Appendix E: Cylinder Growth Vapor Diffusion Calculations

This appendix contains details and results of the calculation of $h^c (1, 0)$, $h^a (1, 1)$, and $h^c (0, 1)$.

The technique used here is separation of variables. To use the separation of variables technique, I will break the region into 2 such that the general solution is easily obtainable in each region. I will then require that the 2 solutions agree on their common boundary. These continuity conditions result in 2 linear matrix equations which are then solved.

There are 2 obvious ways to separate the region. I will only discuss the details of the way illustrated in Figure E.1, but I will use results from both. The technique for solving $\nabla^2 \Delta N = 0$ in the diffusive boundary layer is as follows:

1. break up the layer into 2 regions, 1 and 2 (Figure E.1), and define

$$\begin{align*}
\Delta N_1' &= \Delta N_2' \\
\Delta N_1 &= \Delta N_2 \\
h^c_1 &= h^c_2 \\
h^a_1 &= h^a_2
\end{align*}$$

2. solve $\nabla^2 \Delta N_1, 2 = 0$ in each region

3. match the solutions at the boundary between regions 1 and 2.

Figure E.1: separation of regions for the cylinder vapor diffusion calculation

$$\Delta N = \Delta N_{1,2} \text{ in each region}$$

(2) solve $\nabla^2 \Delta N_{1,2} = 0$ in each region

(3) match the solutions at the boundary between regions 1 and 2.
If $F_{v}^{a,c} = 0$, then $\Delta N = 0$. Therefore, one would expect that the solution has a form analogous to that of EQ 3.8;

$$\Delta N' (p', z') = -F_{v}^{c} h_{c} (p', z') - F_{v}^{a} h_{a} (p', z').$$  

Equation E. 1

Therefore

$$\Delta N_{1}' (p', z') = -F_{v}^{c} h_{1}^{c} (p', z') - F_{v}^{a} h_{1}^{a} (p', z')$$  

EQS E. 2

$$\Delta N_{2}' (p', z') = -F_{v}^{c} h_{2}^{c} (p', z') - F_{v}d h_{2}^{a} (p', z').$$

The above result will be derived in the following section.

1. Method 1
   a) General Solution

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![Diagram](image_url)

**Figure E.2**: boundary conditions for region 1

I will now divide the solution $\Delta N_{1}'$ into 2 parts (Figure E.2), such that each has only 1 non-homogeneous boundary condition. Let

$$\Delta N_{1}' = \Delta N_{10}' + F_{v}^{a} \Delta N_{11}'$$  

Equation E. 3

where $\Delta N_{10}'$ has zero flux to the 'a' face, and $\Delta N_{11}'$ has a unit gradient to the crystal (Figure E.3).

The first solution is
Figure E.3: the 2 simpler conditions for region 1

\[ \Delta N'_{10} = \sum_{n=1}^{\infty} B_n^1 \cdot [P_n^1 (\rho') Z_n^1 (z')] = B^1 \cdot E^1 (\rho', z'). \]

where

\[ P_n^1 (\rho') = J_0 (\beta_n^{10} \rho') Y_0 (\beta_n^{10} A') - J_0 (\beta_n^{10} A') Y_0 (\beta_n^{10} \rho') \]

and \( \beta_n^{10} \) are the roots of

\[ J_1 (\beta_n^{10}) Y_0 (\beta_n^{10} A') - J_0 (\beta_n^{10} A') Y_1 (\beta_n^{10}) = 0. \]

The second solution is

\[ \Delta N'_{11} = \frac{2}{\Gamma} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} I_0 (\beta_n^{11} \rho') + K_n^{11} K_n (\beta_n^{11} \rho')}{[\beta_n^{11}]^2} \frac{I_1 (\beta_n^{11}) - K_n^{11} K_n (\beta_n^{11})}{I_1 (\beta_n^{11}) - K_n^{11} K_n (\beta_n^{11})} \cos [\beta_n^{11} (\rho - \rho')]. \]

where,
$B^1$ is calculated below.

**Region 2**

![Diagram showing boundary conditions for region 2]

The solution in region 2 is

$$\Delta N^*_2 = \sum_{n=1}^{\infty} B^2_n \cdot [P^2_n(p') Z^2_n(z')] = B^2 \cdot E^2(p',z').$$

where

$B^2_n$ will be calculated on both sides of the equation later.
Now, by requiring that $\Delta N'$ and $\partial\Delta N'/\partial z'$ be continuous across the boundary, $B^{1,2}$ will be calculated.

Since $\Delta N'_{1}(\rho', 1) = \Delta N'_{2}(\rho', 1)$,

$$\sum_{n=1}^{\infty} B_{n}^{1} \cdot [P_{n}^{1}(\rho')Z_{n}^{1}(1)] = \sum_{n=1}^{\infty} B_{n}^{2} \cdot [P_{n}^{2}(\rho')Z_{n}^{2}(1)].$$

EQE. 7

Now operate on both sides of the equation by

$$\frac{1}{N^{1}_{m}}\int d\rho' \rho' P_{m}^{1}(\rho'),$$

where

$$N^{1}_{m} = \int d\rho' \rho' [P_{m}^{1}(\rho')]^{2}.$$  

Def E. 5

Therefore

$$B_{m}^{1} = \sum_{n=1}^{\infty} B_{n}^{2} \cdot \left[\frac{Z_{n}^{2}(1) A'}{N^{1}_{m}}\int d\rho' \rho' P_{m}^{1}(\rho') P_{n}^{2}(\rho')\right].$$

EQE. 8

in matrix notation,

$$B^{1} = M^{12}B^{2},$$

EQE. 9

where

$$M_{m, n}^{12} = \frac{Z_{n}^{2}(1) A'}{N^{1}_{m}}\int d\rho' \rho' P_{m}^{1}(\rho') P_{n}^{2}(\rho').$$

Def E. 6
Matching the flux at $z' = 1$ for $\Delta N_n$,

$$F_v^c \quad (0 \leq \rho' \leq 1)$$

$$\sum_{n=1}^{\infty} B_n^2 \cdot P_n^2 (\rho') = \sum_{n=1}^{\infty} B_n^1 \cdot P_n^1 (\rho') \frac{d}{dz} Z_n^1 (1) + F_{v1}^a \frac{d}{dz} \Delta N_{11} \quad (1 \leq \rho' \leq A')$$

EQ E. 10

Now operate on both sides by

$$\frac{1}{N_m^2} \int_0^{A'} \! \! d\rho' \rho' P_m^2 (\rho').$$

where

$$N_m^2 = \int_0^{A'} \! \! d\rho' \rho' [P_m^2 (\rho')]^2.$$ 

Def E. 7

Therefore

$$B_m^2 = F_v^c B_m^{2c} + F_v^a B_m^{2a} + \sum_{n=1}^{\infty} B_n^1 \cdot D_{mn}^{21},$$

EQ E. 11

where

$$B_m^{2c} = \frac{1}{N_m^2} \int_0^{A'} \! \! d\rho' \rho' P_m^2 (\rho').$$

Def E. 8

$$B_m^{2a} = \frac{1}{N_m^2} \int_0^{A'} \! \! d\rho' \rho' P_m^2 (\rho') \frac{d}{dz} \Delta N_{11} (\rho', 1)$$

$$D_{mn}^{21} = \frac{d}{dz} Z_n^1 (1) \frac{A'}{N_m^2} \int_0^{A'} \! \! d\rho' \rho' P_m^2 (\rho') P_n^1 (\rho').$$

In matrix form,

$$B^2 = F_v^c B^{2c} + F_v^a B^{2a} + D^{21} B^1.$$  

EQ E. 12
Inserting EQ E. 9 into EQ E. 12, and solving

\[ B^1 = F^c_v M^{12} (1 - D^{21} M^{12})^{-1} B^{2c} + F^a_v M^{12} (1 - D^{21} M^{12})^{-1} B^{2a} \]  
\[ B^2 = F^c_v (1 - D^{21} M^{12})^{-1} B^{2c} + F^a_v (1 - D^{21} M^{12})^{-1} B^{2a} \]  
EQS E. 13

Finally, inserting EQS E. 13 into EQ E. 4, and EQ E. 6

\[ \Delta N'_1 = F^c_v M^{12} (1 - D^{21} M^{12})^{-1} B^{2c} \cdot E^1 (\rho', z') + \]
\[ + F^a_v (M^{12} (1 - D^{21} M^{12})^{-1} B^{2a} \cdot E^1 (\rho', z') + \Delta N'_{11} (\rho', z')) \]  
EQS E. 14

\[ \Delta N'_2 = F^c_v (1 - D^{21} M^{12})^{-1} B^{2c} \cdot E^2 (\rho', z') \]
\[ + F^a_v (1 - D^{21} M^{12})^{-1} B^{2a} \cdot E^2 (\rho', z') \]

Comparing EQS E. 14 and EQS E. 2,

\[ h^c_1 (\rho', z') = -M^{12} (1 - D^{21} M^{12})^{-1} B^{2c} \cdot E^1 (\rho', z') \]
\[ h^a_1 (\rho', z') = -M^{12} (1 - D^{21} M^{12})^{-1} B^{2a} \cdot E^1 (\rho', z') - \Delta N'_{11} (\rho', z') \]  
EQS E. 15

\[ h^c_2 (\rho', z') = -(1 - D^{21} M^{12})^{-1} B^{2c} \cdot E^2 (\rho', z') \]
\[ h^a_2 (\rho', z') = -(1 - D^{21} M^{12})^{-1} B^{2a} \cdot E^2 (\rho', z') \]

Now the elements of the matrices and vectors will be calculated.

b) Evaluation of Terms

Inspection of EQS E. 15 shows that there are 2 matrices, 4 vectors, and one function whose components need to be calculated. These are listed in Table E.1. Once these are known, I will use Mathematica to calculate the matrix and vector operations.

<table>
<thead>
<tr>
<th>type</th>
<th>component</th>
</tr>
</thead>
<tbody>
<tr>
<td>vectors</td>
<td>(B^2c, B^2a, E^1, E^2)</td>
</tr>
<tr>
<td>matrices</td>
<td>(M^{12}, D^{21})</td>
</tr>
<tr>
<td>function</td>
<td>(\Delta N'_{11} ((a)))</td>
</tr>
</tbody>
</table>

Table E.1: components needed for evaluation of \(h^{a, c}\)

Since \(E^{1, 2}\) are already known we only need to simplify \(B^{2c}, B^{2a}, M^{12}, D^{21}\) and
\[ \Delta N'_{11} \left( (a) \right) \].

From Defs E. 4, Def E. 7, and Defs E. 8

\[ B^{2c}_m = \frac{1}{A^2 J_1^2 (\beta_m^2 A')} \int d\rho' \rho' P_m^2 (\rho') \frac{\rho}{2} \frac{J_1 (\beta_m^2)}{\beta_m^2} . \]

Therefore

\[ B^{2c}_m = \frac{2 J_1 (\beta_m^2)}{\beta_m^2 A^2 J_1^2 (\beta_m^2 A')} . \quad \text{EQ E. 16} \]

From Defs E. 4, Def E. 7, Defs E. 8, and EQ E. 5

\[ B^{2a}_m = \frac{1}{A^2 J_1^2 (\beta_m^2 A')} \int d\rho' \rho' P_m^2 (\rho') \frac{\rho}{2} \Delta N'_{11} (\rho', 1) = \frac{2}{A^2 J_1^2 (\beta_m^2 A')} \int d\rho' \rho' J_0 (\beta_m^2 \rho') . \]

Therefore

\[ \frac{2}{A^2 J_1^2 (\beta_m^2 A')} \int d\rho' \rho' J_0 (\beta_m^2 \rho') \left[ \frac{\rho}{2} \Delta N'_{11} (\rho', 1) \right] . \]

\[ = \frac{4}{\frac{1}{A^2 J_1^2 (\beta_m^2 A')} \sum_{n=1}^{\infty} \frac{1}{\beta_n^{11} (I_1 (\beta_n^{11}) - K_n^{11} K_1 (\beta_n^{11}))} \cdot \int d\rho' \rho' J_0 (\beta_m^2 \rho') \left[ I_0 (\beta_n^{11} \rho') + K_n^{11} K_0 (\beta_n^{11} \rho') \right] . \]

The latter integral is a special case of Int E. 3 and Int E. 4.
\[
\int d\rho' \rho' J_0(\beta^2 m \rho') [I_0(\beta^{11}_n \rho') + K^{11}_n K_0(\beta^{11}_n \rho')] \\
= \frac{1}{(\beta^{11}_n)^2 + (\beta^{2}_m)^2} \left[ A' \{\beta^2 m J_1(\beta^{2}_m A') - I_0(\beta^{11}_n A') + \beta^{11}_n J_0(\beta^{2}_m A') \cdot I_1(\beta^{11}_n A') \} \right. \\
\left. \quad - \{\beta^2 m J_1(\beta^{2}_m) \cdot I_0(\beta^{11}_n) + \beta^{11}_n J_0(\beta^{2}_m) \cdot I_1(\beta^{11}_n) \} + \\
\quad + K^{11}_n A' \{\beta^2 m J_1(\beta^{2}_m A') - I_0(\beta^{11}_n A') - \beta^{11}_n J_0(\beta^{2}_m A') \cdot K_1(\beta^{11}_n A') \} \right] \\
\quad - K^{11}_n \{\beta^2 m J_1(\beta^{2}_m) \cdot K_0(\beta^{11}_n) - \beta^{11}_n J_0(\beta^{2}_m) \cdot K_1(\beta^{11}_n) \} \\
\Rightarrow B^{2a}_m = \\
\frac{4}{A^2 J^2_1(\beta^{2}_m A')} \sum_{n=1}^{-\infty} \frac{-1}{(I_1(\beta^{11}_n) - K^{11}_n K_1(\beta^{11}_n))} \cdot \frac{1}{(\beta^{11}_n)^2 + (\beta^{2}_m)^2} \cdot \\
\quad \{\beta^2 m J_1(\beta^{2}_m) \cdot [I_0(\beta^{11}_n) + K^{11}_n K_0(\beta^{11}_n)] + \beta^{11}_n J_0(\beta^{2}_m) \cdot [I_1(\beta^{11}_n) - K^{11}_n K_1(\beta^{11}_n)] \} \\
\text{Therefore} \\
B^{2a}_m = -\frac{4 \beta^2 m J_1(\beta^{2}_m)}{A^2 J^2_1(\beta^{2}_m A')} \sum_{n=1}^{-\infty} \frac{I_0(\beta^{11}_n) + K^{11}_n K_0(\beta^{11}_n)}{I_1(\beta^{11}_n) - K^{11}_n K_1(\beta^{11}_n)} \cdot \frac{1}{(\beta^{11}_n)^2 + (\beta^{2}_m)^2} + \\
\quad \frac{4 J_0(\beta^{2}_m)}{A^2 J^2_1(\beta^{2}_m A')} \sum_{n=1}^{-\infty} \frac{1}{(\beta^{11}_n)^2 + (\beta^{2}_m)^2} \\
\text{The second sum can be evaluated exactly} \\
\sum_{n=1}^{-\infty} \frac{1}{(\beta^{11}_n)^2 + (\beta^{2}_m)^2} = \frac{\Gamma}{2 \beta^2_m} \text{Tanh} [\beta^2_m \Gamma]. \quad \text{EQ. 18} \\
\text{The first sum can be re-arranged such that it needs fewer terms for a given accuracy. Since the first factor in the first sum approaches -1 as n increases,}
This re-arrangement is useful because the first sum converges rapidly, while the second sum may be summed exactly

\[
\sum_{n=1}^{\infty} \left[ \frac{I_0(\beta_{n1}) + K_{n1} K_0(\beta_{n1})}{I_1(\beta_{n1}) - K_{n1} K_1(\beta_{n1})} + 1 \right] \cdot \frac{1}{\beta_{n1}^2 \left[ (\beta_{n1})^2 + (\beta_{m2})^2 \right]} + \sum_{n=1}^{\infty} \frac{-1}{\beta_{n1}^2 \left[ (\beta_{n1})^2 + (\beta_{m2})^2 \right]}
\]

where \( \gamma \) is Euler’s constant \((- 0.5772)\), and \( \psi \) is the Digamma function \([36]\).

Therefore

\[
B_{2a}^2 = -\frac{4B_m^2 J_1(\beta_m^2)}{A^2 J_1^2(\beta_m^2 A')} \sum_{n=1}^{\infty} \left[ \frac{I_0(\beta_{n1}) + K_{n1} K_0(\beta_{n1})}{I_1(\beta_{n1}) - K_{n1} K_1(\beta_{n1})} + 1 \right] \cdot \frac{1}{\beta_{n1}^2 \left[ (\beta_{n1})^2 + (\beta_{m2})^2 \right]} + \frac{4J_1(\beta_m^2)}{A^2 J_1^2(\beta_m^2 A')} \frac{\Gamma}{\pi} \left\{ \gamma + 2Ln [2] + Re \left[ \psi \left( \frac{1}{2} + i\beta_m^2 \frac{\Gamma}{\pi} \right) \right] \right\} \cdot \text{EQ E.20}
\]

\[
\frac{\Delta N_{11}^a (a)}{A^2 J_1^2(\beta_m^2 A')} = B_{2a}^2 \frac{\Gamma}{2\beta_m^2} \left\{ \gamma + 2Ln [2] + Re \left[ \psi \left( \frac{1}{2} + i\beta_m^2 \frac{\Gamma}{\pi} \right) \right] \right\} \cdot \text{EQ E.20}
\]

This will be evaluated at 2 points \( (a) = \{ (1, 1), (1, 0) \} \) (the corner and the center). \( \Delta N_{11}^a (1, 1) = 0 \). When \( (a) = (1, 0) \),
Rearranging the terms,
\[ \Delta N_{11}^*(1, 0) = 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\beta_n^{11}} \left[ \frac{I_0(\beta_n^{11}) + K_{n1}^{11}K_0(\beta_n^{11})}{I_1(\beta_n^{11}) - K_{n1}^{11}K_1(\beta_n^{11})} + 1 \right] \]

Summing the second sum
\[ \Delta N_{11}^*(1, 0) = -2 \sum_{n=1}^{\infty} \frac{(-1)^{n}}{\beta_n^{11}} \left[ \frac{I_0(\beta_n^{11}) + K_{n1}^{11}K_0(\beta_n^{11})}{I_1(\beta_n^{11}) - K_{n1}^{11}K_1(\beta_n^{11})} + 1 \right] - \frac{8\gamma_c}{\pi^2}. \text{ EQS E. 21} \]

where \( \gamma_c \) is Catalan's constant \( \approx 0.9159. \)

Using Int E. 5, and simplifying
\[ M_{m,n}^{12} = \frac{\pi^2}{2} \left[ \beta_m^{10} \right]^2 \left[ J_1(\beta_m^{10}) \right]^2 \frac{Tanh \left[ \beta_n^{2} \Gamma (1 - C') \right]}{\beta_n^{2} \Gamma} \int dp' \left[ \left( J_0(\beta_m^{10}p')Y_0(\beta_m^{10}A') - J_0(\beta_m^{10}A')Y_0(\beta_m^{10}p') \right) \cdot \left( J_0(\beta_n^{2}p') \right) \right] \text{ EQ. E. 22} \]
\[
D_{mn}^{21} = \frac{d}{dz} Z_n^1(1) \Lambda^1 \int dp' \rho' P_m^2(p') P_n^1(p') = \frac{1^{10} \Gamma \tanh[1^{10} \Gamma]}{A^{2} J_1^2(\beta_m^2 A')} \cdot \frac{N_n^1}{Z_m^2(1) M_{nm}^{12}}
\]

\[
= \frac{1^{10} \Gamma \tanh[1^{10} \Gamma]}{A^{2} J_1^2(\beta_m^2 A')} \cdot \frac{\beta_m^2 J_1(\beta_m^2) P_n^1(1)}{(1^{10} \beta_m^2 - (\beta_m^2)^2)}
\]

Therefore

\[
D_{mn}^{21} = \frac{1^{10} \Gamma \tanh[1^{10} \Gamma]}{A^{2} J_1^2(\beta_m^2 A')} \cdot \frac{\beta_m^2 J_1(\beta_m^2) P_n^1(1)}{(1^{10} \beta_m^2 - (\beta_m^2)^2)}
\]

EQE. 23

2. Method 2

Given the amount of algebra in the previous calculation, it is important to search for alternative methods to compare it with. Dividing up the diffusive boundary layer differently is one such way, and is described in this section.

The region is now divided up as shown in Figure E.5.

The algebra and evaluation of terms is very much like that in method 1. I will simply give the results here.

\[
h_1^c(p', z') = -(1 - D^{12} M^{21})^{-1} B^{1c} \cdot E^1(p', z')
\]

\[
h_1^a(p', z') = -(1 - D^{12} M^{21})^{-1} B^{1a} \cdot E^1(p', z')
\]

\[
h_2^c(p', z') = -M^{21} (1 - D^{12} M^{21})^{-1} B^{1c} \cdot E^2(p', z') - \Delta N_{21}^c(p', z')
\]

\[
h_2^a(p', z') = -M^{21} (1 - D^{12} M^{21})^{-1} B^{1a} \cdot E^2(p', z')
\]

where

\[
M_{mn}^{21} = \frac{2}{(C - 1)} \frac{\sin[\beta_n^1 \Gamma]}{\beta_n^2^2 - [\beta_n^1]^2} \cdot \frac{I_0(\beta_n^1) + K_n^1 K_0(\beta_n^1)}{I_1(\beta_n^1) - K_n^1 K_1(\beta_n^1)}
\]

EQE. 25
Figure E.5: dividing up the diffusive boundary layer in method 2

\[
D_{mn}^{12} = \frac{2}{C} \frac{\beta_n^{20} \beta_m^{11} \sin[\beta_m^{11} \Gamma]}{[\beta_n^{20}]^2 - [\beta_m^{11}]^2} I_1(\beta_n^{20}) \tag{Eq. E.26}
\]

\[
B_m^{1a} = \frac{2}{C} \frac{\sin[\beta_m^{11} \Gamma]}{\beta_m^{11}} \tag{Eq. E.27}
\]

\[
B_m^{1c} = \frac{-4}{C \Gamma^2} \sum_{n=1}^{\infty} \frac{\beta_n^{21} \cos[\beta_m^{11} \Gamma] + \beta_m^{11} \sin[\beta_m^{11} \Gamma] \tanh[\beta_n^{21} \Gamma(1 - C)]}{\beta_n^{21} \{[\beta_m^{11}]^2 + [\beta_n^{21}]^2\}} \tag{Eq. E.28}
\]

\[
\Delta N_{21}(\rho', z') = \frac{2}{\Gamma} \sum_{n=1}^{\infty} \frac{J_0(\beta_n^{21} \rho')}{{J_1(\beta_n^{21})}} \frac{\sinh[\beta_n^{21} \Gamma(z' - C)]}{2 \cosh[\beta_n^{21} \Gamma(1 - C)]} \tag{Eq. E.29}
\]

While

\[
E_n^1(\rho', z') = \frac{I_0(\beta_n^{11} \rho') + K_0(\beta_n^{11} \rho')}{I_1(\beta_n^{11}) - K_1(\beta_n^{11})} \cdot \cos[\beta_n^{11} \Gamma z'] \tag{Defs E.9}
\]

\[
E_n^2(\rho', z') = \frac{I_0(\beta_n^{20} \rho')}{I_0(\beta_n^{20})} \cdot \cos[\beta_n^{20} \Gamma(z' - 1)]
\]
The infinite sum in EQ E. 28 for $B^{1c}_m$ can be expressed in a form requiring less terms in a similar manner as $B^{2a}_m$ in EQ E. 20

$$B^{1c}_m = \frac{-4 \cos \left[ \beta^1_m \Gamma \right]}{C \Gamma^2} \left\{ \sum_{n=1}^{\infty} \left[ \frac{1}{\left[ \beta^1_m \right]^2 + \left[ \beta^{21}_n \right]^2} - \frac{1}{\left[ \beta^1_m \right]^2 + \left[ \beta^{21}_n \right]^2} \right] \right\}$$

$$+ \frac{\beta^1_m \tan \left[ \beta^1_m \Gamma \right]}{\beta^{21}_n} \left( \frac{\tanh \left[ \beta^{21}_n \Gamma (1 - C') \right]}{\left[ \beta^1_m \right]^2 + \left[ \beta^{21}_n \right]^2} - \frac{\beta^{21}_n}{\beta^1_m} \left\{ \frac{\left[ \beta^1_m \right]^2 + \left[ \beta^{21}_n \right]^2}{\left[ \beta^1_m \right]^2 + \left[ \beta^{21}_n \right]^2} \right\} \right)$$

$$+ \frac{1}{\pi \beta^1_m} \text{Im} \left[ \Psi \left( \frac{3}{4} + i \cdot \frac{\beta^1_m}{\pi} \right) \right]$$

$$+ \beta^1_m \tan \left[ \beta^1_m \Gamma \right] \frac{1}{\pi \left[ \beta^1_m \right]^2} \left( \gamma + 3 \ln 2 - \frac{\pi}{2} + \text{Re} \left[ \Psi \left( \frac{3}{4} + i \cdot \frac{\beta^1_m}{\pi} \right) \right] \right)$$

where $\beta^{21}_n = (n - 1/4) \pi$ is the large $n$ approximation to $\beta^{21}_n$.

3. Results

Choosing particular values of $\rho'$, $z'$, $\Gamma$, $A'$, $C'$ for the terms in EQ E. 16, EQ E. 20, EQS E. 21, EQ E. 22 and EQ E. 23, and inserting them into EQS E. 15, the $h$ functions are calculated. Similarly for method 2. It was found that the accuracy of these 2 methods decreased as:

1. the size of the matrices are reduced (I found that large matrices are needed for accurate results; 800 by 800 matrices were used to obtain the results given below.)

2. $A'$, $C'$ increased (see Figure E.6 and Figure E.8)
(3) $|\log[T]|$ increased (compare Figure E.7 with Figure E.6 and Figure E.8).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure.png}
\caption{$h_c^e(1,0)$ as a function of $C'$ for $\Gamma = 100$}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure.png}
\caption{$h_c^e(1,0)$ as a function of $A' = C'$ for $\Gamma = 1$}
\end{figure}

The exact nature of diffusive boundary layer thickness is unknown, except that it increases as the crystals decrease in size and fall through the air at smaller speeds. Therefore I will focus my attention on the limit $A', C' \to \infty$. For this limit, the observation
Figure E.8: \( h^a(0, 1) \) as a function of \( A' \) for \( \Gamma = 0.01 \)

\[
h(\rho', z', \Gamma, A', C = \frac{A'}{\Gamma}) = h(\rho', z', \Gamma, \infty, \infty) \cdot (1 - \frac{s^A}{A'}) \tag{EQ E.31}
\]

is used for estimating \( h(\rho', z', \Gamma, \infty, \infty) \), where \( s^A \) is a function of \( \rho', z', \Gamma \). The curves with the form of EQ E.31 which best fit the calculated points in Figure E.6, Figure E.7, and Figure E.8 are shown as solid curves.

Shown in Figure E.9 and Figure E.10 are the two \( h \) functions used most frequently in chapter 3 \( h^{c,a}(1, 1) \). These functions can be fairly well represented by the following functions:

\[
h^c(1, 1) = \frac{1}{\Gamma} \left\{ \frac{1.15}{\pi} + \frac{0.85}{\pi} \left( 1 - \text{Tanh} \left[ 0.9576 \left( 0.486 + \text{Log}_{10} [\Gamma] \right) \right] \right) \right\} \tag{EQS E.32}
\]

\[
h^a(1, 1) = 0.608 \cdot \text{Ln} \left[ 1 + 2.11 \cdot \Gamma^{0.861} \right]
\]

The other \( h \) functions are shown below.

4. Checks on Accuracy

The accuracy of the method used here was checked several ways. First of all, numerical values were calculated using both method 1 and method 2. The 2 results generally were consistently within 1% of each other. For a given method, there are additional checks. For instance, the values of \( h^{a,c}(1, 1) \) are calculated 2 ways, \( h_1^{a,c}(1, 1) \), and \( h_2^{a,c}(1, 1) \). These values differed from each other by 1% or less. In the results given here, the average of the 2 values were used. In addition, the values for \( h^c(1, 1) \) and \( h^c(0, 1) \) match those of the thin disc functions when
Figure E.9: $h^c(1, 1)$ as a function of $\Gamma$ for $\Lambda', C \to \infty$. The approximation in EQS E. 32 is shown as a solid curve.

Figure E.10: $h^a(1, 1)$ as a function of $\Gamma$ for $\Lambda', C \to \infty$. The approximation in EQS E. 32 is shown as a solid curve.

$\Gamma \to 0$. It is felt that the numerical results given here are accurate.
Figure E.11: \( h^c(1,0) \) for \( A', C' \to \infty \)

Figure E.12: \( h^a(1,0) \) for \( A', C' \to \infty \)
Figure E.13: $h^c(0, 1)$ for $A', C \to \infty$

Figure E.14: $h^\sigma(0, 1)$ for $A', C \to \infty$
The data points plotted in the figures above are listed in Table E.2.

### Table E.2: calculated values of the $h$ functions for an infinite thickness diffusive boundary layer

<table>
<thead>
<tr>
<th>$\Gamma$</th>
<th>$h^c(1, 1)$</th>
<th>$h^a(1, 1)$</th>
<th>$h^c(1, 0)$</th>
<th>$h^a(1, 0)$</th>
<th>$h^c(0, 1)$</th>
<th>$h^a(0, 1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>.01</td>
<td>61.96</td>
<td>0.02332</td>
<td>60.90</td>
<td>0.03080</td>
<td>99.70</td>
<td>0.009799</td>
</tr>
<tr>
<td>.05</td>
<td>11.73</td>
<td>0.09001</td>
<td>11.03</td>
<td>0.1209</td>
<td>19.71</td>
<td>0.04607</td>
</tr>
<tr>
<td>.1</td>
<td>5.618</td>
<td>0.1547</td>
<td>5.042</td>
<td>0.2140</td>
<td>9.744</td>
<td>0.08674</td>
</tr>
<tr>
<td>.2</td>
<td>2.650</td>
<td>0.2575</td>
<td>2.205</td>
<td>0.3674</td>
<td>4.786</td>
<td>0.1575</td>
</tr>
<tr>
<td>.5</td>
<td>0.9595</td>
<td>0.4693</td>
<td>0.6641</td>
<td>0.7072</td>
<td>1.853</td>
<td>0.3183</td>
</tr>
<tr>
<td>1</td>
<td>0.4450</td>
<td>0.6903</td>
<td>0.2432</td>
<td>1.086</td>
<td>0.902</td>
<td>0.5064</td>
</tr>
<tr>
<td>2</td>
<td>0.2081</td>
<td>0.9632</td>
<td>0.08162</td>
<td>1.576</td>
<td>0.439</td>
<td>0.7536</td>
</tr>
<tr>
<td>5</td>
<td>0.07807</td>
<td>1.377</td>
<td>0.01660</td>
<td>2.360</td>
<td>0.1712</td>
<td>1.145</td>
</tr>
<tr>
<td>10</td>
<td>0.03735</td>
<td>1.705</td>
<td>0.004577</td>
<td>3.016</td>
<td>0.08437</td>
<td>1.466</td>
</tr>
<tr>
<td>20</td>
<td>0.01836</td>
<td>2.048</td>
<td>0.001199</td>
<td>3.695</td>
<td>0.04192</td>
<td>1.804</td>
</tr>
</tbody>
</table>
5. Notes to Appendix E

a) Some Useful Integrals

\[
\int_0^\beta dx \cdot x \cdot J_0(x) = \beta J_1(\beta) \quad \text{Int E. 1}
\]

\[
\int_0^\beta dx \cdot x \cdot [J_0(x)]^2 = \frac{\beta^2}{2} \left[ [J_0(x)]^2 + [J_1(x)]^2 \right] \quad \text{Int E. 2}
\]

In the following 2 integrals, \( \zeta_0 \) and \( \Upsilon_0 \) can be either \( J_0 \) or \( Y_0 \).

\[
(c^2 + d^2) \int_1^A dx \cdot x \cdot \zeta_0(cx) \cdot I_0(dx) = \\
= A \left\{ c\zeta_1(cA) \cdot I_0(dA) + d\zeta_0(cA) \cdot I_1(dA) \right\} \\
- \left\{ c\zeta_1(c) \cdot I_0(d) + d\zeta_0(c) \cdot I_1(d) \right\} 
\quad \text{Int E. 3}
\]

\[
(c^2 + d^2) \int_1^A dx \cdot x \cdot \zeta_0(cx) \cdot K_0(dx) = \\
= A \left\{ c\zeta_1(cA) \cdot K_0(dA) - d\zeta_0(cA) \cdot K_1(dA) \right\} \\
- \left\{ c\zeta_1(c) \cdot K_0(d) - d\zeta_0(c) \cdot K_1(d) \right\} 
\quad \text{Int E. 4}
\]

\[
(c^2 - d^2) \int_1^A dx \cdot x \cdot \zeta_0(cx) \cdot \Upsilon_0(dx) = \\
= A \left\{ c\zeta_1(cA) \cdot \Upsilon_0(dA) - d\zeta_0(cA) \cdot \Upsilon_1(dA) \right\} \\
- \left\{ c\zeta_1(c) \cdot \Upsilon_0(d) - d\zeta_0(c) \cdot \Upsilon_1(d) \right\} 
\quad \text{Int E. 5}
\]
Vita

Jon Thomas Nelson received his bachelor of science degree in Physics from the University of Washington in August of 1985 and a master of science degree in Physics from the University of Illinois at Urbana-Champaign in October of 1988.