Orientation statistics and settling velocity of ellipsoids in
decaying turbulence

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**ABSTRACT**

Motivated by applications in technology as well as in other disciplines where the motion of particles in a turbulent flow field is important, the orientation and settling velocity of ellipsoidal particles in a spatially decaying isotropic turbulent flow are numerically investigated. With respect to cloud microphysics ellipsoidal particles of various shapes are interpreted as archetypes of regular ice crystals, i.e., plates and columns approximated by oblate and prolate ellipsoids. The motion of 19 million small and heavy ellipsoidal particles is tracked by a Lagrangian point-particle model based on Stokes flow conditions. Five types of ellipsoids of revolution such as prolaters, spheres, and oblates are considered. The orientation and settling velocity statistics are gathered at six turbulence intensities characterized by the turbulent kinetic energy dissipation rate ranging from 30 to 250 cm²s⁻³. It is shown that the preferential orientation of ellipsoids is disturbed by the turbulent fluctuations of the fluid forces and moments. As the turbulence intensity increases the orientation probability distribution becomes more and more uniform. That is, the settling velocity of the ellipsoids is influenced by the turbulence level since the drag force is dependent on the orientation. The effect is more pronounced, the longer the prolate or the flatter the oblate is. The theoretical settling velocity based on the orientation probability of the non-spherical particles is smaller than that found in the simulation. The results show the existence of the preferential sweeping phenomenon also for non-spherical particles. These two effects of turbulence on the motion of ellipsoids change the settling velocity and as such the swept volume, that is expected to result in modified collision probabilities of ellipsoid-shaped particles.

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1. Introduction

During the last decades the influence of in-cloud turbu-
lence on the formation of precipitation was a field of intensive research (Devenish et al., 2012). The collision probabilities of cloud particles are described physically by collision kernels (Beheng, 2010). One major effect of turbulence is the modi-
fication of the collision kernel by modifying the relative velocities of the cloud particles (Grabowski and Wang, 2013). Whereas many investigations about the collision probabilities in warm clouds have been conducted, only relatively little is known about mixed-phase clouds where supercooled water drops and ice particles coexist (Pinsky and Khain, 1998). This is mainly due to the complex shapes of ice particles depending on the temperature and the available supersatura-
tion (Pruppacher and Klett, 1997). Due to the diverse shapes of the ice particles, velocities and relative velocities depend on their orientation, which is why the orientation of non-spherical particles in turbulent atmospheric flow was part of investigations which are mentioned in the following. However, the findings of these studies vary regarding the orientation behavior. Using dimensional arguments Cho et al. (1981) concluded that turbulence intensities typically found in clouds cannot destroy the preferential orientation. This is supported by the orientation model of Klett (1995). Both
studies conclude that ellipsoids are in a stable orientation with their long axes horizontal as measured under quiescent conditions (Newsom and Bruce, 1994). On the other hand, Krushkal and Gallily (1988) concluded from their numerical results that the preferential orientation effect decreases at increasing turbulent dissipation energy. The measurements of Newsom and Bruce (1998) support this conclusion.

Also in more technical applications Fan and Ahmadi (1995a) as well as Olson (2001) numerically studied the motion of fibers in a modeled isotropic turbulent field. However, they did not consider gravity. Thus, these studies only provide time scales for turbulent translation and rotation of fibers.

Recently, the motion of rods and fibers was analyzed by several research groups. Parsheh et al. (2005) measured the orientation of fibers in a planar contraction. Shin and Koch (2005) numerically investigated the motion of fibers in isotropic turbulence. Wilkinson et al. (2009) investigated the orientation of fibers experimentally and numerically. Parsa et al. (2011) reported measured rod alignment in 2D flows. Pumir orientation of fibers experimentally and numerically. Parsa et al. (2012) compared experimental results with numerical findings. What all these studies have in common is that they investigated the motion and orientation of fibers at the same density as the carrier fluid. That is, the particle translation motion is the same as of fluid tracers. The studies found that fibers rotate like material lines and are more aligned with the direction of the vorticity vector than with the principal axis of shear. Thus, it can be concluded that the knowledge about the Lagrangian properties of turbulence is extended towards the rotational degree of freedom.

However, the presence of inertia and gravity results in an even more complex behavior. For spherical particles several additional mechanisms are known. For example it was first shown by Wang and Maxey (1993) that particles with mass sediment faster in a turbulent flow than in quiescent conditions. This result is due to the fact that if the time scale of the particles and the turbulent eddies match, the particles preferentially choose the downward-moving side of the eddy such that they experience a downward fluid motion on average. Dávila and Hunt (2001) predicted theoretically that this effect of preferential sweeping should be maximal if the so-called particle Froude number is unity. This was confirmed numerically by Ayala et al. (2008) and Siewert et al. (2013). The higher velocity increases the swept volume which alters the relative velocities between pairs of particles and as such the collision probabilities, which might promote rain formation.

The proof of the existence of the preferential sweeping effect for non-spherical particles is not trivial. The drag and as such the sedimentation velocity of a non-spherical particle depend on its orientation even in quiescent conditions. In general, the drag might even be unknown, although the most common ice particle shapes, namely hexagonal plates and columns, can be reasonably well approximated by oblate and prolate ellipsoids (Pruppacher and Klett, 1997), for which the drag is known. Anyway the orientation will be influenced by the presence of turbulence. Therefore, the orientation under the influence of turbulence must be known before the settling velocities are investigated. The results of Mallier and Maxey (1991) investigating non-spherical particles in laminar cellular flow suggest that the preferential sweeping effect might also exist for non-spherical particles. However, Gavze et al. (2012) have recently shown by analyzing the governing equation of the rotational motion of prolate ellipsoids that the time scales the particle needs to reach a stable orientation are larger than the shear time scale of the turbulent flow. Therefore, a Lagrangian simulation of ellipsoids in a turbulent flow is required.

Until today all the numerical studies have utilized flow models, i.e., a turbulence parameterization to study the effect of isotropic turbulence on the motion of heavy non-spherical particles. In contrast, the current study is the first to investigate the motion of heavy ellipsoids using a direct numerical simulation of isotropic turbulence, i.e., a time-dependent simulation without any turbulence modeling of the flow phase. Additionally, to the authors’ knowledge, this is the first numerical study of the motion of plate-like particles in any turbulent flow. While the motion of elongated particles like rods and fibers in a turbulent flow have been the subject of many investigations, only a few theoretical studies considered disk- or plate-like shaped particles, e.g. Klett (1995). With the present method it is possible to investigate the existence of the preferential sweeping effect for non-spherical particles.

This study is mainly attributed to the motion of ice particles in turbulent clouds, i.e., all parameters are chosen to match cloud conditions as closely as possible. However, the general setup and the results also are of interest for many other scientific and engineering fields where the orientation of non-spherical particles plays a role in, e.g., rheology, aerosol science, papermaking, drag reduction in turbulent multiphase pipe flows and so forth.

This paper is organized as follows. First, the governing equations of the motion of ellipsoids in the limit of Stokes flow are presented in general form. Afterwards, the setup for this investigation is briefly introduced including the description of the types of ellipsoids. Then, the orientation statistics of ellipsoids in turbulent flow are presented. Resulting from these statistics the theoretical sedimentation velocities are determined and compared with ensemble-averaged velocities to get an insight into the preferential sweeping effect of non-spherical particles. At the end a physical interpretation for the occurrence of the preferential orientation and sweeping is given.

2. Equations of motion for ellipsoids

The numerical method used to simulate the motion of small and heavy ellipsoids is similar to that used by Fan and Ahmadi (1995b) and subsequent investigators (Marchioli et al., 2010; Mortensen et al., 2008a; Zhang et al., 2001). They investigated the deposition rates of fibers in turbulent channel flows. However, the method is generalized to deal with all kinds of ellipsoids of revolution similar to Gallily and Cohen (1979) as well as Krushkal and Gallily (1988). Hence, the equations of motion are presented in detail.

2.1. Kinematics

The half-axes of an ellipsoid are denoted by a, b, and c. The coordinate system is defined such that it is fixed at the particle center of mass and rotates with the particle, such that, the axes of the coordinate system denoted by a hat
always coincide with the principal axes of the ellipsoid. The orientation of the ellipsoid, i.e., the particle fixed coordinate system relative to the inertial system, is tracked by quaternions because of their numerical superiority over the Euler angles (Fan and Ahmadi, 1995b). Using the sketch shown in Fig. 1, the four quaternions \( \epsilon_1, \epsilon_2, \epsilon_3, \eta \) are related to the axes of rotation \( \mathbf{e} \) and the angle of rotation \( \Omega \):

\[
\begin{pmatrix}
\epsilon_1 \\
\epsilon_2 \\
\epsilon_3 \\
\eta
\end{pmatrix}
= \mathbf{e} \sin \left( \frac{\Omega}{2} \right), \quad \eta = \cos \left( \frac{\Omega}{2} \right).
\]

Thus, the coordinates given in the particle fixed coordinate system \((x, y, z)\) can be converted into the inertial coordinate system \((x, y, z)\) by a translation operation using the position of the particle center of mass \(x_c\), and a rotation operation using the rotation matrix (Goldstein, 1980):

\[
A = \begin{pmatrix}
1 - 2\left(\epsilon_2^2 + \epsilon_3^2\right) & 2(\epsilon_1\epsilon_2 - \epsilon_3\eta) & 2(\epsilon_1\epsilon_3 + \epsilon_2\eta) \\
2(\epsilon_2\epsilon_1 - \epsilon_3\eta) & 1 - 2\left(\epsilon_1^2 + \epsilon_3^2\right) & 2(\epsilon_2\epsilon_3 + \epsilon_1\eta) \\
2(\epsilon_3\epsilon_1 + \epsilon_2\eta) & 2(\epsilon_3\epsilon_2 - \epsilon_1\eta) & 1 - 2\left(\epsilon_2^2 + \epsilon_1^2\right)
\end{pmatrix}.
\]

The orientation of an ellipsoid changes over time due to its angular velocities \(\omega\) by:

\[
\frac{d\epsilon_1}{dt} \quad \frac{d\epsilon_2}{dt} \quad \frac{d\epsilon_3}{dt} \quad \frac{d\eta}{dt}
= \frac{1}{2} \begin{pmatrix}
+\eta & -\epsilon_3 & +\epsilon_2 \\
+\epsilon_3 & +\eta & -\epsilon_1 \\
-\epsilon_2 & +\epsilon_1 & +\eta \\
-\epsilon_1 & -\epsilon_2 & -\epsilon_3
\end{pmatrix} \begin{pmatrix}
\omega_x \\
\omega_y \\
\omega_z
\end{pmatrix}.
\]

To prevent error accumulation the quaternions are normalized each time step to fulfill (Mortensen et al., 2008a)

\[
\sqrt{\epsilon_1^2 + \epsilon_2^2 + \epsilon_3^2 + \eta^2} = 1.
\]

The position of the center of mass \(x_c\) depends on the velocity \(v\):

\[
\frac{dx_c}{dt} = v.
\]

\[\text{Fig. 1. Ellipsoidal coordinate system. Coordinates in the ellipsoidal coordinate system } (\hat{x}, \hat{y}, \hat{z}) \text{ can be converted into the inertial coordinate system } (x, y, z) \text{ by two steps. First, a rotation around the ellipsoid center of mass determined by the axes of rotation } \mathbf{e} \text{ and the angle of rotation } \Omega. \text{ Second, a translation determined by with the position vector of the ellipsoid center of mass } x_c.\]

2.2. Dynamics

In the limit of very small and heavy particles the translation and rotation equations of motion can be significantly simplified. In this study, the ellipsoid half-axes \(a, b, c\) are at least one order of magnitude smaller than the Kolmogorov scale \(\eta_k\). Nevertheless, the ellipsoids are large and heavy enough to neglect the influence of Brownian motion (Foss et al., 1989). The density of the ellipsoids \(\rho_e\) is nearly three orders of magnitude larger than the fluid density \(\rho_f\). In these limits, the translation acceleration only depends on the gravity \(g\), the ellipsoid mass \(m = \rho_e \frac{4}{3} \pi abc\), and the hydrodynamic drag force \(\mathbf{F}\) (Brenner, 1964). Under creeping flow or Stokes conditions the hydrodynamic drag force depends only on the fluid viscosity \(\nu\), the fluid density \(\rho_f\), the particle resistance tensor \(\mathbf{K}\), and the difference of the fluid velocity at the particle position \(\mathbf{u}\) and the particle velocity \(\mathbf{v}\):

\[
\frac{dv}{dt} = g + \frac{F}{m} = g + \frac{\nu \Omega}{m} \mathbf{A}^{-1} \mathbf{K} \mathbf{A} (\mathbf{u} - \mathbf{v}).
\]

Note that the particle resistance tensor is a dimensional quantity. To extend the range of the Stokes approximation towards higher particle Reynolds numbers \(Re_p = \frac{\nu a}{\nu}\), several non-linear drag laws were proposed for spherical particles (Clift, 1978). Using perturbation methods forces (Brenner and Cox, 1963) and moments (Cox, 1965) of particles of arbitrary shapes have been generalized by an expansion in the Reynolds number. However, as pointed out by Chester (1990): ”although the evaluation for ellipsoids is straightforward in principle, it is not so in execution”. So, to the authors' knowledge, a non-linear correction for general ellipsoids is not available in the literature. However, since \(Re_p < 1\) it seems reasonable to use the Stokes approximation in this investigation.

The resistance tensor of an ellipsoid \(\mathbf{K}\) is a diagonal matrix in its principal axes. A first solution attempt can be found in Oberbeck (1876):

\[
\mathbf{K} = \mathbf{E} \begin{pmatrix}
\frac{1}{x_0 + a^2 \alpha_0} \\
\frac{1}{x_0 + b^2 \beta_0} \\
\frac{1}{x_0 + c^2 \gamma_0}
\end{pmatrix} = 16\pi abc \mathbf{E} \begin{pmatrix}
\frac{1}{x_0 + a^2 \alpha_0} \\
\frac{1}{x_0 + b^2 \beta_0} \\
\frac{1}{x_0 + c^2 \gamma_0}
\end{pmatrix}.
\]

The quantity \(\mathbf{E}\) denotes the identity matrix. The shape parameters \(x_0, \alpha_0, \beta_0, \text{ and } \gamma_0\) are given by Brenner (1964) by the following integrals:

\[
\begin{aligned}
\alpha_0 &= abc \int_0^\infty \frac{d\lambda}{\Delta} \\
\beta_0 &= abc \int_0^\infty \frac{d\lambda}{(a^2 + \lambda) \Delta} \\
\gamma_0 &= abc \int_0^\infty \frac{d\lambda}{(b^2 + \lambda) \Delta} \\
\end{aligned}
\]

with \(\Delta = \sqrt{(a^2 + \lambda)(b^2 + \lambda)(c^2 + \lambda)}\).
The rotational motion of an ellipsoid about its principal axes is governed by

\[
\begin{pmatrix}
\frac{d\omega_x}{dt} \\
\frac{d\omega_y}{dt} \\
\frac{d\omega_z}{dt}
\end{pmatrix} = \begin{pmatrix}
\omega_y \omega_z I_y - I_z \\
\omega_z \omega_x I_z - I_x \\
\omega_x \omega_y I_x - I_y
\end{pmatrix} + \frac{1}{\rho} \begin{pmatrix}
T_x \\
T_y \\
T_z
\end{pmatrix},
\tag{9}
\]

The moments of inertia of an ellipsoid about its principal axes constitute a diagonal matrix

\[
\begin{pmatrix}
I_x & 0 & 0 \\
0 & I_y & 0 \\
0 & 0 & I_z
\end{pmatrix} = \frac{m}{5} \begin{pmatrix}
b^2 + c^2 \\
c^2 + a^2 \\
a^2 + b^2
\end{pmatrix} = \frac{4}{15} \pi \rho abc E \begin{pmatrix}
b^2 + c^2 \\
c^2 + a^2 \\
a^2 + b^2
\end{pmatrix}.
\tag{10}
\]

The hydrodynamic torques \( \mathbf{T} \) for ellipsoids in general shear flow in the limit of Stokes flow are given in the classical paper by Jeffery (1922). Although it is only a leading-order approximation, the formulation is widely used, since only for special cases more accurate torques are known, e.g., nearly spherical particles from perturbation methods Cox (1965) or very long fibers from the slender body theory (Schiby and Gallily, 1980; Shin et al., 2009). Hence, the most general approximation of Jeffery (1922) is used here:

\[
\begin{pmatrix}
T_x \\
T_y \\
T_z
\end{pmatrix} = \rho V \frac{16}{3} abc \begin{pmatrix}
b^2 + c^2 & T_{yz} & T_{zx} \\
T_{yz} & c^2 + a^2 & T_{xy} \\
T_{zx} & T_{xy} & a^2 + b^2
\end{pmatrix} + \frac{2}{3} \begin{pmatrix}
\omega_x \omega_z I_y - I_z \\
\omega_z \omega_y I_z - I_x \\
\omega_x \omega_y I_x - I_y
\end{pmatrix}.
\tag{11}
\]

The torque components depend on the fluid shear stress and the fluid vorticity in the particle fixed coordinate system

\[
\begin{align*}
\tau_{xy} &= \frac{1}{2} \left( \frac{\partial u_z}{\partial y} + \frac{\partial u_y}{\partial z} \right) \\
\tau_{yz} &= \frac{1}{2} \left( \frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x} \right) \\
\tau_{zx} &= \frac{1}{2} \left( \frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y} \right) \\
\omega_{xy} &= \frac{1}{2} \left( \frac{\partial u_z}{\partial x} - \frac{\partial u_x}{\partial z} \right) \\
\omega_{yz} &= \frac{1}{2} \left( \frac{\partial u_x}{\partial y} - \frac{\partial u_y}{\partial x} \right) \\
\omega_{zx} &= \frac{1}{2} \left( \frac{\partial u_y}{\partial z} - \frac{\partial u_z}{\partial y} \right).
\end{align*}
\tag{12}
\]

In brief, 13 coupled partial differential equations have to be solved to describe the motion of an ellipsoid, three for the position, four for the orientation, and three for the translation and the rotation velocities.

3. Setup

3.1. Flow field

The flow field used for this investigation is the same as in Kunnen et al. (2013). The influence of the particles on the flow is neglected because of the small volume loading. Hence, the flow field can be calculated independent of the type of suspended particles. The computational domain is an elongated box (Fig. 2).

The inlet is at the lower side, where turbulence is generated and advected by a constant mean flow velocity \( \mathbf{U} = \|\mathbf{U}\| = 1.5 \) m/s to the top, which is opposite to the gravity \( g \). The Reynolds number \( Re = \frac{UL}{\nu} = 80,000 \) is based on the mean flow velocity \( U \), the length of the domain \( L \), and the viscosity \( \nu = 1.5 \times 10^{-3} \) m²/s. All the values are chosen to be representative for cloud turbulence (Devenish et al., 2012; Pruppacher and Klett, 1997; Siebert et al., 2006). Note, however, that the length scale and as such the Reynolds number are restricted by the numerical capacity due to the computational effort, i.e., an acceptable overall computing time.

To simulate this flow the domain contains about 53 million Cartesian cells at a cell length of 0.00125 Δx/L. The resolution is chosen such that all turbulent scales down to the Kolmogorov length are resolved such that a direct numerical simulation of the flow field is performed. The full compressible Navier–Stokes equations are solved using a finite volume solver (Hartmann et al., 2008). Since the flow problem is essentially incompressible the Mach number is set to 0.1. At the outflow boundary, the pressure is kept constant and the streamwise velocity and density gradients are zero. On the sidewalls, free-slip boundary conditions are used and the wall-normal pressure and density gradients are zero. A sponge layer is imposed on all boundaries except the inflow. In these layers the flow is increasingly forced towards the prescribed mean velocity \( \mathbf{U} = (U,0,0) \) (Hartmann et al., 2008). They act as a smooth transition region such that the

![Fig. 2. Domain setup. At the lower side inlet synthetic turbulence \( \mathbf{u}' \) is added to the constant mean flow \( \mathbf{U} \) opposite to gravity \( g \). The six statistic volumes are located in the upper part of the domain.](image-url)
domain can be bounded without major effects on the internal flow. Synthetic turbulence is generated at the inlet according to the formulation of Batten et al. (2004), which is an extension of the method of Kraichnan (1970). The turbulence evolves according to the Navier–Stokes equations and reaches a physical state after an initial transition region (Siewert et al., 2013). Since neither mean shear nor perturbations are present except at the inflow, the turbulence decays due to viscous damping. Hence, the flow field consists of isotropic spatially decaying turbulence which mimics grid-generated wind tunnel turbulence (e.g. Bateson and Aliseda, 2012; Bordás et al., 2013). Under these conditions the turbulent kinetic energy dissipation rate \( \epsilon \) satisfies

\[
\epsilon(t) = \epsilon_0 \left( \frac{t}{t_0} \right)^{n-1}
\]

with \( n = -1.3 \) (Pope, 2000). The temporal evolution can be related to a spatial dependence using the constant mean flow velocity \( U \). Fig. 3 shows \( \epsilon \) as a function of \( x/L \) in a log–log plot. Near the inlet, \( \epsilon \) is almost constant and then further downstream the turbulent flow reaches a physical state where \( \epsilon \) decays with the theoretical exponent of \(-2.3\). The turbulent kinetic energy dissipation rate is important for the transport of small and heavy particles since it is proportional to the local shear rates. An advantage of the presented setup is that the dependence on \( \epsilon \) can be investigated within one single computation since the particles are advected through the decaying turbulence field. For this reason, the statistics are gathered in six different volumes distributed consecutively in the \( x \)-direction, see Fig. 2. The streamwise locations are chosen to match dissipation rates relevant for in-cloud turbulence (Devenish et al., 2012). The streamwise extent of the volumes is a tradeoff between the streamwise changing turbulence properties within the volumes and the computing time needed to obtain accurate statistics, see Kunnen et al. (2013) for more details. It has to be kept in mind that streamwise decaying turbulence is considered and the given dissipation rates are streamwise averages within the statistic volumes. As indicated in Fig. 3 the averaged dissipation rates within the statistic volumes are 250 to 30 cm\(^2\)s\(^{-3}\). Note that the Reynolds number dependence cannot be checked since it is restricted by the computational effort to values of approximately \( Re_s = 20 \) based on the Taylor length scale. It is expected that higher Reynolds numbers also affect the result, particularly due to intermittency. Nevertheless, Rosa et al. (2011) recently reported saturation of the collision probabilities of spherical particles at relatively modest Reynolds numbers.

3.2. Particle phase

Only ellipsoids at the same length \( a \) and \( b \) of the semi-axes are investigated. These are called spheroids or ellipsoids of revolution. The third half-axis \( c \) points in the \( z \) direction. Using the aspect ratio \( \beta = c/a \), prolate spheroids at \( \beta > 1 \), the special case \( \beta = 1 \) is a sphere, and defines \( \beta < 1 \) an oblate shape. The shapes of the different types of ellipsoids are visualized in Fig. 4. The solutions of the shape factor integrals (Eq. (8)) for the special case of an ellipsoid of revolution with \( c = a \beta \) are listed in Table 1.

After the flow field has reached a statistical stationary state, i.e., it is independent from the initial conditions, the ellipsoids are constantly released at random positions at the inflow with their velocity equal to the local instantaneous fluid velocity. The orientation is chosen randomly: all four quaternions are chosen from a standard normal distribution and normalized according to Eq. (4). Five types of ellipsoids are considered whose properties are listed in Table 2.

They are distinguished by the aspect ratio \( \beta \) ranging from 4.0 to 0.25. The lengths of the semi-axes are chosen such that the mass is equivalent to a sphere at radius 30 \( \mu \text{m} \). The density ratio of the spheroid to the fluid is \( \rho_c/\rho_f = 992 \) (Pruppacher and Klett, 1997). A constant number of 19 million ellipsoids, equally distributed among the five types of spheroids, is advanced in time. A predictor–corrector method is used for the time integration. The necessary fluid velocities as well as the derivatives at the position of the spheroid center are interpolated by a tri-cubic least square method. The time step is the same as for the fluid solution, i.e., \( 5.5 \times 10^{-5} \) s, which is a factor 200 smaller than the translation time scale of the spherical particle. Thus, it is expected that also the ellipsoidal motions are accurately resolved. After an initial time period of \( 1 \times L/U \) the statistics are gathered over a time span of \( 5 \times L/U \). To ensure converged statistics, the values are additionally ensemble averaged over all ellipsoids of a specific type in the region of interest. As explained in Section 3.1 the six regions of interest are distinguished by the streamwise averaged turbulence intensity or dissipation rate, respectively (see Fig. 3). Due to the inertia the particles might memorize the turbulent vortices they were in contact before.

---

**Fig. 3.** The turbulent energy dissipation rate \( \epsilon \) as a function of the streamwise coordinate \( x/L \) in a log–log plot. Additionally, the scaling for decaying isotropic turbulence is plotted with \( x^{-2.3} \) (Pope, 2000). The symbols show the locations of the statistic volumes, see also Fig. 2. The symbols have the same meaning throughout the paper.
Hence, the results might not be solely dependent on the local mean dissipation rate.

4. Results

4.1. Orientation distribution function (ODF)

To present the orientation of an ellipsoid in a turbulent flow it is convenient to introduce a spherical coordinate system (Fig. 5). The angle between the direction of the third semiaxis $c$ and the $x$ axis in the inertial system is the polar angle $\theta$, such that

$$\cos(\theta) = \frac{c_x}{|c|}. \quad (14)$$

The angle between the projection of $c$ into the $y$-$z$ plane and the $y$ axis is the azimuth angle $\phi$, such that

$$\tan(\phi) = \frac{c_z}{c_y}. \quad (15)$$

The orientation of a type of ellipsoid can now be characterized by a $\theta - \phi$ joint probability density function $\Psi(\theta, \phi)$, also known as orientation distribution function. The ODF fulfills the identity

$$\int_0^{2\pi} \int_0^\pi \Psi(\theta, \phi) \sin(\theta) d\theta d\phi = 1. \quad (16)$$

Fig. 6 is color coded by $\Psi(\theta, \phi)$ over $\theta$ and $\phi$ for the spheroid type with $\beta = 0.25$ at a dissipation rate of $50 \text{ cm}^2\text{s}^{-3}$. It can be seen that the orientation is not uniformly distributed ($\Psi \neq \frac{1}{4\pi}$) but intermediate values of $\theta$ are more likely. Additionally, it can be seen that $\Psi$ is independent of $\phi$ and symmetric in $\theta$. The independence of $\phi$ is expected because ellipsoids of revolution are considered. The symmetry regarding $\theta$ is due to the fore-aft symmetry of the ellipsoids. Although not explicitly shown, these two statements hold for all types of ellipsoids and all dissipation rates. Hence, the definition of the ODF $\Psi$ is simplified as

$$\int_0^{\pi/2} \Psi(\theta) \sin(\theta) d\theta = 1. \quad (17)$$

In Fig. 7 the calculated ODFs for all five types of ellipsoids are plotted. Additionally, the dependence on the dissipation rate is depicted. The $\theta$ bins are chosen such that they correspond to equal surface areas on the orientation sphere such that the statistical accuracy is maximized. Hence, the data point density increases at increasing $\theta$. It can be seen in Fig. 7(a) and (b) that low values of $\theta$ are more likely for prolate ellipsoids. Thus, they are preferentially aligned with gravity or the mean flow direction, respectively. As stated in the Introduction there is no study concerning isotropic

Table 1
Solution of the shape factor integrals (Eq. (8)) for ellipsoids of revolution with $a = b$ and $c = a/\beta$ for the three cases of prolaters ($\beta > 1$), spheres ($\beta = 1$), and oblates ($\beta < 1$). Similar expressions can be found in Happel and Brenner (1965, chaps. 5-11).

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>$\chi_0$</th>
<th>$\alpha_0$</th>
<th>$\gamma_0$</th>
<th>$\kappa$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta &gt; 1$</td>
<td>$-16\beta^2 \pi^2 - 1$</td>
<td>$\frac{8\beta}{\sqrt{1-\beta^2}}$</td>
<td>$-\frac{2}{\sqrt{1-\beta^2}}$</td>
<td>$\log\left(\frac{\beta^{1/2} - 1}{\beta^{1/2} + 1}\right)$</td>
</tr>
<tr>
<td>$\beta = 1$</td>
<td>$-16\pi^2$</td>
<td>$0$</td>
<td>$0$</td>
<td>$1$</td>
</tr>
<tr>
<td>$0 &lt; \beta &lt; 1$</td>
<td>$-16\beta^2 \pi^2 - 1$</td>
<td>$\frac{8\beta}{\sqrt{1-\beta^2}}$</td>
<td>$-\frac{2}{\sqrt{1-\beta^2}}$</td>
<td>$\log\left(\frac{1}{\beta^{1/2} + 1}\right)$</td>
</tr>
</tbody>
</table>

Table 2
Definition of the five types of spheroids used in this study. The aspect ratio $\beta$ ranges from 4 to 0.25. The length of the semiaxis is chosen such that all types of spheroids have the same mass.

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>$a$ [\text{nm}]</th>
<th>$c$ [\text{nm}]</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.0</td>
<td>19.44</td>
<td>77.76</td>
</tr>
<tr>
<td>2.0</td>
<td>24.49</td>
<td>48.99</td>
</tr>
<tr>
<td>1.0</td>
<td>30.86</td>
<td>30.86</td>
</tr>
<tr>
<td>0.5</td>
<td>38.88</td>
<td>19.44</td>
</tr>
<tr>
<td>0.25</td>
<td>48.99</td>
<td>12.25</td>
</tr>
</tbody>
</table>
turbulence to directly compare to, however, the results seem to be consistent with the measurements of Bernstein and Shapiro (1994) and the numerical analyses of Zhang et al. (2001). They found that in a turbulent channel flow the fibers are preferentially aligned with the mean flow direction. This will be further discussed in Subsection 4.3. The spherical particles do not show a preferential orientation as expected (Fig. 7(c)). However, the ODFs and Klett (1995) only showed averaged orientation distributions. The terminal velocity is then often expressed via a particle response time \( \tau_p \).

\[
vt_{\text{sphere}} = \frac{2\rho_p a^2}{9\rho_f \nu} g. \tag{18}
\]

To define an equivalent ellipsoidal response time different formulations have been introduced in the literature. Shapiro and Goldenberg (1993) defined the equivalent response time under the assumption of isotropic particle orientation leading to

\[
vt_{\text{Shapiro}} = \frac{\tau_{eq\text{Shapiro}} g}{2n a \beta} = 2n a \beta \left( \frac{1}{k_{xx}} + \frac{1}{k_{yy}} + \frac{1}{k_{zz}} \right) \frac{2\rho_p a^2}{9\rho_f \nu} g. \tag{19}
\]

Fan and Ahmadi (1995b) averaged the translation tensor resulting in

\[
vt_{\text{Fan}} = \frac{\tau_{eq\text{Fan}} g}{2n a \beta} = 2n a \beta \left( \frac{1}{k_{xx}} + \frac{1}{k_{yy}} + \frac{1}{k_{zz}} \right) \frac{2\rho_p a^2}{9\rho_f \nu} g. \tag{20}
\]

In general, the settling velocity of a non-spherical particle depends on its orientation even in stagnant air because of the orientation dependent drag force. Only for the sphere \( \beta = 1 \) the translation resistance tensor (Eq. (7)) is isotropic and can be inverted unambiguously. The terminal velocity \( v_t \) is then often expressed as

\[
v_t(\theta) = 6 n a \beta \left( \frac{1}{k_{xx}} + \frac{1}{k_{zz}} - \frac{1}{k_{yy}} \cos^2(\theta) + \frac{1}{k_{zz}} \right) \frac{2\rho_p a^2}{9\rho_f \nu} g. \tag{21}
\]

Thereby the rotation matrix \( A \) is composed of the intrinsic Euler angles as defined in Fig. 5 and the resulting equation is simplified by exploiting \( k_{xx} = k_{yy} = k_{zz} \) for ellipsoids of revolution. Hence, Eq. (21) reflects the independence of the results from the angle \( \phi \) and the symmetry regarding the angle \( \theta \) as mentioned already in Section 4.1. Using the ODF \( \Psi(\theta) \) defined in Eq. (17) and depicted in Fig. 7 the theoretical mean turbulent settling velocity can be calculated as

\[
v_{t,\text{ODF}}(\epsilon) = \int_0^{\pi/2} \Psi(\theta, \epsilon)v_t(\theta)\sin(\theta)d\theta. \tag{22}
\]

In Table 3 the values of the different velocities defined in Eqs. (18) to (22) are listed. In the first two rows, the minimum and maximum settling velocities are reported corresponding to the \( \theta \) values of 0 and \( \pi/2 \). Then, the velocities defined by Shapiro and Goldenberg (1993) and...
Fan and Ahmadi (1995b) are given. Next, the theoretical settling velocities using Eq. (22) are calculated for the lowest and highest investigated turbulence dissipation rates. Finally, the $x$-velocities found in the simulation are time and ensemble averaged and the mean flow velocity is subtracted. The resulting ensemble-averaged turbulent settling velocities $v_{x,\text{turb}}(\epsilon)$ are also given for the lowest and highest dissipation rates. The comparison of all these values shows that $v_{x,Fan}$ is always smaller than $v_{x,\text{ODF}}$. Hence, considering the parameter range of this study, the equivalent response time defined by Shapiro and Goldenberg (1993) fits better. This is especially true for higher turbulence intensities since the spheroids are more randomly orientated. Finally, it should be pointed out that the settling velocities of all ellipsoids are smaller than those of the sphere with the equivalent mass. This is important since several cloud microphysics schemes treat ice particles as spheres (Gavze et al., 2012).

**Fig. 7.** Orientation distribution function for all five types of spheroids over $\theta$. Additionally, the dependence of the ODF on the dissipation rate $\epsilon$ is depicted.
In Fig. 9 the ensemble-averaged velocity $v_{x,\text{turb}}/C_15$ is compared with the corresponding $v_{t,\text{ODF}}/C_15$ for all considered dissipation rates. It can be seen that $v_{x,\text{turb}}/C_15 = 250\text{cm}^2\text{s}^{-3}/C_0/C_1$ is significantly larger than the corresponding $v_{t,\text{ODF}}/C_15 = 250\text{cm}^2\text{s}^{-3}/C_0/C_1$ calculated from the orientation based drag force. The increase is comparable with the one found by Ayala et al. (2008) for spherical particles. Thus, in addition to the effect that the drag and as such the velocity of an ellipsoid dependent on its orientation depends on the turbulence intensity. The preferential orientation effect slows down the spheroids in a turbulent flow. In contrast, the preferential sweeping effect accelerates the spheroids and turns out to be dominant. These combined effects change the relative velocities of the particles in turbulent flows. As such the aggregation probability of regular ice crystals as well as the riming probability of regular ice crystals and supercooled droplets is expected to be changed in the presence of cloud turbulence.

In Fig. 9 the ensemble-averaged velocity $v_{x,\text{turb}}(\epsilon)$ is compared with the corresponding $v_{t,\text{ODF}}(\epsilon)$ for all considered dissipation rates.

It can be seen that $v_{x,\text{turb}}(\epsilon = 250\text{cm}^2\text{s}^{-3})$ is significantly larger than the corresponding $v_{t,\text{ODF}}(\epsilon = 250\text{cm}^2\text{s}^{-3})$ calculated from the orientation based drag force. The increase is comparable with the one found by Ayala et al. (2008) for spherical particles. Thus, in addition to the effect that the drag and as such the velocity of an ellipsoid dependent on its orientation depends on the orientation, there is also preferential sweeping, making the velocity in the direction of gravity dependent on the turbulence intensity. The preferential orientation effect slows down the spheroids in a turbulent flow. In contrast, the preferential sweeping effect accelerates the spheroids and turns out to be dominant. These combined effects change the relative velocities of the particles in turbulent flows. As such the aggregation probability of regular ice crystals as well as the riming probability of regular ice crystals and supercooled droplets is expected to be changed in the presence of cloud turbulence.

### Table 3

| Settling velocity for all investigated types of spheroids. |
|----------------|----------------|----------------|----------------|----------------|----------------|
|                | $\beta = 4$   | $\beta = 2$   | $\beta = 1$   | $\beta = 0.5$ | $\beta = 0.25$ |
| $v_t(\theta = 0)$ | 13.64         | 14.37         | 13.73         | 12.04         | 9.97           |
| $v_t(\theta = \pi/2)$ | 10.60         | 12.55         | 13.73         | 13.75         | 12.68          |
| $v_{t,\text{Shapiro}}$ | 11.61         | 13.15         | 13.73         | 13.18         | 11.78          |
| $v_{t,\text{Fan}}$ | 11.45         | 13.10         | 13.73         | 13.13         | 11.63          |
| $v_{t,\text{ODF}}(\epsilon = 30\text{cm}^2\text{s}^{-3})$ | 11.78         | 13.27         | 13.73         | 13.35         | 12.13          |
| $v_{t,\text{ODF}}(\epsilon = 250\text{cm}^2\text{s}^{-3})$ | 11.62         | 13.19         | 13.73         | 13.26         | 11.92          |
| $v_{t,\text{Shapiro}}(\epsilon = 30\text{cm}^2\text{s}^{-3})$ | 11.92         | 13.30         | 13.72         | 13.37         | 12.22          |
| $v_{t,\text{Fan}}(\epsilon = 250\text{cm}^2\text{s}^{-3})$ | 12.09         | 13.45         | 13.91         | 13.50         | 12.34          |

4.3. Physical Interpretation

After the presentation of the ellipsoidal motion under the influence of gravity and isotropic turbulence some physical interpretation is given on the coherence of the preferential orientation and the preferential sweeping. For this purpose the orientation of the ellipsoids is investigated for three different cases: without turbulence but with gravity, with turbulence but without gravity, and the already studied case with turbulence and gravity.

In quiescent laminar air, or in laminar air moving with a constant uniform velocity $U$ the ellipsoids would not rotate while falling under the influence of gravity since the torques based on creeping flow conditions depend only on the fluid velocity gradient tensor. The case without gravity but with a turbulent background flow is more complex. At first glance, a uniformly distributed, random orientation of the ellipsoids would be expected since from an Eulerian point of view the fluid velocity gradient tensor in isotropic turbulence provides a uniform distribution whose average is zero. This also holds for the Lagrangian statistics of fluid tracers. The recent numerical investigations of turbulent channel flow suggest that random orientation can also be expected for inertial particles. For instance, Mortensen et al. (2008b) found that the orientation becomes more and more isotropic going from the wall region to the core region of a turbulent channel where the turbulence gets more and more isotropic. However, for spherical particles with inertia it was shown that these particles show effects like clustering in an isotropic turbulent flow (Ayala et al., 2008). The particles are expected to have a higher probability of presence in regions of low vorticity and high strain. Hence, there could exist a preferential orientation of ellipsoids in a turbulent flow without the influence of gravity due to the inertial bias. However, following the theoretical arguments of Dávila and Hunt (2001) preferential sweeping and preferential clustering
are only weakly correlated for a particle Froude number $F_p > 1$. In this study, the particle Froude number of the spherical particle is $F_p \sim 9$. On the other hand, also non-spherical particles are expected to cluster, although the effect should be weaker than at spherical particles due to the variable drag force depending on the orientation. Thus, an effect of the preferential clustering on the preferential orientation cannot be completely ruled out. Therefore, an additional simulation without gravity but with isotropic turbulence is conducted. The results are shown in Fig. 10. Comparing Fig. 10 with Fig. 7 the ellipsoid orientation is found to be random for all investigated ellipsoids and dissipation rates without gravity. Hence, it is likely that the alignment of the ellipsoids with the mean flow direction found in studies of turbulent channel flows mentioned in Subsection 4.1 is not due to the mean flow itself but the higher streamwise velocity fluctuations, i.e., the anisotropy of the turbulent flow, which was also suggested by Mortensen et al. (2008a) and Marchioli et al. (2010). It can be concluded that clustering has no effect on the orientation of the ellipsoids, at least for the particle mass examined in this study.

However, in the investigated case the ellipsoids are subject to both gravity and turbulence. From the orientation statistics it is now clear that gravity has a pronounced impact. Since the underlying mechanisms are very complex and could be influenced by many different parameters like the particle inertia, the translation and rotation resistance, the vortex time and length scales etc., we cannot conclusively explain the underlying mechanisms of preferential orientation at the moment. However, we want to give a plausible explanation, on why the randomizing effect of the turbulent fluctuations is broken and a preferential orientation effect exists. Given that we have proven the preferential settling of ellipsoids in the last subsection, it is reasonable to assume that ellipsoids also preferentially sample the downward-moving sides of the turbulent eddies (Dávila and Hunt, 2001). Thereby the ellipsoids predominantly encounter a certain subset of velocity gradients (the shear induced by a turbulent eddy in its direct vicinity) that tends to orient their longest axis parallel to gravity. When the turbulence has a higher dissipation rate $\epsilon$ this effect is reduced given that more and smaller eddies contribute to the local velocity gradient, partially destroying the preferential orientation but enhancing preferential settling. In conclusion, preferential orientation and preferential settling are both thought to be driven by the specific interactions of gravity and turbulence with the ellipsoids.

5. Conclusion

The motion of ellipsoids which in this study are interpreted as archetypes of regular ice crystals under the influence of gravity and isotropic turbulence has been studied by a DNS of spatially decaying isotropic turbulence. Within this flow the motions of 19 million ellipsoids of revolution are tracked by a Lagrangian point model. Five types of ellipsoids at constant mass are considered, among them prolaters, spheres, and oblates. Using a one-way coupling the forces and moments on the ellipsoids are evaluated assuming creeping flow conditions. Statistics have been determined for six turbulence intensities characterized by turbulent kinetic energy dissipation rates ranging from 30 to 250 $\text{cm}^2\text{s}^{-3}$. First, the orientation probabilities have been investigated. A preferential orientation regarding to the direction of gravity is found. It is increasingly pronounced for both larger and smaller aspect ratios. The higher the dissipation rate is the closer the orientation probability is to a uniform distribution. Thus, the turbulence influences the settling velocity of ellipsoids, since their drag is dependent on their orientation. Additionally, the settling velocities are found to be higher than those calculated from

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**Fig. 10.** Orientation distribution function of spheroids with $\beta = 4, 2, \text{ and } 1$ over $\theta$ in the turbulent flow without gravity. Additionally, the dependence of the ODF on the dissipation rate $\epsilon$ is depicted.
the orientation probabilities. Hence, the existence of the preferential sweeping effect was shown for non-spherical particles. Just like spheres, non-spherical particles do fall faster, the higher the turbulence intensities are. They might preferentially choose the downward-moving sides of the turbulent eddies. These two turbulence effects change the settling velocity of ellipsoids. It is likely that this will affect the mean radial relative velocity, which is generally determined in large part by the differential settling velocities of two particles, although the connection might not be linear (Grabowski and Wang, 2013). Thereby, also the collision probability of non-spherical particles such as ice crystals is expected to change due to the turbulence in clouds.

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