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Table C.

<table>
<thead>
<tr>
<th>Before lubricating.</th>
<th>After lubricating with vaseline.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of determinations</td>
<td>9.</td>
</tr>
<tr>
<td>Maximum total resistance</td>
<td>0.00248 ohm</td>
</tr>
<tr>
<td>Minimum &quot;</td>
<td>0.00168 &quot;</td>
</tr>
<tr>
<td>Mean &quot;</td>
<td>0.00217 &quot;</td>
</tr>
<tr>
<td>Variation &quot;</td>
<td>± 0.0016 &quot;</td>
</tr>
<tr>
<td>Percentage &quot;</td>
<td>± 7.4 &quot;</td>
</tr>
</tbody>
</table>

In this case the plugs had been fitted with great care: consequently we find that the variations in the resistance both before and after applying the lubricant are much smaller than in the cases previously cited.

XIX. Dispersion and the Size of Molecules of Hydrogen, Oxygen, and Nitrogen. By L. Silberstein, Ph.D., Lecturer at the University of Rome*.

The molecular refractivity of an isotropic substance whose molecules consist of two equal atoms is given by

\[ N = \frac{2}{3} N_0 \left[ 2 + \frac{1}{1 - \alpha N_0 / 2\pi R^3} \right] \]

where \( N_0 \) is the atomic refractivity,

\[ \alpha = \frac{3 m \mu}{1.008 R \cdot 10^{-24}} \]

and \( R \) the mutual distance of the "centres" of the two atoms composing each molecule, \( i.e. \) the positions of equilibrium of the dispersive particles within their atoms†.

We shall henceforth use the abbreviation

\[ \sigma = \frac{\alpha}{2\pi R^3} \]

and write, therefore,

\[ N = \frac{2}{3} N_0 \left[ 2 + \frac{1}{1 - \sigma N_0} \right] \]

It will be remembered that the dimensions of

\[ N = \frac{\mu^2 - 1}{\mu^2 + 2 \tilde{d}} \]

are cm.\(^3\) gr.\(^{-1}\), so that \( \sigma N_0 \) is a pure number. When the

* Communicated by the Author.
† Cf. Phil. Mag. vol. xxxiii. (1917), pp. 92-128. Throughout that paper, \( R^3 \) is to be replaced by \( 4\pi R^3 \), our charges e, in formula (4) et seq., being in rational units. Keeping this in mind, the above formula will be found identical with (24 b) of the quoted paper. Another erratum is on p. 113, line 13, repeated in formula (30), where \( N_0 \) in the denominator should read \( 2N_0 \); this, however, affects only the numbers of pp. 113-114 concerning a purely fictitious example. The subject will be taken up at a later opportunity in connexion with the real compounds \( C_2H_4 \) and \( C_2H_2 \).
interatomic distance \( R \) is large enough for \( \sigma N_0 \) to become a negligible fraction, then \( N = 2N_0 \), as in the usual theory. The expression for \( N_0 \) itself is, in our previous symbols,

\[
N_0 = \frac{B/a}{\gamma_0 - \gamma} = \frac{B}{4\pi^2c^2a} \cdot \frac{\lambda_0^2}{1 - (\lambda_0/\lambda)^2},
\]

where \( \lambda \) is the incident, and \( \lambda_0 \) the free wave-length belonging to each of the atoms, when undisturbed by its neighbours; \( B = e^2/m \), \( m = \) mass, \( e \) charge of dispersive particle, in rational units.

In general, the atomic \( \lambda_0 \) will fall into the extreme ultraviolet, and since, in all cases to be treated hereafter, we shall limit ourselves to the visible region of the spectrum, the fourth and the higher powers of \( \lambda_0/\lambda \) will be negligible, so that

\[
N_0 = b_0 + g_0/\lambda^2,
\]

where \( b_0, g_0 \), two constant attributes of the atom, are defined by

\[
b_0 = \frac{B\lambda_0^2}{4\pi^2c^2a}, \quad g_0 = b_0 \lambda_0^2,
\]

whence also

\[
\frac{b_0^2}{g_0} = \frac{B}{4\pi^2c^2a} = \frac{1.008}{12\pi^2c^2} \left( \frac{e}{m} \right) \left( \frac{e}{m_H} \right),
\]

independent of the free wave-length. The latter formula will be useful in the sequel. Introducing (4) into (2), developing the denominator and rejecting the second and the higher powers of \( g_0/\lambda^2 \), we have, for the molecular refractivity of the substance in question,

\[
N = b + g/\lambda^2,
\]

where

\[
b = \frac{2}{3} b_0 \left[ 2 + \frac{1}{1 - \sigma b_0} \right],
\]

\[
g = \frac{2}{3} g_0 \left[ 2 + \frac{1}{(1 - \sigma b_0)^2} \right].
\]

Thus the refraction and the dispersion-coefficient of the substance, as \( b, g \) may be called, appear as simple functions of the atomic coefficients \( b_0, g_0 \) and of the interatomic distance involved in \( \sigma \). Notice in passing that, the denominator \( 1 - \sigma b_0 \) being a fraction (as in all concrete cases to be treated in the following sections), the dispersion will show a stronger departure from additivity than the refraction, which is a well-known feature of this class of phenomena. As to the range of \( \sigma b_0 \) in connexion with the condition of stability, we shall return to it a little later on.
The simple problem to be treated in the present note, as an application of the general ideas laid down in the paper quoted above, will be this:

Given the molecular refractivity \( N \) of the diatomic substance, that is, given its coefficients \( b, g \), find the atomic coefficients \( b_0, g_0 \), and therefore the atomic refractivity \( N_0 \), and also the interatomic distance \( R \) involved in \( \sigma \).

These are three unknowns, while we have in (8) but two equations *. Thus far, therefore, the problem is indeterminate. In order to convert it into a determinate one, we shall have to make some assumption about the relation between \( b_0 \) and \( g_0 \) in each atom. For this purpose let us choose the obvious assumption that the value of \( b_0^2/g_0 \) belonging to each "dispersive particle" or atomic resonator is equal to an exact multiple of the electronic value of the right-hand member of (6). More definitely, if \( \varepsilon \) be that electronic value, let us assume that, for any atom,

\[
\frac{b_0^2}{g_0} = \kappa \varepsilon, \quad \ldots \ldots \ldots \ldots \ldots \quad (9)
\]

where \( \kappa \) is in each case the smallest integer compatible with the conditions of the problem. By "electron" we mean here the electron proper. Remembering that our charges \( e \) are in rational units, and taking for the specific charge

\[
\frac{1}{\sqrt{4\pi}} \frac{e/c}{m} \quad \text{that value,} \quad 1.77 \cdot 10^7 \quad \text{c.g.s.,} \quad \text{which suits best the observed typical cases of the Zeeman phenomenon and represents, at the same time, the mean of the more recent}
\]

* Theoretically one could say that \( b_0, g_0, \sigma \) are to be determined from three observed values of \( N \), corresponding to three different wavelengths. But having once neglected higher powers of \( g_0/\lambda^2 \) and having therefore adopted the form (7), linear in \( \lambda^{-2} \), we have \( c \), in principle, reduced any number of observations to two independent data only, \( b \) and \( g \). The same would be the case if instead of the straight line (7), representing \( N \) as function of \( \lambda^{-2} \) we had taken a hyperbola analogous to (3). For this would again involve only two independent data. The rigorous curve of \( N \) is, by (25 b) loc. cit., the superposition of two hyperbolas, viz.

\[
N = 2B \frac{2}{3a} \left( \frac{2}{\gamma_0 - \gamma} + \frac{1}{\gamma' - \gamma} \right), \quad \gamma' = \gamma_0 - B/2\pi R^3,
\]

so that, theoretically speaking, the three unknowns \( \gamma_0, B, R \) could be determined from three observed values of \( N \). The actually available observations, however, are, notably in the case of the gases to be treated here, quite insufficient to distinguish the dispersion curve from a straight line. Thus, for the time being, we are driven by necessity to the linear form (7), amounting to two independent experimental data only, and the third datum must be supplied by some plausible assumption.
direct measurements by Classen, Wolz, and others *, we find
\[
\epsilon = 1 \cdot 008 \frac{e/c}{3\pi m} \frac{e/c}{\sqrt{4\pi m_H}} = 0 \cdot 107 \cdot 1 \cdot 77 \cdot 10^7 \cdot 9650
\]
\[= 183 \cdot 10^8 \text{ cm.} \quad \text{gr.} \quad \ldots \quad (10)
\]
This is the numerical value to be substituted in (9). The assumption (9) can be put in words by saying that each atomic resonator consists of a certain number \( \kappa \) of electrons. In fact, provided that the distance of these electrons apart is large as compared with their radii \( \dagger \), the mutual mass will be negligible, so that their total mass will be \( \kappa \) times the mass, and their total charge \( \kappa \) times the charge of a single electron. Thus \( e/m \) will retain its value and \( e/m_H \) will be \( \kappa \) times greater. We need not enter into the mutual action of these \( \kappa \) electrons, but can treat them summarily by attributing to the whole system a single relevant free frequency \( \sqrt{\gamma_0} \), which may already be the outcome of their co-operation together with the usual restitutive force.

But it seems safer to abstain from any such interpretation and to take our assumption as it is written down in (9). Merely for the sake of convenient language, this can be read as "atomic resonator consisting of \( \kappa \) electrons," or "atom containing \( \kappa \) dispersive electrons." The clause that \( \kappa \) should be the smallest integer compatible with the special conditions of the problem will be made clear presently.

Equations (8) with (9) are now sufficient for the determination of the two attributes \( b_0, \gamma_0 \) of the atoms and of their mutual distance in the molecule. This will be done conveniently, in each of the concrete cases, in the following manner. To abbreviate, write
\[
k = \frac{k^2}{\gamma}, \quad \text{and} \quad k_0 = \frac{b_0^2}{\gamma_0}. \quad \ldots \quad \ldots \quad (11)
\]
Then \( k \) is known from experience, and \( k_0 = \kappa \epsilon \). Dividing the square of the first by the second of (8), we have
\[
k = 2k_0 \sqrt{3 - 4\sigma b_0 + \frac{4}{3} \sigma^2 b_0^2},
\]
whence
\[
\sigma b_0 = 2k_0 - k + \sqrt{k(k_0 - \frac{1}{2}k)}. \quad \ldots \quad \ldots \quad (12)
\]
Now, \( k \) is essentially positive; thus the smallest value of

* Cf. Zeeman's 'Researches in Magneto-optics' (1913), p. 68 and passim.

† I. e. large compared with \( 10^{-13} \) cm., which, however, may still be a very small distance in comparison with atomic dimensions, \( 10^{-8} \) cm.
\( \kappa \) making \( \sigma b_0 \) real will be the smallest integer \( \kappa \) satisfying the inequality

\[
k_0 = \kappa \epsilon > \frac{1}{2} k. \quad \ldots \quad (13)
\]

This is the condition hinted at. It will be applied numerically in each of the cases to be treated presently. As to the sign of the square root it will be easily decided, remembering that both \( \sigma \) and \( b_0 \) are essentially positive. Thus, \( b, \sigma, \) and therefore \( k, \) being known from observation, the product \( \sigma b_0 \) will be found. Inserting this in (8) we shall find the atomic coefficients \( g_0, b_0, \) and therefore also \( \sigma \) and the interatomic distance.

Finally, the condition of stability is (loc. cit., p. 115),

\[
\gamma_0 > \frac{B}{2\tau H^3}, \quad \text{i.e.,} \quad \frac{B\sigma}{\alpha} < \frac{4\pi^2 e^2}{\lambda_0^2},
\]

that is, in terms of \( b_0, \) by formula (5),

\[
\sigma b_0 < 1. \quad \ldots \quad (14)
\]

This necessary and sufficient condition of optical stability will be tested in each of the particular cases to which we shall now pass.

**Hydrogen.**

The observed values of refractivity of hydrogen gas at 0° C. and pressure 760 mm. are, for the lines \( \text{H}_\alpha, \text{D}, \text{H}_\beta, \) and \( \text{H}_\gamma \) respectively,

\[
\mu - 1 = 1.387 \quad 1.392 \quad 1.406 \quad 1.412 \cdot 10^{-4}.
\]

The molecular weight of hydrogen gas (\( \text{H}_2 \)) is \( M = 2.016, \) and its density, at the above temperature and pressure, \( d = 8.9873 \cdot 10^{-5}. \) Whence the corresponding values of the molecular refractivity,

\[
N = \frac{\mu^2 - 1}{\mu^2 - 2} \frac{M}{d} = \frac{2}{3} (\mu - 1) \frac{M}{d},
\]

\[
N = 2.074 \quad 2.082 \quad 2.103 \quad 2.112.
\]

These observed values can be represented with sufficient accuracy by

\[
N = 2.044 + \frac{.01279}{\lambda^2}, \quad \lambda \text{ in microns.}
\]

In fact, the latter formula gives, for \( \text{H}_\alpha, \text{D}, \text{H}_\beta, \text{H}_\gamma, \) respectively, 2.074, 2.081, 2.099, 2.112. Thus, in c.g.s. units,

\[
b = 2.044; \quad g = 1.279 \cdot 10^{-10}, \quad \ldots \quad (\text{H}_2)
\]

in the visible region of the spectrum at least. Whence the required ratio

\[
k = b^2/g = 3.268 \cdot 10^{10},
\]

i.e., \( \frac{1}{2} k = 1.634 \cdot 10^{10}. \) Now \( \epsilon = 1.83 \cdot 10^{10}. \) Thus the smallest
integer satisfying the condition (13) is \( \kappa = 1 \). Consequently we shall attribute one dispersive electron to each hydrogen atom, i.e. put

\[ k_0 = \epsilon = 1.83 \times 10^{10}. \] (H)

Introducing these values of \( k, k_0 \) into formula (12) we find (as the unique positive root)

\[ \sigma b_0 = 0.4932. \] (15)

Thus the condition of stability, (14), is amply satisfied. Substituting this value of \( \sigma b_0 \), and the above \( b, g \) for \( \text{H}_2 \) in (8), we find, for the atomic coefficients of hydrogen,

\[ b_0 = 0.7719; \quad g_0 = 0.3255 \times 10^{-10} \text{ c.g.s.} \] (16)

Thus the atomic refractivity \( N_0 \) of hydrogen, which may conveniently be denoted by \( H \) itself, will be

\[ H = 0.7719 + 0.00326 \frac{1}{\lambda^2} (\lambda \text{ in microns}). \] (16 a)

Notice in passing that the free wave-length \( \lambda_0 = \lambda_H \) belonging to a hydrogen atom is equal \( (g_0/b_0)^{1/2} \), by (5), that is, by (16),

\[ \lambda_H = 6.494 \times 10^{-6} \text{ cm.} = 649.4 \text{ Å.U.}. \] (17)

much beyond the Lyman region.

Applications of the above atomic refractivity of hydrogen will be given at a later opportunity. In the present note I should like to draw the reader's attention chiefly to the interatomic distance which follows from the above results. From (15) and the first of (16) we have \( \sigma = \alpha/2\pi R^2 = 0.6389 \), where \( \alpha = 4.88 \times 10^{-24} \). This gives for the central distance of the two atoms in a molecule of hydrogen

\[ R = 1.067 \times 10^{-8} \text{ cm.} \] (18)

Now, the remarkable thing about this distance is that it approaches very nearly the values of the semidiameter of a molecule of hydrogen (considered as an elastic sphere), obtained by various methods based on the kinetic theory of gases. In fact, these values are *

\begin{align*}
\text{Deviation from} & \quad \text{Heat-} \\
\text{Boyle's Law.} & \quad \text{conduction.} \\
\text{Viscosity.} & \quad \text{Diffusion.} \\
\text{Semidiameter} = 1.025 & \quad 1.024 \\
1.024 & \quad 0.995 \\
1.01 \times 10^{-8} \text{ cm.} & \quad 1.01 \times 10^{-8} \text{ cm.}
\end{align*}

If we were to judge from this single case the coincidence could claim to be one of the order of magnitude only. But the following two cases, of oxygen and nitrogen, will show that there is more than this.

Oxygen.

Determining the coefficients $b$, $g$ from the observed refractivities $\mu = 1$ of oxygen gas for the lines $H_\alpha$, $H_\gamma$, which, at normal temperature and pressure, are $2.697$ and $2.747 \cdot 10^{-4}$, I find, with $M=32$ and $d=1.4294 \cdot 10^{-3}$,

$$b = 3.967; \quad g = 2.50 \cdot 10^{-10}, \quad \ldots \quad (O_2)$$

whence

$$k = b^2 / g = 6.297 \cdot 10^{10},$$

i.e. $k = 3.149 \cdot 10^{10}$. Thus the smallest integer fulfilling the condition (13) is, in the present case, $\kappa = 2$ (which happens to coincide with the "valency" of oxygen). Consequently we shall attribute two dispersive electrons to each oxygen atom,

$$k_0 = 2 \epsilon = 3.66 \cdot 10^{10}. \quad \ldots \quad (O)$$

Substituting these values of $k$ and $k_0$ in (12), we have

$$\sigma b_0 = 0.5446, \quad \ldots \quad \ldots \quad (19)$$

again satisfying amply the condition of stability (14). This value, together with the above $b$, $g$ for $O_2$, substituted in (8) gives, for the atomic coefficients of oxygen,

$$b_0 = 1.418; \quad g_0 = 0.549 \cdot 10^{-10}, \quad \ldots \quad (20)$$

and for the free wave-length belonging to an oxygen atom

$$\lambda_0 = 6.226 \cdot 10^{-6} \text{ cm.} = 622.6 \text{ Å.U.} \quad \ldots \quad (21)$$

From (19) and the first of (20) we have $\omega = 0.3840$, whence, the distance of the two atoms in a molecule of oxygen,

$$R = 1.265 \cdot 10^{-8} \text{ cm.} \quad \ldots \quad \ldots \quad (22)$$

This again coincides with the semidiameter of the oxygen molecule*, as found by the methods of the kinetic theory of gases, viz.:

<table>
<thead>
<tr>
<th>Viscosity</th>
<th>Heat-conduction</th>
<th>Diffusion</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.405</td>
<td>1.29</td>
<td>1.35 \cdot 10^{-8} cm.</td>
</tr>
</tbody>
</table>

In this case the distance $R$ differs from the heat-conduction value even much less than this does from the viscosity and the diffusion value of the semidiameter.

Nitrogen.

The observed refractivities for nitrogen gas ($N_2$) at normal conditions are $2.982$ and $3.020 \cdot 10^{-4}$ for the lines $H_\alpha$ and $H_\gamma$, respectively; $M=28.02$, $d=1.2507 \cdot 10^{-2}$. Proceeding as in the former case, I find from these data

$$b = 4.409; \quad g = 1.91 \cdot 10^{-10}; \quad k = \frac{b^2}{g} = 10.20 \cdot 10^{10}, \quad (N_2)$$

i.e. $k = 5.10 \cdot 10^{10}$. The smallest integer, therefore,

* As it should do if, say, the spherical atoms are in contact with one another. It will be remembered that $R$ is the mutual distance of the "centres" of the atoms.
satisfying (13) is now \( \kappa = 3 \) (which, again, happens to coincide with the "valency" of nitrogen). Thus

\[
k_0 = 3\varepsilon = 5.49 \times 10^{10}
\]

(N)

Proceeding exactly as in the above two cases, I find

\[
\sigma b_0 = 0.4158,
\]

and the atomic attributes of nitrogen

\[
b_0 = 1.782 \; \text{c.c.} \quad g_0 = 0.580 \times 10^{-10} \text{ c.c.s.} \quad \lambda_N = 570.5 \text{ Å}.
\]

(24)

Whence \( \sigma = 0.2334 \) and, the interatomic distance in a molecule of nitrogen,

\[
R = 1.493 \times 10^{-8} \text{ cm.}
\]

(25)

while the values of the molecular semidiameter furnished by the kinetic theory of gases are:

<table>
<thead>
<tr>
<th>Boyle's Law</th>
<th>Viscosity</th>
<th>Heat-conduction</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.56</td>
<td>1.448</td>
<td>1.37</td>
</tr>
</tbody>
</table>

Here again the distance \( R \) differs much less from the viscosity-value than the several kinetic values do among one another.

Collecting the above scattered results, together with those of the kinetic theory of gases, we have the following Table, in which it has seemed convenient to replace the atomic coefficients \( b_0, g_0 \) by the values of \( k_0 \) and the free wavelengths \( \lambda_0 \) characterizing equally well the optical properties of the atoms.

<table>
<thead>
<tr>
<th>Theory of Dispersion</th>
<th>Kinetic Theory of Gases.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Atom</td>
<td>( \lambda_0 = \left(\frac{g_0}{b_0}\right)^{1/2} )</td>
</tr>
<tr>
<td>H ...</td>
<td>649.4</td>
</tr>
<tr>
<td>O ...</td>
<td>622.6</td>
</tr>
<tr>
<td>N ...</td>
<td>570.5</td>
</tr>
</tbody>
</table>

All lengths are in \( 10^{-8} \) cm., and \( \varepsilon \), the electronic value of

\[
1.008 \frac{e}{12\pi^2e^2m/m_H},
\]

is equal to \( 183 \times 10^8 \) cm. gr.\(^{-1}\), as in (10).