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XCVII. Remarks concerning Fourier's Theorem as applied to
Physical Problems. By Lord Rayleigh, O.M., F.R.S.*

FOURIER'S theorem is of great importance in mathematical
physics, but difficulties sometimes arise in
practical applications which seem to have their origin in the
aim at too great a precision. For example, in a series of
observations extending over time we may be interested in
what occurs during seconds or years, but we are not con-
cerned with and have no materials for a remote antiquity or a
distant future; and yet these remote times determine whether
or not a period precisely defined shall be present. On the
other hand, there may be no clearly marked limits of time
indicated by the circumstances of the case, such as would
suggest the other form of Fourier's theorem where every-
thing is ultimately periodic. Neither of the usual forms of
the theorem is exactly suitable. Some method of taking off
the edge, as it were, appears to be called for.

The considerations which follow, arising out of a physical
problem, have cleared up my own ideas, and they may
perhaps be useful to other physicists.

A train of waves of length $\lambda$, represented by

$$\psi = e^{2\pi i (ct+\xi)/\lambda}, \quad \ldots \ldots \quad (1)$$

advances with velocity $c$ in the negative direction. If the
medium is absolutely uniform, it is propagated without dis-
turbation; but if the medium is subject to small variations,
a reflexion in general ensues as the waves pass any place $x$.
Such reflexion reacts upon the original waves; but if we
suppose the variations of the medium to be extremely small,
we may neglect the reaction and calculate the aggregate
reflexion as if the primary waves were undisturbed. The
partial reflexion which takes place at $x$ is represented by

$$d\psi = e^{2\pi i (ct-\xi)/\lambda} \phi(x)dx \cdot e^{4\pi i \xi/\lambda}, \quad \ldots \ldots \quad (2)$$

in which the first factor expresses total reflexion supposed to
originate at $x=0$, $\phi(x)dx$ expresses the actual reflecting
power at $x$, and the last factor gives the alteration of phase
incurred in traversing the distance $2x$. The aggregate re-
flexion follows on integration with respect to $x$; with
omission of the first factor it may be taken to be

$$C + iS, \quad \ldots \ldots \ldots \quad (3)$$

* Communicated by the Author.
where
\[ C = \int_{-\infty}^{+\infty} \phi(v) \cos uv \, dv, \quad S = \int_{-\infty}^{+\infty} \phi(v) \sin uv \, dv, \quad (4) \]
with \( u = 4\pi/\lambda \). When \( \phi \) is given, the reflexion is thus determined by (3). It is, of course, a function of \( \lambda \) or \( u \).

In the converse problem we regard (3)—the reflexion—as given for all values of \( u \) and we seek thence to determine the form of \( \phi \) as a function of \( x \). By Fourier’s theorem we have at once
\[ \phi(x) = \frac{1}{\pi} \int_0^{\infty} du \{ C \cos ux + S \sin ux \}. \quad (5) \]

It will be seen that we require to know \( C \) and \( S \) separately. A knowledge of the intensity merely, viz. \( C^2 + S^2 \), does not suffice.

Although the general theory, above sketched, is simple enough, questions arise as soon as we try to introduce the approximations necessary in practice. For example, in the optical application we could find by observation the values of \( C \) and \( S \) for a finite range only of \( u \), limited indeed in eye observations to less than an octave. If we limit the integration in (5) to correspond with actual knowledge of \( C \) and \( S \), the integral may not go far towards determining \( \phi \). It may happen, however, that we have some independent knowledge of the form of \( \phi \). For example, we may know that the medium is composed of strata each uniform in itself, so that within each \( \phi \) vanishes. Further, we may know that there are only two kinds of strata, occurring alternately. The value of \( \int \phi \, dx \) at each transition is then numerically the same but affected with signs alternately opposite. This is the case of chlorate of potash crystals in which occur repeated twinnings\(^*\). Information of this kind may supplement the deficiency of (5) taken by itself. If it be for high values only of \( u \) that \( C \) and \( S \) are not known, the curve for \( \phi \) first obtained may be subjected to any alteration which leaves \( \int \phi \, dx \), taken over any small range, undisturbed, a consideration which assists materially where \( \phi \) is known to be discontinuous.

If observation indicates a large \( C \) or \( S \) for any particular value of \( u \), we infer of course from (5) a correspondingly important periodic term in \( \phi \). If the large value of \( C \) or \( S \) is limited to a very small range of \( u \), the periodicity of \( \phi \)

\* Phil. Mag. vol. xxvi. p. 256 (1888); Scientific Papers, vol. iii. p. 204.
extends to a large range of \( x \); otherwise the interference of components with somewhat different values of \( u \) may limit the periodicity to a comparatively small range. Conversely, a prolonged periodicity is associated with an approach to discontinuity in the values of \( C \) or \( S \).

The complete curve representing \( \phi(x) \) will in general include features of various lengths reckoned along \( x \), and a feature of any particular length is associated with values of \( u \) grouped round a corresponding centre. For some purposes we may wish to smooth the curve by eliminating small features. One way of effecting this is to substitute everywhere for \( \phi(x) \) the mean of the values of \( \phi(x) \) in the neighbourhood of \( x \), viz.

\[
\frac{1}{2a} \int_{x-a}^{x+a} \phi(x) \, dx, \quad \ldots \ldots \quad (6)
\]

the range \((2a)\) of integration being chosen suitably. With use of (5) we find for (6)

\[
\frac{1}{2a} \int_{x-a}^{x+a} \phi(x) \, dx = \frac{1}{\pi} \int_0^\infty du \frac{\sin ua}{ua} \{ C \cos ux + S \sin ux \}, \quad (7)
\]

differing from the right-hand member of (5) merely by the introduction of the factor \( \sin ua - ua \). The effect of this factor under the integral sign is to diminish the importance of values of \( u \) which exceed \( \pi/a \) and gradually to annul the influence of still larger values. If we are content to speak very roughly, we may say that the process of averaging on the left is equivalent to the omission in Fourier's integral of the values of \( u \) which exceed \( \pi/2a \).

We may imagine the process of averaging to be repeated once or more times upon (6). At each step a new factor \( \sin ua - ua \) is introduced under the integral sign. After a number of such operations the integral becomes practically independent of all values of \( u \) for which \( ua \) is not small.

In (6) the average is taken in the simplest way with respect to \( x \), so that every part of the range \( 2a \) contributes equally (fig. 1). Other and perhaps better methods of smoothing may be proposed in which a preponderance is given to the central parts. For example we may take (fig. 2)

\[
\frac{1}{a^2} \int_0^a (a - \xi) \{ \phi(x + \xi) + \phi(x - \xi) \} \, d\xi. \quad \ldots \ldots \quad (8)
\]
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From (5) we find that (8) is equivalent to

$$\frac{2}{\pi} \int_{0}^{\infty} du \frac{1 - \cos ua}{u^2 a^2} \{C \cos ux + S \sin ux\}, \ldots \ (9)$$

reducing to (5) again when $a$ is made infinitely small. In comparison with (7) the higher values of $ua$ are eliminated more rapidly. Other kinds of averaging over a finite range may be proposed. On the same lines as above the formula next in order is (fig. 3)

$$\frac{3}{4a} \int_{-a}^{+a} \left(1 - \frac{\xi^2}{a^2}\right) \phi(x + \xi) d\xi$$

$$= \frac{1}{\pi} \int_{0}^{\infty} du \frac{\sin au - au \cos au}{\frac{1}{3} a^2 u^3} \{C \cos ux + S \sin ux\} dx. \ (10)$$

In the above processes for smoothing the curve representing $\phi(x)$, ordinates which lie at distances exceeding $a$ from the point under consideration are without influence. This may or may not be an advantage. A formula in which the integration extends to infinity is

$$\frac{1}{a \sqrt{\pi}} \int_{-\infty}^{+\infty} \phi(x + \xi) e^{-\xi^2/a^2} d\xi$$

$$= \frac{1}{\pi} \int_{0}^{\infty} du e^{-u^2 a^2/4} \{C \cos ux + S \sin ux\}, \ldots \ (11)$$

In this case the values of $ua$ which exceed 2 make contributions to the integral whose importance very rapidly diminishes.

The intention of the operation of smoothing is to remove from the curve features whose length is small. For some purposes we may desire on the contrary to eliminate features of great length, as for example in considering the record of an instrument whose zero is liable to slow variation from some extraneous cause. In this case (to take the simplest formula) we may subtract from $\phi(x)$ — the uncorrected record — the average over a length $b$ relatively large, so obtaining

$$\phi(x) - \frac{1}{2b} \int_{x-b}^{x+b} \phi(x) dx$$

$$= \frac{1}{\pi} \int_{0}^{\infty} du \left\{1 - \frac{\sin ub}{ub}\right\} \{C \cos ux + S \sin ux\}. \ (12)$$
Here, if \( u b \) is much less than \( \pi \), the corresponding part of the range of integration is approximately cancelled and features of great length are eliminated.

There are cases where this operation and that of smoothing may be combined advantageously. Thus if we take

\[
\frac{1}{2a} \int_{-a}^{x+a} \phi(x) \, dx - \frac{1}{2b} \int_{-b}^{x+b} \phi(x) \, dx
\]

\[
= \frac{1}{\pi} \int_0^\infty du \left\{ \sin \frac{ua}{u} - \sin \frac{ub}{u} \right\} \left\{ C \cos ux + S \sin ux \right\}, \quad (13)
\]

we eliminate at the same time the features whose length is small compared with \( a \) and those whose length is large compared with \( b \). The same method may be applied to the other formulæ (9), (10), (11).

A related question is one proposed by Stokes *, to which it would be interesting to have had Stokes' own answer. What is in common and what is the difference between \( C \) and \( S \) in the two cases (i.) where \( \phi(x) \) fluctuates between \( -\infty \) and \( +\infty \) and (ii.) where the fluctuations are nearly the same as in (i.) between finite limits \( \pm a \) but outside those limits tend to zero? When \( x \) is numerically great, \( \cos ux \) and \( \sin ux \) fluctuate rapidly with \( u \); and inspection of (5) shows that \( \phi(x) \) is then small, unless \( C \) or \( S \) are themselves rapidly variable as functions of \( u \). Case (i.) therefore involves an approach to discontinuity in the forms of \( C \) or \( S \). If we eliminate these discontinuities, or rapid variations, by a smoothing process, we shall annul \( \phi(x) \) at great distances and at the same time retain the former values near the origin. The smoothing may be effected (as before) by taking

\[
\frac{1}{2a} \int_{-a}^{u+a} C \, du, \quad \frac{1}{2a} \int_{-a}^{u+a} S \, du
\]

in place of \( C \) and \( S \) simply. \( C \) then becomes

\[
\int_{-\infty}^{+\infty} dv \, \phi(v) \cos \frac{uv}{av} \sin \frac{uv}{av},
\]

\( \phi(v) \) being replaced by \( \phi(v) \sin \frac{uv}{av} \). The effect of the added factor disappears when \( av \) is small, but when \( av \) is large, it tends to annul the corresponding part of the integral. The new form for \( \phi(x) \) is thus the same as the old one near the origin but tends to vanish at great distances on either

side. Case (ii.) is thus deducible from case (i.) by the application of a smoothing process to $C$ and $S$, whereby fluctuations of small length are removed.

We may sum up by saying that a smoothing of $\phi(x)$ annuls $C$ and $S$ for large values of $u$, while a smoothing of $C$ and $S$ (as functions of $u$) annuls $\phi(x)$ for values of $x$ which are numerically great.

Terling Place, Witham,
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XCIII. On the Gas-equation.
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In the famous gas-equation of van der Waals:

$$\left( P + \frac{a}{v^2} \right)(v - b) = R\theta.$$  

(1)

$a$ and $b$ are small constants depending, $a$ on the mutual attraction of the molecules, and $b$ on the volume of the molecules.

$P$ is the pressure of the gas, $v$ its volume, and $\theta$ denotes the absolute temperature. $R$ is the gas-constant.

In reality, however, the quantities $a$ and $b$ are not constant. They vary both with temperature and pressure.

In the following I will regard $a$ and $b$ as functions of the temperature only; thus getting a gas-equation that covers the case of Clausius and several others.

It is the object of this paper to transform the gas-equation into a more convenient form by means of series, and to study the quantities $a$ and $b$, expressing them in known and measurable quantities.

The equation (1) can be transformed into

$$Pv\left(1 - \frac{b}{v}\right) = R\theta\left[1 - \frac{a}{vR\theta} \left(1 - \frac{b}{v}\right)\right].$$

We express $P$ in atmospheres;

$v$ in the volume of one gram-molecule of the ideal gas at one atmosphere and $0^\circ$ centigrade;

$\theta$ in degrees centigrade.

By index zero I mean the value at $0^\circ$ C.

* Communicated by the Author.