Modeling Ice Crystal Aspect Ratio Evolution during Rimming:
A Single-Particle Growth Model

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(Manuscript received 6 October 2014, in final form 19 February 2015)

ABSTRACT

This paper describes and tests a single-particle ice growth model that evolves both ice crystal mass and shape as a result of vapor growth and riming. Columnar collision efficiencies in the model are calculated using a new theoretical method derived from spherical collision efficiencies. The model is able to evolve mass, shape, and fall speed of growing ice across a range of temperatures, and it compares well with wind tunnel data. The onset time of riming and the effects of riming on mass and fall speed between -3°C and -16°C are modeled, as compared with wind tunnel data for a liquid water content of 0.4 g m⁻³. Under these conditions, riming is constrained to the more isometric habits near -10°C and -4°C. It is shown that the mass and fall speed of riming dendrites depend on the liquid drop distribution properties, leading to a range of mass–size and fall speed–size relationships. Rimming at low liquid water contents is shown to be sensitive to ice crystal habit and liquid drop size. Moreover, very light riming can affect the shape of ice crystals enough to reduce vapor growth and suppress overall mass growth, as compared with those same ice crystals if they were unrimed.

1. Introduction

Ice crystal habit evolution is an indispensable part of clouds microphysics. The dynamics and lifetime of ice-containing clouds depend on intricate pathways among ice nucleation, growth, sedimentation, and thermal forcings. Ice commonly coexists with liquid in persistent mixed-phase clouds despite ice’s glaciating ability (Curry et al. 1996; Pinto 1998; Morrison et al. 2012). The liquid in these clouds dominantly controls the cloud radiative cooling that drives buoyant dynamics, while ice grows rapidly at the expense of liquid by vapor diffusion and riming. Subsequent precipitation results in a net loss of liquid, which can decrease the dynamic driver and cause the cloud to spin down and dissipate (Ovchinnikov et al. 2011). Therefore, phase partitioning in these clouds can control cloud lifetime (Harrington et al. 1999), potentially influencing Arctic climate (Curry et al. 1996; Morrison et al. 2012). Even in deep convection, graupel is a pertinent precursor for precipitation (Rutledge and Hobbs 1984). Besides its influence on quantitative precipitation forecasts (Gilmore et al. 2004; Milbrandt and Morrison 2013), precipitation is inseparably linked to storm dynamics (Bryan and Morrison 2012). The job of microphysical models is to simulate these multiphase growth processes to help us understand cloud dynamics and precipitation.

Modeling ice growth consists of equations for increase in mass. However, the basic equations, capacitance for vapor growth and collection for riming and aggregation, generally contain no explicit information regarding changes in shape. These equations provide instantaneous growth rates as a function of ice particle properties, including size and fall speed, but also require shape. To evolve shape, the mass-growth equations must be supplemented with equations or parameterizations for how mass is distributed over a particle. Many models assume a spherical shape with a reduced density to account for the larger spherical volume of any nonspherical ice. Other models use mass–dimensional (m–D) relationships derived from in situ data (Mitchell 1996, their Table 1). These power laws, of the form \( m_x = \alpha_x D_x^{\beta_x} \) for some ice species \( x \), relate mass \( m_x \) to the maximum dimension of the particle \( D_x \) and so implicitly contain shape and density information. Consequently, \( m–D \) relationships provide a way to distribute mass and, indirectly, change particle shape. But the vast range of ice species in clouds, from quasi-spherical graupel to thin, vapor-grown dendrites,
means numerous $m-D$ relationships are needed to represent ice growth.

Early and current ice models (e.g., Ferrier 1994) rely on separating ice species based on roughly defined nucleation and growth categories, such as cloud ice and snow for vapor-grown ice, graupel for rimed ice, and aggregates for ice collection. Each species is characterized using independent $m-D$ relationships, allowing, for example, graupel and dendrites of the same size to have different masses and different fall speeds (usually defined by a separate fall speed–size relationship). This method of modeling, although practical, is limited and somewhat artificial for two main reasons. First, the way different ice category sizes are modeled to grow is predetermined by their $m-D$ relationships. Hence, the particle size is constrained to follow the $m-D$ relationship independent of any changes that occur in the environmental conditions or the ice particle itself. Though $m-D$ relationships are useful for separating growth characteristics of distinct ice categories, they constrain ice growth of those individual categories themselves. For instance, most ice particles are nearly isometric when they nucleate and develop various shapes over time. But $m-D$ relationships do not allow these smooth transitions; they either implicitly assume a branched dendrite always has a branched shape, or they must be instantaneously changed to account for a change in growth characteristics. Some methods use spherical ice for sizes less than some threshold, with nonspherical $m-D$ relations used beyond the threshold size. However, because the mass–size coefficients ($\alpha$, $\beta$) depend on ice particle growth (Chen and Lamb 1994), it is difficult to know if joining two mass–size relations in this way is justified. Second, converting between ice categories happens instantaneously in models that rely on $m-D$ relationships requiring artificial thresholds between categories (e.g., Walko et al. 1995; Woods et al. 2005). In contrast, real transitions between particles happen gradually, for example, from an isometric crystal to a dendrite or from an ice crystal to graupel. These transitional states include lightly rimed and heavily rimed crystals and are generally ignored in models. Instead, rimed snow retains its snow characteristics, such as fall speed, until it meets the criterion to become graupel. Differences in snow, graupel, and conversion parameterizations can lead to significant precipitation differences in models, specifically for weakly forced and light riming cases, such as orographic precipitation (Colle and Zeng 2004; Colle et al. 2005; Milbrandt et al. 2008).

A new paradigm in microphysical modeling predicts particle properties. For instance, Morrison and Grabowski (2008) predict rime fraction by fitting $m-D$ relationships together for vapor-grown and rimed ice. This removes discontinuities in mass that arise from an instantaneous change of $m-D$ relationships, but it limits particle transitions depending on chosen $m-D$ relationships. Newer approaches take advantage of current theory that allows models to simultaneously evolve aspect ratio (shape) as well as mass and fall speed by vapor deposition (Chen and Lamb 1994; Hashino and Tripoli 2007; Harrington et al. 2013). Predicting aspect ratio requires modeling two ice dimensions. Because ice has a sixfold hexagonal symmetry, many studies characterize ice by two dimensions, denoted the $a$ axis and $c$ axis (see, for example, Fig. 4). The $a$ axis is half the major axis for platelike crystals and half the minor axis for columnlike crystals, while the $c$ axis is half the minor axis for platelike crystals and half the major axis for columnlike crystals. Although atmospheric ice retains this hexagonal structure, the resulting shapes can be complex. Therefore, ice crystals are sometimes modeled as spheroids characterized by an aspect ratio ($b = c/a$). In general, not all crystal shapes can be represented by an aspect ratio alone. However, predicting an aspect ratio does account for a higher-order measure of nonsphericity, which is important for capturing the effect of particle shape on diffusional growth (Sulia and Harrington 2011). But particle transitions from vapor-grown to rimed crystals are not well understood, and modeling this becomes challenging when evolving aspect ratio. Though Hashino and Tripoli (2011) model vapor diffusion with evolving aspect ratio, they still diagnose hydrometeor type and certain properties based on process rates. They then assume that the aspect ratio of graupel evolves as a power law limited by a chosen mass threshold. For better consistency with the particle property approach, categorization and limited transitions should be replaced with a natural evolution of particle shape. The shape of vapor-grown ice is temperature dependent, and models can capture this (Chen and Lamb 1994; Zhang and Harrington 2014), but because riming depends on ice crystal shape and size, it also exhibits a temperature dependence (Fig. 1). Wind tunnel data show ice crystals grow fairly isometrically near $-10^\circ$C and preferentially rime. This is because of the lower drag and, therefore, higher fall speed of compact ice crystals, as compared with ice crystals having more extreme aspect ratios (Fukuta and Takahashi 1999). This temperature dependence of riming is missing from nearly all models.

Our approach is to construct a model that remains as independent of empirical relations as possible while evolving ice crystal shape during riming in a physical manner accounting for mass and shape transitions. We attempt to capture the temperature dependence of riming by allowing ice crystals to evolve using mass-growth and mass-distribution equations for both vapor growth and riming. This means we need to predict, as accurately as
possible, ice crystal properties, such as size, aspect ratio, and density, because of simultaneous vapor growth and riming. Evolving particle properties removes artificial conversions between snow and graupel. We compare our model to wind tunnel studies to test our model against controlled measurements.

2. Model development

a. Vapor depositional growth

The rate at which mass is added to an ice crystal by vapor deposition is

\[ \frac{dm_v}{dt} = 4\pi C \rho_i G_i s_i, \]  

(1)

where \( C \) is the capacitance of the ice crystal; \( G_i \) is an effective diffusion coefficient that accounts for vapor diffusion to, and thermal energy transport away from, the growing crystal; \( s_i \) is the ice supersaturation; and \( \rho_i \) is the density of bulk ice and is approximately 920 kg m\(^{-3}\) [Lamb and Verlinde 2011, p. 343, their Eq. (8.40)]. We use the Chen and Lamb (1994) method for vapor growth, their mass distribution hypothesis being that the ratio of the \( c \)-axis to \( a \)-axis growth is aspect ratio dependent, where \( dc/da = \Gamma(T)\phi \). The ratio of deposition coefficient of the \( c \) axis to the deposition coefficient of the \( a \) axis is \( \alpha_c/\alpha_a = \Gamma(T) \), is temperature dependent, and is derived from laboratory data. The individual deposition coefficients are effectively growth efficiencies ranging between zero and one that quantify the fraction of vapor molecules that incorporate into the surface. Consequently, the aspect ratio evolves, affecting diffusional growth, allowing our model to capture the temperature dependence of ice crystal growth. We use spheroids as a general shape to model ice crystals for reasons mentioned above.

To relate the mass-growth rate to a change in ice crystal volume \( V \),

\[ \frac{dm_v}{dt} = \rho_\Delta \frac{dV}{dt}, \]  

(2)

where \( \rho_\Delta \) is the deposition density. Chen and Lamb (1994) show that this change in volume can be related to a change in \( \phi \), allowing changes in mass to alter shape.

b. Deposition density

The deposition density can be less than the density of bulk ice and accounts for secondary habits, such as branching commonly observed in platelike crystals. In spherical ice growth models, this deposition density accounts for both the primary and secondary habits. Because ice crystal density affects both particle size, through growth, and fall speed, it is important for both vapor growth and riming. Estimating \( \rho_\Delta \) and, subsequently, the effective ice crystal density \( \rho \) can be done in several ways. Wind tunnel data from Takahashi et al. (1991) and published in Fukuta and Takahashi (1999, their Fig. 12) provide laboratory
values of \( \rho \) as a function of temperature that can be used as deposition densities in models. Chen and Lamb (1994) use a functional form of \( \rho_a \) that is empirical.

We propose a method to eliminate the empirical deposition density data as input to the model. To find a theoretical minimum effective density limit, we note that the growth of the \( a \) axis is determined by an appropriate flux \( F_a \) divided by a deposition density \( \rho_{a,a} \) (Sulia and Harrington 2011) such that

\[
\frac{da}{dt} = \frac{\alpha_a F_a}{\rho_{a,a}}.
\]

A similar formulation exists for \( c \)-axis growth using an appropriate flux \( F_c \) and the deposition density onto the \( c \) axis \( \rho_{c,c} \). The more efficiently molecules can adsorb and attach to a surface, the faster the axis growth rate is but also the lower the effective density of a given axis direction as a result of lacunas created during the crystal growth process (Frank 2009). Therefore, we expect that the ratio of the deposition coefficients will scale inversely with the ratio of the deposition densities on each axis:

\[
\frac{\alpha_c}{\alpha_a} \propto \frac{\rho_{a,a}}{\rho_{c,c}}.
\]

This scaling works in the correct manner because for \( \alpha_c/\alpha_a < 1 \), for which growth would result in a branching plate (at high-enough ice supersaturation), \( \rho_{a,a}/\rho_{c,c} < 1 \), meaning the density along the \( a \) axis or branching direction is smaller than along the \( c \) axis. If we assume that the decrease in effective density is due only to branching for plates and only to hollowing or sheathlike growth for columns while the other axis grows at the bulk density of ice, then for plates \( \rho_{a,a} = \rho_a \) and \( \rho_{c,c} = \rho_c \), and for columns \( \rho_{c,c} = \rho_c \) and \( \rho_{a,a} = \rho_a \). Relating our hypothesis to Chen and Lamb’s (1994) definition of \( \Gamma \) means \( \rho_a = \Gamma(T) \rho_c \) for plates and \( \rho_a \propto \rho_c / \Gamma(T) \) for columns. Allowing \( \rho_a \to \rho \) when \( \Gamma \to 1 \) means that modeled spherical (isometric) ice cannot branch or hollow, though isometric ice can show some hollowing. Assuming the effective density for plates is only reduced because of branching means, we need to know when branching starts. Using the theory of Mason (1953), we assume branching will not occur until \( v \omega^2 > \pi D_{a,c} \), where \( v \) is the terminal fall speed of the ice crystal and \( D_v \) is the water vapor diffusion coefficient. This occurs when ice crystals reach a radius (\( a \)-axis length) of approximately 100 \( \mu \)m. Therefore, before crystals reach this limit, we assume the deposition density to be that of bulk ice. This method is physically plausible and provides a way to evolve density along a preferred growth direction.

An advantage of our approach is that it links the deposition density to the deposition coefficients, therefore reducing the number of free parameters in the model. As an approximation, our method is reasonable because large differences in the axis growth efficiencies should lead to growth instabilities (Nelson 2001) and, therefore, the formation of branching and hollowing. Comparing wind tunnel measurements with both the Chen and Lamb (1994) parameterization and ours (Fig. 2) reveals our method’s greater accuracy in capturing the increase in \( \rho \) of thicker plates (from \(-9^\circ\) to \(-12^\circ\))C. Our method also captures both local minima in \( \rho \) near \(-6^\circ\) and \(-15^\circ\)C. Although we do not exactly capture the temperature dependence of \( \rho \), our method is an improvement over Chen and Lamb (1994).

Ice crystals grown at \(-15^\circ\)C have an inherent growth ratio \( \Gamma \) of 0.27 (Chen and Lamb 1994), which means that \( \rho \) falls to about 250 kg m\(^{-3}\) with our method (Fig. 3a). Our parameterization of deposition density leads to accurate dimension and density evolution compared with wind tunnel data. Using a lower deposition density like that from the Chen and Lamb (1994) parameterization leads to an underpredicted density and, therefore, a larger ice crystal at 30 min of vapor growth. Ice crystal size depends on growth time as well as on both habit evolution and assumed deposition density. Although their parameterization yields reasonable crystal properties, ours better captures the transitional growth for branching crystals and, therefore, the fall speed evolution (Fig. 3b) and fall speed transitions when branching starts.

For columns grown at \(-6^\circ\)C, the density loss due to hollowing is assumed to occur immediately, which is consistent with wind tunnel data (Takahashi et al. 1991). Laboratory data for needles at \(-5.3^\circ\)C show their limiting effective density to be approximately 300 kg m\(^{-3}\) (Jayaweera and Ohtake 1974; Takahashi et al. 1991). Chen and Lamb (1994) show a maximum inherent growth ratio of \( \Gamma = 2.3 \), meaning the minimum density for long hollow columns or sheath needles would be \( \rho \approx 400 \) kg m\(^{-3}\).

c. Rimming growth

Rimming is the collection of liquid drops on an ice crystal and depends on both ice crystal and liquid drop distribution properties. Rimming rates depend on ice crystal area and fall speed causing changes in shape that subsequently alter both riming and vapor growth. This interdependence between growth processes and particle properties, such as size and fall speed, provides a physical check for our model, because any errors should compound and lead to systematic inaccuracies in comparison with laboratory data.

\( ^1 \)See appendix for further justification.
The rate at which mass is added to an ice crystal by collecting supercooled liquid drops is

$$\frac{dm_r}{dt} = \sum_{l} EA_\| u_l - v_l |m_l n_l,$$

where $E$ is the efficiency at which ice collects drops of radius $r_l$, $A_\|$ is the geometric cross-sectional area occupied by both the ice crystal and the liquid drop, $u_l$ is the ice crystal fall speed calculated from Mitchell and Heymsfield (2005), $v_l$ is the drop fall speed, $m_l$ is the mass of drops of radius $r_l$, and $n_l$ is the number concentration of drops of radius $r_l$. We next turn to our methods of computing the variables in Eq. (5).

1) CROSS-SECTIONAL AREA

The cross-sectional area of a platelike crystal is $A = \xi \pi a^2$, where $\xi$ is the ratio of the area to the circumscribed area $\pi a^2$ (Böhm 1989). As long as a plate is sufficiently thin such that gaps in branches (not hollowing) cause the reduction in effective density, then those gaps also cause a reduced ratio of area to circumscribed area ($\xi = \rho/\rho_i$). However, for plates, $\xi$ varies with $\phi$. To account for shape variation for plates, we write $\xi$ as a function of both $\rho/\rho_i$ and $\phi$. Therefore, for platelike ice crystals $\xi = (1 - \phi)(\rho/\rho_i) + \phi$ so that, when $\phi \to 1$ (from riming or isometric growth), then $\xi \to 1$, and $\xi \to \rho/\rho_i$ when $\phi \to 0$ (for branched growth). This allows the cross-sectional area to be dependent on particle properties, though the dependence is linear. Because there are few observations of cross-sectional area, we rely on a simple linear method that has the correct limiting behavior. We show later that our method produces good agreement with laboratory-determined riming rates.

Columnar ice crystals typically hollow inwards along the basal face, which results in columns having a cross-sectional area perpendicular to their fall direction of $A = \pi a c$, and, therefore, they always have $\xi = 1$. Thick platelike crystals do not tend to have branches, but they usually grow with hollowing prism faces (Frank 2009, their Fig. 9). Therefore, like columns, they have an effective density less than $\rho_i$ without a reduction in cross-sectional area.

2) COLLISION EFFICIENCIES

We calculate riming collision efficiencies for both spherical and platelike ice particles with spherical drops. Spherical collision efficiencies come from Beard and Grover (1974), while collision efficiencies for plates come from Hall (1980). Although Hall (1980) used an aspect ratio cutoff to determine whether to use spherical or platelike collision efficiencies, we use an aspect ratio–weighted collision efficiency between spherical and platelike efficiencies for oblate spheroids. This frees us from having to pick a cutoff aspect ratio. In reality, we expect a transition in collision efficiencies from platelike
to spherical as, for example, a riming dendrite becomes isometric.

For columns, we use spherical collision efficiencies reasonably modified (because of a lack of data) for spheroids in the following way: a cross section of a prolate spheroid perpendicular to the $c$ axis is a circle. Assuming the flow field around a column depends on the $a$ axis (Sclamp et al. 1975) allows us to obtain prolate collision efficiencies from spheres of radius $a$. The collision efficiency of a sphere with radius $a$ collecting a drop of a certain size is approximately 

$$E_{\text{sphere}} = \frac{\pi y_c^2}{\pi a^2},$$

where $y_c$ is the radius of the collisional cross section, and we assume the collected drop radius is small compared with $a$ (see Fig. 4). We can compute $E_{\text{sphere}}$ numerically (see preceding paragraph), and we therefore can evaluate $y_c$. Imagine extending our sphere along the $c$ axis an amount $dc$ such that it becomes a prolate spheroid (i.e., from Fig. 4a to Fig. 4b). It now has a major axis ($c$ axis) radius of $c = a + dc$, but the minor axis ($a$ axis) radius is unchanged. If we assume that the collisional cross-sectional major axis changes by the same amount as the major axis (i.e., from gray shaded area in Fig. 4a to Fig. 4b), then our prolate spheroid collision efficiency becomes

$$E \approx \frac{\pi y_c(y_c + dc)}{\pi a(a + dc)} = \frac{\pi y_c(y_c + c - a)}{\pi ac}. \quad (6)$$

Mathematically, this is equivalent to extending the collisional cross-sectional major axis ($y_{c,e}$ in Fig. 4) the same distance from the geometrical edge along the $c$ axis as along the $a$ axis. Because we know $y_c$ as a function of $E_{\text{sphere}}$ of radius $a$, we can rearrange the collision efficiency for our prolate spheroid as $E = \sqrt{E_{\text{sphere}}} (1 - \phi^{-1} + \phi^{-3/2})$.

This has the correct limit, because when $\phi \to 1$, $E \to E_{\text{sphere}}$. For columns with Reynolds numbers of approximately 1 or lower, we extend the collision area all the way to the end of the $c$ axis by setting $y_{c,e} = c$. This leads to an efficiency of $E = \sqrt{E_{\text{sphere}}}$, which is consistent with the theory of Sclamp et al. (1975). Our collision efficiencies are consistent with values calculated by the
numerical schemes of Wang and Ji (2000) (see Fig. 5). The reason our efficiencies cut off at a certain small-enough drop size, whereas Wang and Ji’s (2000) do not, is because of our spherical collision efficiency parameterization.

3) RIME MASS DISTRIBUTION

Observations show that riming increases the minor axis, while the major axis remains roughly constant (Heymsfield 1982). Riming increases the minor axis, because ice crystals generally fall with their major axis oriented perpendicular to their fall direction, a result of drag. This is a simplification, because ice crystals change their orientation while falling. However, including a detailed treatment of crystal oscillations is not possible at present. Therefore, in our model, rime accumulates only along the $c$ axis for platelike crystals and only along the $a$ axis for columns (Chen 1992) until the aspect ratio reaches a limiting value. Graupel particles have an aspect ratio of approximately 0.8 (Heymsfield 1978), because graupel tumbles as it falls and rimes, becoming quasi spherical. We therefore let plates rime to an aspect ratio of 0.8 and columns rime to an aspect ratio of $1/0.8 = 1.25$. Once graupel reaches this aspect ratio, it grows by riming at a constant aspect ratio.

Distributing rime over a particle requires a rime density $\rho_r$. Rime density accounts for trapped air between the frozen drops during riming that leads to a reduced density of, for example, spherical graupel compared with a spherical frozen drop. Parameterizing rime density as a function of mean drop size, temperature, and impact velocity is possible (Macklin 1962; Pflaum and Pruppacher 1979), but we use constant rime density for simplicity. For exploring the effects of riming at $-15^\circ C$ (Figs. 6–11), we use $\rho_r = 250 \text{ kg m}^{-3}$, calculated using the formula of Pflaum and Pruppacher (1979). For all other simulations, we use $\rho_r = 100 \text{ kg m}^{-3}$, which is consistent with values derived from Pflaum and Pruppacher (1979) using wind tunnel drop spectra characteristics at $-10^\circ C$. This value is also consistent with our estimate of the final graupel density at $-10.5^\circ C$ from Takahashi and Fukuta (1988, their Fig. 5).

3. Results

Our method is a simplified growth model in which we assume spheroidal particles, use effective densities, and ignore changes in crystal orientation during fall. Nevertheless, the model is designed to capture the shape and density evolution effects required to successfully model vapor and riming growth. We test our new method by simulating vapor and riming growth over a large parameter space and compare with laboratory observations from Takahashi and Fukuta (1988) and Takahashi et al. (1991) when possible. We compare our model with
these more recent laboratory observations because earlier wind tunnel studies only grew ice crystals for 1–3 min (Fukuta 1969; Ryan et al. 1974), while Takahashi and Fukuta (1988) and Takahashi et al. (1991) grew ice crystals for up to 30 min. We use lognormal drop spectra, and, for the laboratory comparisons, the distribution parameters are chosen based on observed spectra (Takahashi and Fukuta 1988). At a liquid water content (LWC) of 0.4 g m$^{-3}$, we can accurately match the observed drop size spectra, but at 2.0 g m$^{-3}$, we cannot match their spectra and are forced to modify our spectra. We focus on riming at moderate LWCs, because our main goal is modeling transitions from vapor-grown crystals to rimed crystals. We provide some tests at higher LWCs but do not focus on them, because crystals riming at 2.0 g m$^{-3}$ can turn to graupel in minutes (Pflaum et al. 1978). Riming is especially sensitive to large drops, because the collision efficiencies are sensitive to drop size (Fig. 5). We therefore remove the tail of our lognormal liquid distribution for comparisons at 0.4 g m$^{-3}$ by setting a maximum liquid bin radius at 15 μm. We get similar results by slightly narrowing the distribution and leaving the tail of large drops. The main goal of the wind tunnel comparison is to see if we capture the impacts of ice crystal growth by vapor deposition and riming.

a. Rimming dendrites

Our modeled effective density and fall speed for vapor-grown dendrites compare well with wind tunnel data (Fig. 3), encouraging us to explore the parameter space of riming dendrites. We simulate growth at $-15^\circ$C for LWCs from 0.1 to 0.6 g m$^{-3}$ using mean drop radii from 2 to 12 μm. These values are characteristic of mixed-phase clouds (Verlinde et al. 2007) as well as orographic clouds (Prasad et al. 1989). Because our model predicts aspect ratio, density, and size, it effectively predicts mass–dimensional relationships.

Comparing our predicted mass–size relations (Figs. 6–8) to typical $m$–$D$ relationships (Mitchell 1996, their Table 1), our modeled parameter space leads to results that (compared with observation) would be considered unrimed crystals, lump graupel (completely rimed, quasi-spherical ice), and transitions between typical $m$–$D$ categories. In our model, transitions depend on growth processes, environmental conditions, and ice crystal properties, not predetermined $m$–$D$ relationships and conversion thresholds. These transitional states, which can persist depending on the environmental conditions, are both important and missing from cloud models, specifically ones that are tied to $m$–$D$ relationships. The higher LWCs (Fig. 8) result in more lump-graupel-like mass–size relations, but depending on the liquid distribution, dendrites can exist as unrimed crystals, lightly rimed crystals, densely rimed crystals, or graupel for all LWCs between 0.3 and 0.6 g m$^{-3}$. Liquid water contents of 0.2 g m$^{-3}$ can lead to densely rimed dendrites but not lump graupel in 30 min (Fig. 7 dashed lines). There is also a strong effect of the mean drop distribution size.

![Fig. 5. Collision efficiencies for prolate spheroids derived using spherical collision efficiencies at different Reynolds numbers (solid lines). Numerically calculated columnar collision efficiencies (points connected by dashed lines) at different Reynolds numbers are from Wang and Ji (2000).](image-url)
from collision efficiencies. Simulations with average liquid drop radii below 5 \( \mu \text{m} \) (not shown) do not lead to significant riming because the drop sizes are too small. All of the simulations run with 11–12-\( \mu \text{m} \) drops and LWCs from 0.3 to 0.6 \( \text{g m}^{-3} \) result in what would be considered lump graupel by 30 min. Also at high-enough LWC and large-enough drop size—for example, 0.4 \( \text{g m}^{-3} \) and 12 \( \mu \text{m} \) (violet solid line, Fig. 8)—the

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**Fig. 6.** Mass vs maximum dimension for ice crystals grown by vapor deposition and riming at \(-15^\circ\text{C}\) with 0.3 \( \text{g m}^{-3} \) of liquid water and mean liquid drop radii ranging from 5 to 12 \( \mu \text{m} \) (different colors). The filled circles represent values at 30 min of growth. Traditional \( m-D \) relationships are also shown (solid gray).

**Fig. 7.** Mass vs maximum dimension for ice crystals grown by vapor deposition and riming at \(-15^\circ\text{C}\) with 0.1–0.2 \( \text{g m}^{-3} \) of liquid water (different line types) and mean liquid drop radii ranging from 5 to 12 \( \mu \text{m} \) (different colors). Traditional \( m-D \) relationships are also shown (solid gray).
mass accelerates past the lump-graupel \(m-D\) value but then maintains a slope similar to the \(m-D\) curve. This is because the maximum dimension of riming dendrites initially does not change as riming only fills in the crystal area and changes the \(c\) axis. Consequently, under certain conditions, riming rates will be large enough to increase mass rapidly with little change in size until the particle becomes quasi spherical, after which riming can extend the maximum dimension.

This mass evolution has an important impact on fall speeds (Figs. 9–11). Our simulations show that transitions from dendrites to graupel can lead to rapid changes in fall speed as particles rime. Note that our model captures the densely rimed dendrite data from Locatelli FIG. 8. As in Fig. 7, but for higher liquid water contents.

![Figure 8](image)

**Fig. 8.** Fall speed vs maximum dimension for ice crystals grown by vapor deposition and riming at \(-15^\circ C\) with \(0.3 \, g \, m^{-3}\) of liquid water and mean liquid drop radii ranging from 5 to 12 \(\mu m\) (different colors). The filled circles represent values at 30 min of growth. Fall speed–size relationships (solid gray) are from Locatelli and Hobbs (1974) (lump graupel) and Kajikawa (1972) (dendrites). Densely rimed dendrite data (hexagons) and the fit (dashed gray) are from Locatelli and Hobbs (1974).

![Figure 9](image)

**Fig. 9.** Fall speed vs maximum dimension for ice crystals grown by vapor deposition and riming at \(-15^\circ C\) with \(0.3 \, g \, m^{-3}\) of liquid water and mean liquid drop radii ranging from 5 to 12 \(\mu m\) (different colors). The filled circles represent values at 30 min of growth. Fall speed–size relationships (solid gray) are from Locatelli and Hobbs (1974) (lump graupel) and Kajikawa (1972) (dendrites). Densely rimed dendrite data (hexagons) and the fit (dashed gray) are from Locatelli and Hobbs (1974).
and Hobbs (1974); however, only some simulations at low LWCs—for example, 0.2 g m$^{-3}$ and 11 μm (Fig. 10)—follow a curve almost parallel to their data. At higher LWCs, our fall speeds evolve rapidly through their data while transitioning to lump graupel. At an LWC of 0.3 g m$^{-3}$ with a mean drop radius of 9 μm, 30 min of growth produces a densely rimed dendrite (Figs. 6 and 9 green filled circles). But beyond this growth time, the crystal begins to transition toward lump graupel, as it should, with a larger mass at a given size than a densely rimed dendrite and a rapidly increasing fall speed. Our model results indicate that observations of quickly riming or densely rimed dendrites are snapshots of transitions. Several of our fall speed–size curves pass through the Locatelli and Hobbs (1974) data, meaning that, for certain liquid drop distribution properties and ice crystal growth times, snapshots of our model would lie near the Locatelli and Hobbs (1974) parameterization line. These results provide a cautionary tale regarding fall speed–size relationships used in models: One may be tempted to parameterize densely rimed dendrite fall speeds with a curve through the data.

**Fig. 10.** Fall speed vs maximum dimension for ice crystals grown by vapor deposition and riming at −15°C with 0.1–0.2 g m$^{-3}$ of liquid water (different line types) and mean liquid drop radii ranging from 5 to 12 μm (different colors). Fall speed–size relationships (solid gray) are from Locatelli and Hobbs (1974) (lump graupel) and Kajikawa (1972) (dendrites). Densely rimed dendrite data (hexagons) and the fit (dashed gray) are from Locatelli and Hobbs (1974).

**Fig. 11.** As in Fig. 10, but for higher liquid water contents.
However, this would not capture the correct fall speed evolution. Employing fall speed–size relationships not only ignores the natural spread in the fall speed of real crystals, which depends on ice crystal properties, but these relationships also ignore a large part of fall speed–size parameter space that exists between slower-falling dendrites and faster-falling graupel at larger sizes. We are able to capture the fall speed–size parameter space of ice crystals that grow by vapor deposition to large sizes and then begin to rime (Fig. 11, solid orange line).

b. Wind tunnel comparison

1) −10.5°C

Wind tunnel data of rimed ice crystals, while lacking breadth, are useful as a way to compare our model to growth rates and fall speeds of actual ice crystals. If such comparisons are favorable, then we can make inferences about other, unobserved, particle properties based on our model. Ice crystals grow fairly isometrically at −10.5°C and have been studied in depth (Takahashi and Fukuta 1988; Fukuta and Takahashi 1999) because their shape allows them to preferentially rime. This riming leads to an accelerated mass-growth rate and increased fall speed, which further accelerates riming. Our model captures the mass evolution, including the transition when riming starts to dominate after about 10 min of growth (Fig. 12a). We also pick up the increase in fall speed observed approximately 15 min into the total growth time (Fig. 12b). Our model captures the difference in fall speed evolution between light (0.4 g m$^{-3}$) and heavy (2.0 g m$^{-3}$) riming. This is an important distinction because, for isometric crystals, light riming increases the mass compared with the no-riming simulation, but not enough to significantly alter the slope of the fall speed (Fig. 12b). It is only at higher LWCs (2 g m$^{-3}$) that significant riming of isometric particles leads to fall speeds over 1 m s$^{-1}$ at 30 min. Because our model simultaneously evolves mass, fall speed, and aspect ratio and because each of those particle properties depends on one another, any errors in predicting one variable should show up in the others. For example, overpredicting fall speed means riming rates thus will increase overpredicting mass. The coupling among the predicted particle properties provides some confidence in predicted quantities that were not measured, such as aspect ratio evolution (Fig. 12c). This aspect ratio evolution suggests the

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![Figure 12](https://example.com/figure12.png)

FIG. 12. Model output of (a) mass growth, (b) fall speed, and (c) aspect ratio as functions of time at −10.5°C for no riming (dashed black line), light riming (0.4 g m$^{-3}$; blue line), and heavy riming (2.0 g m$^{-3}$; red line). Data from Takahashi and Fukuta (1988) are plotted (circles), where the colors correspond to the same liquid water contents as the model runs.
transition to graupel takes approximately 10 min at an LWC of 0.4 g m\(^{-3}\).

2) \(-12.2^\circ\text{C}\)

Our model shows negligible influence of riming on mass at \(-12.2^\circ\text{C}\) for low LWCs (0.4 g m\(^{-3}\)) (Fig. 13a) and little influence of riming on fall speed (Figs. 13b). The lack of mass increase during riming compared to unrimed crystals is due to the light riming increasing the aspect ratio of the particle and thus decreasing the capacitance and hence vapor growth, as will be discussed later. Of importance here is that the temperature dependence of vapor growth has enormous implications on riming through particle shape. Plates grown at \(-12.2^\circ\text{C}\) evolve to lightly rimed crystals in 30 min (Fig. 13c), with only slightly larger aspect ratios compared with unrimed ice. At high-enough LWCs (2 g m\(^{-3}\)), sectored plates becomes graupel in a little over 10 min (Fig. 13c), with little distinction between growth at \(-12.2^\circ\text{C}\) and \(-10.0^\circ\text{C}\). Because of the rapidity of riming growth at high LWCs, shape evolution becomes less important.

3) \text{ALL TEMPERATURES}

Our model gives reasonable results for riming at specific temperatures, and we therefore explore a range of temperatures. Figure 14 shows laboratory-determined mass as a function of temperature at different growth times compared with our model. At 10 min, vapor growth is dominant, and two local mass maxima are apparent in the wind tunnel data where aspect ratios become most extreme (\(-6^\circ\text{C}\) and \(-14^\circ\text{C}\); Fig. 14a). The local minimum in mass near \(-10^\circ\text{C}\) is due to the relatively slow vapor-growth rate of isometric crystals. By 15 min of growth, riming begins to increase the mass of crystals between \(-8^\circ\text{C}\) and \(-12^\circ\text{C}\), and our model captures the timing of the onset of riming. The vapor-grown habit effect on mass dominates at around \(-6^\circ\text{C}\) and \(-15^\circ\text{C}\) before 10-min growth for light riming. By 30 min, both our model and wind tunnel show that isometric crystals near \(-10^\circ\text{C}\) overtake needles (\(-6^\circ\text{C}\)) in terms of a local mass maximum. Vapor growth is still the dominant source of mass for the ice crystals near \(-15^\circ\text{C}\) when the LWC is 0.4 g m\(^{-3}\) because of the relatively weak riming rates. We also pick up the riming that Takahashi and Fukuta (1988) observed between \(-3^\circ\text{C}\) and \(-4^\circ\text{C}\), which commences later because of the slower vapor deposition growth rate for those isometric crystals, as compared to isometric crystals at \(-10^\circ\text{C}\). Results at \(-18^\circ\text{C}\) should be considered with caution, as Takahashi and Fukuta (1988) observed polycrystals at these temperatures, which we have not considered, and, in nature, the existence of supercooled liquid drops becomes less probable.
as freezing becomes more likely. Furthermore, note that
the wind tunnel data at around \(-16^\circ\) fall below
the spherical growth limit (cf. Figs. 14a and 14b); therefore,
either the environmental conditions or the mass measure-
ment at those temperatures in the wind tunnel might
be erroneous. Another possibility is that the initial
crystal sizes were significantly different from the rest of
the experiments. Between \(-6^\circ\) and \(-8^\circ\), the model
overpredicts mass. This overprediction begins before
riming starts and might be a result of how prolate
spheroids grow in the capacitance model. Prolate
spheroids inevitably induce sharp vapor gradients along
the \(c\) axis. These gradients could be weaker for real
crystals depending on how the \(c\) axis grows and hollows.

As expected, when using spheres (Fig. 14b), riming
occurs at all temperatures and begins immediately.
While using ice spheres with only vapor depositional
growth can lead to substantial errors in ice mass and
cloud glaciation time (Sulia and Harrington 2011),
adding riming further exacerbates this error: in com-
parison to data, riming starts too early and is only
slightly dependent on temperature through equilibrium
vapor pressure differences. Because ice crystals grow
nearly isometrically at \(-10^\circ\) and \(-3^\circ\), the spherical
results closely match the laboratory data, as expected. In
fact, percentage error in mass at 30 min and \(-10^\circ\)
(compared with wind tunnel data) is 10% for habits and
20% for spheres. The reason assuming spheres leads to a
larger overprediction of mass than assuming isometric
habits is that isometric habits in our model rime to an aspect
ratio of 0.8, meaning they fall slower and rime less efficiently
than spheres. At 30 min, the ratios of the vapor-grown and
rimed mass to the vapor-grown mass (with no riming) for
habits and spheres are 2.5 and 2.8, respectively. But
at \(-14^\circ\), assuming habits only overpredicts mass by 6%,
while assuming spheres leads to a 66% underprediction in
mass. Also while ice at \(-14^\circ\) accumulates little rime, as-
suming spheres leads to a vapor-grown and rimed ice crystal
mass that is 3 times that of just the vapor-grown crystal.
These results point to a potential problem with conversion
from snow to graupel using traditional model parameter-
izations. Instantaneously converting snow to lump graupel
(quasi spherical) should lead to systematic mass errors de-
pending on how the conversion is parameterized. By mov-
ing an ice crystal with a particular shape to a “spherical”
category, riming would be artificially enhanced.

Accurately calculating growth rates is important, but
we also want to accurately model fall speeds, because
that determines the mass flux out of a cloud. Without riming, we expect a sharp decrease in fall speeds with temperature from the maximum at \(-10^\circ C\) to half that value at \(-12^\circ C\) (Fig. 15a). This is because of the branched characteristics that begin to develop near \(-12^\circ C\), thus increasing particle drag. This sharp peak in the fall speed maximum can be seen in Takahashi et al. (1991, their Fig. 10 and our Fig. 15). Riming leads to a broader peak in fall speeds around \(-10^\circ C\), which is consistent with observations at 15 min (Fig. 15a). This enhancement in fall speed for ice grown at temperatures near \(-10^\circ C\) is important to capture, because it can approximately double the fall speed of ice crystals. While traditional models produce an increase in fall speed of rimed spheres as compared with snow (Fig. 15a), they cannot capture the increase in fall speed due to transitional states as particles evolve.

Finally, Takahashi and Fukuta (1988) observed a split in fall speed where rimed isometric crystals have approximately twice the fall speed of ice crystals with distinct habits after 30 min. This split begins at about 5 min, and our model captures this split well, along with the fall speed time series (Fig. 15b). These comparisons suggest that we are evolving particle properties such as mass and aspect ratio accurately.

c. Initial size sensitivity

Ice crystals have a range of initial sizes depending on how they nucleate. Nonspherical ice growth is sensitive to initial size, with initially smaller particles growing more rapidly than larger ones (Sheridan et al. 2009). This effect is the greatest where habit development most influences mass, at \(-6^\circ C\) and \(-15^\circ C\), and has little influence on growth near the isometric-growth temperatures. Figure 16 shows the spread in final mass at 30 min of growth for vapor depositional growth only and riming and vapor growth, using initially spherical ice with radii of 3 and 20 \(\mu m\). Because initially smaller ice crystals develop more extreme aspect ratios faster than initially larger ice, the smaller ice becomes more massive at 30 min of growth. Even with the spread in mass of vapor-grown ice, riming influences mass growth from \(-7^\circ C\) to \(-12^\circ C\) and from \(-3^\circ C\) to \(-5^\circ C\) at 30 min of growth (Fig. 16). This is seen as a distinct separation of the ranges of mass growth due to the initial-size effect between the simulations with vapor growth only and the simulations with riming and vapor growth. Initial size provides a range of mass solutions consistent with wind tunnel data (Fig. 16).
d. Graupel formation: Drop size dependence

To better understand habit and drop size effects on graupel formation, we undertook sensitivity studies. In these studies, we fix the average drop size and LWC for a 30-min simulation and then determine whether graupel formed under those conditions. Figure 17 provides a parameter space indicating whether graupel was observed during the 30-min simulation. As expected, graupel is observed for all LWCs and liquid drop sizes at \(2 \times 10^8\) C (Fig. 17). At an LWC of 0.6 g m\(^{-3}\), all ice crystals become graupel, provided the mean drop radius is 8 \(\mu m\) or larger. The effect of habit is apparent at all modeled LWCs: Larger drops are required to turn ice crystals with pronounced habits (around \(2 \times 6\) and \(2 \times 14\) C) into graupel. Simulations up to 2.0 g m\(^{-3}\) show that ice crystals at \(2 \times 15\) C do not form graupel when the mean drop size is below 5 \(\mu m\). At this temperature, vapor growth dominates such that ice crystals rapidly become dendrites with a low density and fall speed; therefore, small drops are not collected. Harimaya (1975) shows that drops of radius less than 5 \(\mu m\) were not observed as rimed drops on dendrites, while drops as small as a couple of microns in radius could be a substantial part of the rime mass for isometric ice crystals, and these observations support our model predictions.

![Fig. 16. The spread in mass growth at 30 min because of model sensitivity to initial ice crystal size. The red line is the fit through wind tunnel data from Takahashi and Fukuta (1988). The shaded region can be viewed as error bars on mass growth. The temperatures at which riming distinctly increases mass regardless of initial size are where riming is important (from \(-3^\circ\) to \(-5^\circ\)C and from \(-7^\circ\) to \(-12^\circ\)C). This is seen as a displacement of the riming vs no-riming curves between these temperatures. The initial size range is between 3 and 20 \(\mu m\) in radius, where smaller initial sizes lead to larger final mass.](image)

e. Rime suppression of vapor growth

For a small range of temperatures (from \(-14^\circ\) to \(-16^\circ\)C) and for large initial ice crystal sizes, the final mass grown without riming can be larger than with riming (Fig. 16). This is a curious result, which is caused by riming impacts on aspect ratio: light riming can suppress the total mass growth by slowing the increase of the ice crystal capacitance and, thus, decreasing the vapor depositional growth (Fig. 18).

The capacitance is a strong function of the aspect ratio, meaning for platelike crystals, increases in aspect ratio can decrease vapor growth substantially. If riming is not enough to counter the loss in vapor growth due to the increase in aspect ratio for platelike crystals, then riming suppresses overall growth (Fig. 18). While we add the rime mass over the entire face of the ice crystal in our model, in nature, rime mass usually sticks to the edges of branched ice crystals (Harimaya 1975). This will reduce the sharp curvature at the branch tips, weakening the vapor gradient and reducing growth (Marshall and Langleben 1954). As dendrites fill in with rime, there should also be a reduction in vapor growth.

Comparing final ice crystal mass from simulations with riming \(m_{v+r}\) and without riming \(m_v\) by using the ratio \(f_{rv} = (m_{v+r})/m_v\) shows the effect of riming on vapor growth. Because vapor-growth suppression depends on the degree of riming, we plot results as a function of the fraction...
of rime mass. The fraction of rime mass is $m_r/m_{o+r}$, where $m_r$ is the rime mass. As expected, after 30 min of vapor growth and riming at $-10^\circ$C with low LWCs, larger rime-mass fractions lead to larger $f_r$ (Fig. 19a). For isometric crystals, riming enhances total growth at low LWCs. At $-12^\circ$C, riming that produces rime-mass fractions less than 10% has no greater impact on ice growth than vapor diffusion alone. This explains the changing aspect ratio and
fall speed due to riming seen in Fig. 13, which results in no mass increase from riming, as compared to vapor growth. At \(-14^\circ C\), larger rime-mass fraction leads to smaller mass than the same ice crystal grown by vapor deposition alone. A rime fraction of 20% leads to a crystal with about 80% of the mass of a purely vapor-grown crystal grown for the same time. At 30 min of growth, even at 0.2 g m\(^{-3}\), LWC riming can still suppress overall mass growth (Fig. 19b). Rimming an ice crystal for 60 min at 0.1 g m\(^{-3}\) liquid water, provided the liquid drops are large enough, can lead to ice crystals that are 50% rime by mass but only 70% of the mass that the crystal would have without riming. To our knowledge, there are no observations of this effect. However, the effect is plausible based on how vapor gradients are altered near a crystal edge. This effect could be enhanced by the way we distribute mass, which is simplified, but the result is certainly physically plausible.

To show the extent of rim suppression of ice growth, we define \( R_{rv} \) (Fig. 20), where

\[
R_{rv} = \frac{dm_{v+\omega}}{dt} / \frac{dm}{dt},
\]

which is the ratio of the total mass-growth rate from vapor growth and riming to the mass-growth rate for simulations with vapor growth only. We average this over 30 min. Values less than 1 indicate that riming suppresses vapor growth. It takes larger drops for riming to suppress vapor growth, but our model shows the effect exists from \(-12^\circ C\) to \(-18^\circ C\) and up to a LWC of about 0.3 g m\(^{-3}\) (Fig. 20). This effect disappears for smaller liquid drops, as expected. The effect of riming suppressing vapor growth may be prominent in mixed-phase multilayer “seeder-feeder” systems, provided ice crystals grown in one layer grow as dendrites and do not sublime between layers. Data from the Mixed-Phase Arctic Cloud Experiment (M-PACE) show multilayer clouds with LWCs of around 0.1 g m\(^{-3}\) and liquid drop concentrations usually between 10 and 40 cm\(^{-3}\) (Luo et al. 2008). These systems are difficult to model (Luo et al. 2008; Morrison et al. 2009) because of the delicate nature of the forcings that either maintain or dissipate these clouds. If rime suppression of ice growth were to have an important impact in nature, it would most likely be in Arctic mixed-phase clouds.

4. Summary

Microphysical models provide a tool to better understand cloud evolution and precipitation. Growth rates in models must be constrained by theory and
observation and act on appropriate time scales to make models effective. Two important physical processes that are missing from most models are the temperature dependence of riming and the transitions from pristine ice crystals to graupel. Our single-particle model is able to capture the transitions of ice crystals to graupel on appropriate time scales and in a physical manner constrained by wind tunnel data. While sensitivities and uncertainties exist in our model, including rime density and collision efficiencies, our ability to evolve mass, shape, and fall speed while matching wind tunnel data provides confidence that the way we evolve these particle properties is a feasible approach.

Our model captures transitions in both mass–size and fall speed–size parameter spaces as dendrites rim. We show different mass and fall speed evolutions depending on liquid drop distribution properties, and our results are well constrained by observations. Capturing these particle transitions is a direct result of allowing ice particle properties to evolve. We show that riming dendrites transition to lump graupel in such a way that using $m-D$ relationships to model this process would be erroneous: Adding rime mass to a densely rimed dendrite would inaccurately evolve both maximum dimension and fall speed. Our modeled ice growth compares well with wind tunnel data across a temperature range from $-3^\circ$ to $-16^\circ$C, capturing general habit-dependent mass and fall speed evolution. Similar to wind tunnel data, isometric crystals preferentially rime in our model for a liquid water content of 0.4 g m$^{-3}$. As ice crystals grow by vapor deposition, riming begins to occur across a larger range of temperatures. Riming is habit and, therefore, temperature dependent, and we have captured this temperature dependence in our model. Finally, our model shows the importance of evolving aspect ratio from riming dendrites. We show an overall reduction of total mass growth when light riming increases particle aspect ratio enough to reduce vapor growth such that the light riming itself cannot make up for the vapor-growth reduction. The parameter space in which this might occur is reasonable and may be an important feedback in specific mixed-phase environments.

Traditional models may artificially enhance graupel production when there should not be any, especially when drop sizes are small or aspect ratio is far from unity. This occurs because traditional schemes do not treat transitional states as ice particles evolve by riming. Using multiple mass–dimensional relationships cannot easily solve this problem. We parameterize riming in a way that allows us to capture particle property evolution and avoid these issues facing traditional models.

![Figure 20](image-url)

**FIG. 20.** The ratio of the total mass-growth rate from vapor deposition and riming to the mass-growth rate by vapor deposition with no riming averaged over 30 min for different temperatures and liquid water contents. (a) Average liquid drops radius of 10 $\mu$m. (b) Average liquid drops radius of 5 $\mu$m. Values less than one indicate riming suppresses the growth rate. The black and white contours are the rime-mass fractions (%). See text for equation.
should lead to more accurate prediction of both ice growth and sedimentation, two requisite parts of cloud microphysics. Our method can easily incorporate new or different crystal growth theories, such as density and habit evolution, which can be tested and compared with observations. Our method is relatively simple, making it amenable to both bulk and bin development.

Acknowledgments. The authors thank Hugh Morrison and David Mitchell for their comments and advice. The authors are grateful for the insightful comments provided by three anonymous reviewers and thank them for their efforts. This research was supported by the U.S. Department of Energy’s Atmospheric Science Program Atmospheric System Research, an Office of Science, Office of Biological and Environmental Research program, under Grants DE-FG02-05ER64058 and DE-SC0012827.

APPENDIX

Deposition Density

We assume that the deposition density $\rho_a$ can be written as a mass-weighted deposition density between mass added to the $a$ axis $dm_a$ and mass added to the $c$ axis $dm_c$ such that $\rho_a = (\rho_{\Delta a} dm_a + \rho_{\Delta c} dm_c) / dm$. For dendritic growth, most of the depositional mass is added to the $a$ axis such that $\rho_{\Delta a} \approx \rho_{\Delta c}$. Also, in this case, the capacitance becomes $C = (2/\pi)a$, allowing us to write the mass-growth equation as

$$\frac{dm_a}{dt} = 8\rho_a a G_i \dot{s}_i = \rho_a \frac{dV}{dt} \approx \rho_a 8 \pi ac \frac{da}{dt}. \quad (A1)$$

We can solve for $\rho_{\Delta a}$ and divide by $\rho_{\Delta c}$ to obtain

$$\frac{\rho_{\Delta a}}{\rho_{\Delta c}} = \frac{3\rho_i G_i \dot{s}_i}{\pi c (da/dt) \rho_{\Delta c}} = \frac{3\rho_i G_i \dot{s}_i}{\pi c (da/dt) \rho_{\Delta c} \Gamma \alpha_a}. \quad (A2)$$

Therefore,

$$\frac{\alpha_c}{\alpha_a} = C_1 \frac{\rho_{\Delta a}}{\rho_{\Delta c}}, \quad C_1 = \frac{\pi c (da/dt) \rho_{\Delta c}}{3\rho_i G_i \dot{s}_i}. \quad (A3)$$

Applying our previous assumption that any growth on the $c$ axis during dendritic growth is deposited with $\rho_{\Delta c} = \rho_i$ ($c$-axis growth only thickens the crystal) allows us to eliminate $\rho_{\Delta c}$ from $C_1$, meaning that

$$C_1 = \frac{\pi c (da/dt)}{3G_i \dot{s}_i}. \quad (A4)$$

Analysis of Eq. (A4) reveals that, at constant temperature, $C_1$ is constant for any value of $\rho_{\Delta a}$. By constraining the vapor-growth equation to one axis and making an assumption about the deposition density on the other axis, we force $C_1$ to be constant. We have not derived a deposition density but, rather, have shown that our assumptions about the deposition density for dendritic growth do not lead to faulty conclusions or inconsistencies in the capacitance model.

REFERENCES


