi_{rr} \psi_{\theta \theta} - \psi_{\theta \theta} \psi_{rr}) \]
\[ + \frac{2}{r} \psi_{\theta \theta} \psi_{\theta r} + \psi_{r \theta \theta}, \]

where \( \psi_{\theta} = -r \delta, \psi = V_0. \)

It is well known that the gradient wind is a particular symmetric solution of this equation. I have not been able to construct a non-symmetric solution for the case \( \dot{\delta}_0 = \) constant. The failure of certain attempts in this direction is recorded in Appendix II, and intuition suggests that a solution, if such exists, would not represent a meteorological flow pattern. However, the symmetry of the disturbed flow is less important in this Lagrangian solution than in the Eulerian. As has been mentioned, the conventional emphasis on a wave solution arises from the original linearization of the dynamic equations and the consequence thereof that arbitrary disturbances can be constructed from the wave solutions by superposition. In the present study, the solutions are exact and the force fields may be non-linear. Even the linear field (10) can represent pressure patterns which are radially non-symmetric. However, the systematic investigation of this whole problem is a large task and quite beyond the scope of this paper.

5. Conclusions

Two models have been examined from the point of view of the parcel method — that is to say, two pressure patterns which remain stationary in time. One, expressed in terms of the geostrophic wind field, consists of parallel flow with constant shear. Here it was found that non-divergent flow resulted only when the velocity field was disturbed with uniform velocity normal to the geostrophic current, in which case the usual parcel criterion for stability is valid — namely, that the anticyclonic geostrophic vorticity is less than the vorticity of the earth about the local vertical. This is a severe limitation upon the usually completely general application of this criterion, and, if the initial disturbance \( V(a) \) is sinusoidal, it may be said that the parcel criterion is valid only for waves of infinite length. However, the fact that variation of the Coriolis parameter has not been taken into account casts doubt upon this interpretation. Whatever the interpretation, the motion may be described by saying that in
the stable case the parcels follow the oscillatory trajectories (7') through the stationary pressure field.

Comparison with the Kelvin solution was instructive here. If the disturbance $V(a)$ is not uniform, the pressure field does not remain stationary but moves according to a pattern $V(x - u_0 t)$, damped with time. The parcels, however, do not exhibit this pattern but asymptotically approach a finite displacement normal to the direction of flow. The trajectories are completely independent of the earth's rotation.

The second model was that of the circular anticyclonic vortex. Here it was shown that any initially balanced flow remains non-divergent and that the stability criterion is that the anticyclonic geostrophic vorticity is less than one-half the local vertical vorticity of the earth. The simplest balanced initial flow is, of course, the gradient wind. Non-symmetric balanced flows have not been found.

One special pressure pattern common to both the models should be noted: that of uniform pressure. With no force field at all, the parcels exhibit the well-known inertial-frequency oscillation

$$\begin{align*}
x &= a + f^{-1}u_0 \sin ft + 2f^{-1}v_0 \sin^2 \left(\frac{ft}{2}\right), \\
y &= b - 2f^{-1}u_0 \sin^2 \left(\frac{ft}{2}\right) + f^{-1}v_0 \sin ft, \\
\end{align*}$$

whence

$$\begin{align*}
[x(a, b, t) - a - f^{-1}u_0(a, b)]^2 \\
+ [y(a, b, t) - b + f^{-1}u_0(a, b)]^2 \\
= f^{-2} [u_0^2(a, b) + v_0^2(a, b)].
\end{align*}$$

Thus, given a parcel initially at $(a, b)$, the “circle of inertia” is a valid trajectory, provided that the initial field $u_0(a, b), v_0(a, b)$ is in balance with the zero pressure gradient. It cannot, however, be concluded that inertial motion on the sphere is a possible non-divergent flow. The equations of parcel motion are easily solvable in terms of elliptic functions, but it has not been shown that the equation of continuity is satisfied for any non-trivial initial velocity field.

In conclusion, it seems that the parcel method, properly applied, should prove a valuable adjunct to the wave- perturbation technique. Certain instances of meteorologically important flow patterns will be found to satisfy the relevant hypotheses, and in these instances the parcel method gives the added benefit of an exact solution. Even in instances — such as the zonal geostrophic current — in which only trivial results are obtained, the insights gained into the dynamics are of value.

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APPENDIX I

Readers wishing to check the results of Section 4 may be aided by the following intermediate steps:

$$\begin{align*}
\frac{\partial x}{\partial a} &= \cos \left(\frac{t}{2}\right) \cos \lambda t + (2\lambda)^{-1} \sin \left(\frac{t}{2}\right) \sin \lambda t, \\
\frac{\partial x}{\partial b} &= (-2\lambda)^{-1} \cos \left(\frac{t}{2}\right) \sin \lambda t + \sin \left(\frac{t}{2}\right) \cos \lambda t, \\
\frac{\partial x}{\partial u_0} &= \lambda^{-1} \cos \left(\frac{t}{2}\right) \sin \lambda t, \\
\frac{\partial x}{\partial v_0} &= \lambda^{-1} \sin \left(\frac{t}{2}\right) \sin \lambda t, \\
\frac{\partial y}{\partial a} &= -\sin \left(\frac{t}{2}\right) \cos \lambda t + (2\lambda)^{-1} \cos \left(\frac{t}{2}\right) \sin \lambda t, \\
\frac{\partial y}{\partial b} &= \cos \left(\frac{t}{2}\right) \cos \lambda t + (2\lambda)^{-1} \sin \left(\frac{t}{2}\right) \sin \lambda t, \\
\frac{\partial y}{\partial u_0} &= (-\lambda)^{-1} \sin \left(\frac{t}{2}\right) \sin \lambda t, \\
\frac{\partial y}{\partial v_0} &= \lambda^{-1} \cos \left(\frac{t}{2}\right) \sin \lambda t.
\end{align*}$$

$$\begin{align*}
J\left(\frac{x, y}{a, b}\right) = \frac{1}{2} (1 + \cos 2\lambda t) + \frac{1}{8\lambda^2} (1 - \cos 2\lambda t) + \frac{1}{2\lambda} (\frac{\partial u_0}{\partial a} + \frac{\partial v_0}{\partial b}) \sin 2\lambda t + \frac{1}{4\lambda^2} (\frac{\partial v_0}{\partial a} - \frac{\partial u_0}{\partial b}) (1 - \cos 2\lambda t) \\
&+ \frac{1}{2\lambda^2} J\left(\frac{u_0, v_0}{a, b}\right) (1 - \cos 2\lambda t) \\
= \frac{1}{2} (1 + \frac{1}{4\lambda^2}) + \frac{1}{4\lambda^2} \delta_0 + \frac{1}{2\lambda^2} J\left(\frac{u_0, v_0}{a, b}\right) + \frac{1}{2\lambda} \delta_0 \sin 2\lambda t - \frac{\cos 4\lambda t}{4\lambda^2} \left[\delta_0 + 2J\left(\frac{u_0, v_0}{a, b}\right) + 2S\right].
\end{align*}$$
The balance equation to be satisfied has the form
\[ f\left(\frac{\psi_{rr}}{r^2} + \frac{1}{r}\psi_r + \frac{1}{r^2}\psi_{\theta\theta}\right) + \frac{2}{r^2}(\psi_{rr}\psi_{\theta\theta} - \psi_r^2) + \frac{2}{r}\left(\frac{1}{r^2}\psi_{\theta\theta} + \psi_r\psi_{rr}\right) = -2fS = \text{constant}. \]

No function of the form \( \psi(r)e^{i\theta} \) can be a solution of this equation, since the left-hand side will always be a function of \( \theta \). A trial function therefore might be
\[ \psi = \Psi(r)(1 + \epsilon \sin n\theta). \]

If this is substituted into the equation, the result is
\[ -2fS = f\psi'' + \frac{f}{r}\psi' + \frac{2}{r}\psi'\psi'' + \epsilon \sin n\theta\left[ \frac{f\psi'''}{r} + \frac{f}{r^2}\psi' - \frac{fn^2}{r^2}\psi - \frac{2n^2}{r^2}\psi''\psi + \frac{4}{r}\psi'''\psi \right] \]
\[ + \epsilon^2 \sin^2 n\theta\left[ \frac{2}{r}\psi''\left(\frac{\psi'}{r} - \frac{n^2}{r}\psi\right) \right] + \epsilon^2 \cos^2 n\theta\left[ -\frac{2n^2}{r^2}\psi\left(\frac{\psi'}{r} - \frac{1}{r}\psi\right) \right]. \]

No simple function \( \Psi(r) \) appears to satisfy this equation, even for \( n = 1 \).

The trial function
\[ \psi = \Psi(r) + \epsilon e^{i\theta} \]
eliminates all cross-differentiation terms, but it leads to the condition
\[ -2Sf = f\left(\psi'' + \frac{1}{r}\psi'\right) + \frac{2}{r}\psi'\psi'' - \frac{n^2}{r^2}\epsilon e^{i\theta}(f + 2\psi'') \]
which also contains an inconsistency.

REFERENCES