Figure 2.17: condensation coefficient vs. surface supersaturation: ledge nucleation

Contrast to that for spiral ledges, is the sharp cutoff; for \( \sigma_s < \sigma_{cr} \), practically no ledges are produced, while for \( \sigma_s > \sigma_{cr} \), the ledges are closely spaced. The important parameter is \( \sigma_{cr} \), which is a property of the crystal face and may depend on temperature. Some authors [1], [82] have suggested that most ledges on ice crystals in mixed phase clouds are produced from the ledge nucleation mechanism. If this is the case, then \( \sigma_{cr} < \) typical values of \( \sigma_{w} \) (see Figure 2.1).

We can now substitute the relation between \( \alpha \) and \( \sigma_s \) (EQ 2.17, EQ 2.18 or EQS 2.19) in EQ 2.8 (which relates \( \sigma_s / \sigma_{cr} \) and \( \sigma_{w} \)) to calculate \( \alpha \) as a function of the environmental conditions, the crystal size, and the type of ledge source.

Consider the case of spiral ledges where \( \Delta_g \approx x_1 \). Assuming that \( \sigma_{w} / \sigma_{1} < 1 + a'h \),

\[
\alpha \cdot a'h = \frac{1}{4} \sigma_{w} - \frac{1}{2} a'h - \frac{1}{2}.
\]  
(EQ 2.20)

(The inclusion of heat diffusion changes \( a' \rightarrow a' + a'h \).) Note that the left hand side is the ratio of the vapor impedance to the surface impedance. When \( \alpha \cdot a'h \) is small, then surface processes are growth rate controlling, and when it is large, vapor diffusion are growth rate controlling. A expression similar to EQ 2.20 is obtained when \( \Delta_g \ll x_1 \). The magnitude of \( \alpha \) for the latter case is shown in Figure 2.18.

4. Comparison with Experiment
Figure 2.18: condensation coefficient vs. vapor impedance: spiral ledges with $\Delta_x \ll \chi$.

The only experimental data which one may compare to these calculations are those of Shaw and Mason [7]. This is the only study in which the growth rate of each face of an ice crystal growing in air was monitored for a range of supersaturations, temperatures, and crystal sizes. They observed the following:

\[
\frac{dr}{dt} \propto \frac{\sigma_\infty}{r}, \tag{EQ 2.21}
\]

where $r$ is a linear dimension of the crystal. The crystal sizes varied from approximately 20 to 60$\mu$ in diameter ($a' = 35 - 100$), although the size for a particular crystal rarely doubled during the observations. The range of supersaturations over which EQ 2.21 held was approximately -15% to 15% at T=-11.4$^\circ$C and -20% to 35% at T=-17.8$^\circ$C (the range varied from crystal to crystal, but generally increased with a decrease in temperature). Since the flux to each face was significantly less than that predicted by EQ 2.9 with $1/\alpha \to 0$, we may conclude (if the present model applies to their experiment) that $\alpha a'h < 1$. (The heat was conducted to a metal substrate; therefore $a'h$ was probably small.) Since

\[
\frac{da}{dt} = \Omega \cdot \text{flux}_v, \tag{EQ 2.22}
\]

then on the basis of the growth model presented so far:

\[
\frac{da}{dt} \propto \frac{\alpha \cdot \sigma_\infty}{1 + \alpha \cdot a'h}. \tag{EQ 2.23}
\]
I will now compare EQ 2.23 to EQ 2.21 for spiral ledges and nucleated ledges.

Inserting EQ 2.20 into EQ 2.23, we can see that spiral ledges with $\Delta g > x$, cannot satisfy EQ 2.21. The growth rate is $\propto \sigma^2$ at low supersaturations, but it is not inversely proportional to $a$.

The growth rate is shown for the case of spiral ledges with $\Delta g < x$, in Figure 2.19.

Figure 2.19: growth rate for spiral ledges with $\Delta g < x$

The growth rate for this case is proportional to $\sigma^2$ only when $\sigma < = 0.5 \sigma_1$, and
is approximately inversely proportional to \( a' \) when \( \sigma_\infty > 0.2 \sigma_1 \). Although this range is much smaller than that observed by Shaw and Mason, since the crystals were only observed for a short period of time, the growth rate dependence on \( a' \) shown in Figure 2.19 may appear to fit EQ 2.21 over a much larger range of \( \sigma_\infty \).

When ledges are nucleated we have the situation shown in Figure 2.20. Note that the growth rate is not proportional to \( \sigma_\infty^2 \).

In addition to these data, Keller and Hallet [39], and Gonda and Koike [60] found

\[
\frac{dr}{dt} \propto \sigma_\infty^n \quad (n > 1)
\]

EQ 2.24

for ice crystals grown in air and supersaturations in the same range as those of Shaw and Mason.

Although it is clear that surface impedance is comparable in magnitude to vapor impedance (otherwise \( n = 1 \) in EQ 2.24), it is not certain if EQ 2.23 is in agreement with experiment. At very low supersaturations, it almost certainly does not agree, but in this limit, the assumptions which led to EQ 2.23 do not hold. In chapter 4 these assumptions will be relaxed and the resulting model satisfies EQ 2.21. At larger supersaturations, the observations of Shaw and Mason are roughly in agreement with the model presented here if growth occurs via spiral ledges with \( \Delta g \ll x_r \).

5. Summary

In this chapter, I have focussed on the simplest model of ice growth in the atmosphere that includes the following processes:

i) vapor diffusion
ii) heat diffusion
iii) surface diffusion.

I have discussed their roles in the growth process. It was shown that:

i) vapor diffusion is important throughout the atmosphere, but has a greater influence on large crystals
ii) heat diffusion is important at high temperatures, and on large crystals
iii) surface diffusion is important on small crystals and small supersaturations.
This is summarized in Figure 2.21 below. In each corner of the triangle is the dominant growth rate limiting process, while the edges indicate the important variable(s): temperature, size, or supersaturation.

Although the model presented in this chapter contains some of the important features of ice crystal growth in the atmosphere, it does not include non-spherical crystal shapes, it cannot describe crystals with a small number of ledges per face, nor crystals with $\Delta g < y$.

Therefore I have derived a new model which better fits crystal growth in the atmosphere. In the next chapter I introduce the simplest solvable problem of nonspherical crystal growth: namely, the finite cylinder. This case is also unrealistic in several crucial ways, but it leads to the final models, presented in chapter 4.
6. Notes to Chapter 2

a) Radiative Heating and Cooling

A crystal in the atmosphere may undergo heating or cooling due to an imbalance of radiative flux at its surface. The effect this has on growth and evaporation rates is the topic of this note.

The steady state surface temperature of a spherical ice crystal in the atmosphere is determined by the balance of heat conduction, latent heating and radiative heating at the surface;

\[ \frac{k}{h(R')} (T_s - T_m) = \int \frac{kT_m da}{\Omega} + \mathcal{R}, \]

EQ 2.25

where \( \mathcal{R} \) is the net radiative heat flux. To determine \( \mathcal{R} \), consider the idealized situation shown in Figure 2.22. The radiative flux from the upper half plane which is absorbed by the ice crystal is represented by \( \sigma^{sb} T_s^4 \), while the radiative flux from the bottom half plane is represented by \( \sigma^{sb} T_b^4 \), where \( \sigma^{sb} \) is the Stefan-Boltzmann constant. (Solar heating will be neglected here.) The net radiative flux to the crystal
where the last term in EQ 2.26 has been dropped from EQ 2.27 since it is much smaller than the left hand side of EQ 2.25.

The growth rate is

\[ \frac{dr}{dt} = \frac{\bar{\Omega} N_0}{4} \left( \sigma_w + \Delta \sigma_{rad} \right), \]  

where

\[ \Delta \sigma_{rad} = (l' - 1) \frac{\sigma^{sb} T^3 a}{2 \kappa_g} \left( 2 - \left( \frac{T_b}{T_w} \right)^4 - \left( \frac{T_a}{T_w} \right)^4 \right). \]  

Since the air temperature below the crystal will generally be warmer than \( T_w \), the maximum amount of radiative cooling will occur when

\[ T_b = T_w \quad T_a = 0. \]  

For this case

\[ \Delta \sigma_{rad} = (l' - 1) \frac{\sigma^{sb} T^3 a}{2 \kappa_g}. \]  

Note that the effect of radiative cooling is to shift the supersaturation by an amount proportional to the ambient temperature cubed and the crystal radius. The resulting increase in supersaturation for a range of ice crystal sizes is shown in Figure 2.23.

The maximum amount of radiative heating will occur when

\[ T_a = T_w, \quad T_b = T_{ground} = T_w + 7 \cdot h, \]  

where \( h \) is the altitude in kilometers. For this case
Figure 2.23: maximum shift in supersaturation in the atmosphere due to radiative cooling (the initial and mean crystal sizes are calculated using Eqs 1.4)

\[ \Delta \sigma_{rad} = -(\ell' - 1) \frac{\sigma S^3 a}{2 \kappa g} \cdot \frac{h}{T_m} \]  

EQ 2.32

For ice clouds, the maximum change in supersaturation due to radiative heating is approximately the same as the maximum change due to radiative cooling, but opposite in sign.

Depending on the local conditions, the shift in supersaturation will be between EQ 2.30 and EQ 2.32. Given that the uncertainty in the supersaturation in ice clouds is greater than the values given by EQ 2.30 and EQ 2.32 for the smaller crystal sizes, and since the larger crystal sizes will not be treated here, radiative heating and cooling will be ignored. However, it may not always be negligible in the atmospheric context.