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1. In a recent paper in this Magazine (July 1909) Lord Rayleigh discussed some cases of instantaneous propagation of a limited initial disturbance, and pointed out that, although the physical assumption which permits an infinite velocity may be obvious, there is an apparent paradox when the disturbance can be analysed by the Fourier method into simple waves of which the maximum wave-velocity is finite. In the following note two further cases which admit of exact solution are examined; the mathematical expressions are modified by a further Fourier analysis into periods so as to express more suitably a disturbance which has a definite beginning in time, and the result appears to emphasize the connexion of the phenomenon with the dispersive character of the medium.

Let \( y \) denote the effect at a position \( x \) and time \( t \) of an initial disturbance in a dispersive medium; for an initial displacement \( \cos \kappa x \), with no initial velocity, the disturbance is given by

\[
y = \cos (\kappa Vt) \cos \kappa x, \quad \ldots \ldots \quad (1)
\]

where \( V \) is supposed a known function of \( \kappa \).

Generalizing by Fourier's method we have for an initial displacement \( f(x) \),

\[
y = \frac{1}{\pi} \int_{0}^{\infty} d\kappa \cos \kappa Vt \int_{0}^{\infty} f(\omega) \cos \kappa(\omega - x) d\omega. \quad \ldots \ldots \quad (2)
\]

If the initial disturbance is limited practically to a line through the origin it is usual to write

\[
y = \frac{1}{\pi} \int_{0}^{\infty} \cos (\kappa Vt) \cos (\kappa x) d\kappa, \quad \ldots \ldots \quad (3)
\]

where for convergency a factor \( e^{-\kappa y} \) may be introduced under the integral sign, and the limit taken for \( y \) zero.

The chief regular features of the disturbance may be obtained by interpreting (3) in terms of groups of waves travelling out from the origin. The same expression (3) was used by Lord Rayleigh in discussing the initial motion due to a limited disturbance. He showed that in general the effect begins at all points without delay, even when—

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as in water of finite depth—the wave velocity \( V \) has a finite limiting value; a non-dispersive medium (\( V \) constant) appears as a particular case in which the effect may begin abruptly at each point after suitable delay. Lord Rayleigh remarks that in the ordinary theory of waves on water the velocity of propagation of waves of expansion is regarded as infinite, and doubtless some such dynamical assumption can be found in most cases; but the mathematical expressions may be modified and another point of view is suggested. The previous expression (3) gives equal values for negative as for positive values of \( t \); in fact the whole disturbance is built up of standing waves and may be regarded as a vibration in which at the instant \( t = 0 \) the displacement is confined to the region near \( x = 0 \) and the velocity is everywhere zero. The appearance of an effect at every point immediately after this instant appears connected with the occurrence of a disturbance at each point before the instant in question, apart from considering velocity of propagation.

If we attempt to form an expression which shall be zero for negative values of \( t \) and shall give suitable values for \( t \) positive, we find again that the effect begins without delay at all points; but this now appears connected with the fact that for certain periods no wave motion is possible. In other words, a Fourier analysis with respect to period divides the disturbance into two types; to each period there corresponds in the one case a wave motion and in the other a disturbance established in equal phase throughout. Each part of the expression implies an effect beginning without delay at all points, but it is suggested that the existence of both types is essential for this result—the character of the dispersion of the medium being dependent upon the range of period and the manner in which wave motion is impossible.

In order to avoid some complication of formulœ in localizing the initial disturbance, illustrations are taken in propagation along connected systems of similar bodies.

2. Consider the case of a stretched string itself without mass, but carrying loads (\( \mu \)) at equal intervals (\( a \)); let \( T_1 \) be the tension of the string and let \( y_r \) be the transverse displacement of a particle. The potential and kinetic energy functions are

\[
V = \ldots + \frac{T_1}{2a} (y_r - y_{r-1})^2 + \frac{T_1}{2a} (y_{r+1} - y_r)^2 + \ldots \\
T = \sum \frac{1}{2} \mu y_r^2
\]


* * *  
If \( y_r \) is proportional to \( e^{i(\omega t + \kappa r \alpha)} \), the velocity \( V \) of waves of length \( 2\pi/\kappa \) is given by

\[
V = \sqrt{\left( \frac{T_1 \alpha}{\mu} \right) \frac{\sin \frac{1}{2} \kappa a}{\frac{1}{2} \kappa a}}, \quad \cdots \quad (5)
\]

provided \( p^2 < 4T_1/\mu a \). If this condition is not satisfied, simple wave motion is not possible.

Instead of building up solutions from this simple type, we may solve directly a particular problem and then analyse the solution into simple harmonic components.

Suppose that initially a particle \( (y_0) \) is held displaced through unit distance and then released, so that when \( t=0 \) we have \( y_r \) and \( \dot{y}_r \) zero for all values of \( r \) except that \( y_0 = 1 \).

The subsequent disturbance is symmetrical with respect to the particle \( y_0 \), and the equations of motion are

\[
\begin{align*}
\tau^2 \ddot{y}_r &= y_{r-1} - 2y_r + y_{r+1}, \quad r > 0 \\
\tau^2 \ddot{y}_0 &= 2y_1 - 2y_0,
\end{align*}
\]

where \( \tau = a/c \); \( c = \sqrt{T_1 \alpha/\mu} \).

From these equations, with the initial conditions, we may solve for \( y_r \) as a series in \( t \); we find the general result

\[
y_r = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+2r)!} \frac{1}{(\tau)}^{2n+2r}, \quad \cdots \quad (7)
\]

This can be expressed by a Bessel function, as in fact follows directly from the equations (6), and we have for all values of \( r \),

\[
y_r = J_{2r}(2t/\tau). \quad \cdots \quad (8)
\]

Thus the vibrations of the central particle are given by \( J_0(2t/\tau) \), and those of the particles on either side by successive even Bessel functions of the same argument. Using tables of Bessel functions we may represent graphically the motion of the particles. The curves in fig. 1 show how the displacement of certain particles varies with the time, while in fig. 2 we have the positions of the particles on one side of the origin at instants given by \( t/\tau = 0, 2, 4, 6, 11 \). We may notice how the front part of the disturbance remains a crest as it travels out, becoming of greater effective wavelength in the process.
Using an integral expression for $J_{2r}$, and writing $x$ for $ra$ we have from (8),

\[ y = J_{2x/a}(2ct/a) \]

\[ = \frac{2}{\pi} \int_0^{\pi/2} \cos \left\{ \frac{2}{a} (x \phi - ct \sin \phi) \right\} d\phi \]

\[ = \frac{a}{\pi} \int_0^a \cos \left\{ \kappa (x - Vt) \right\} d\kappa, \quad \ldots \quad \ldots \quad (9) \]

where

\[ V = c \frac{\sin \left( \frac{1}{2} \kappa a \right)}{\frac{1}{2} \kappa a}. \]
We have also the group velocity \( U = d(\kappa V) / d\kappa = c \cos \left( \frac{1}{2} \kappa a \right) \). In the range for \( \kappa \) from 0 to \( \pi / a \), both \( U \) and \( V \) are positive and vary between 0 and \( c \); fig. 3 shows the curves for \( U \) and \( V \) as functions of the wave-length \( \lambda \), and their limiting positions for a zero.

When \( a \) is diminished indefinitely we approach as a limit the case of a uniform string with \( V \) and \( U \) equal to \( c \). For an initial displacement \( l \) at the origin, we have \( y \) given by (9) multiplied by \( l \); as \( a \) approaches zero \( l \) must become infinite in order that we may obtain the divergent integral (3) which is used to represent the effect of a concentrated initial disturbance.

In the integral (9) the solution (8) is expressed in terms of simple harmonic components of wave-lengths ranging from

\[
2a \rightarrow \infty
\]

The effect begins without delay at each point, and the solutions give equal values for equal positive and negative values of \( t \). Suppose we regard the central particle \( (y_0) \) as a source of displacement. Then if \( y_0 \) is zero for all negative values of \( t \), and equal to \( f(t) \) for positive values, we can write

\[
y_0 = \frac{1}{\pi} \int_0^\infty d\omega \int_0^\infty f(\omega) \cos n(\omega - t) \, d\omega, \quad \ldots \quad (10)
\]

with \( y_0 = \frac{1}{2} f(0+) \) for \( t = 0 \).

If we put \( f(t) \) equal to \( J_0(2t/\tau) \) we must cut out of the integration with respect to \( n \) the region in the vicinity of \( n = 2t/\tau \); in this way, or writing down the result directly,
we have
\[ y_0 = \frac{1}{\pi} \int_0^{2\pi} \frac{\cos nt}{\sqrt{4\tau^2 - n^2}} \, dn + \frac{1}{\pi} \int_{2\pi}^{\infty} \frac{\sin nt}{\sqrt{n^2 - 4\tau^2}} \, dn \]  
(11)
\[ = \frac{a}{2\pi} \int_0^{\pi} \cos (\kappa Vt) \, d\kappa + \frac{a}{2\pi} \int_0^{\infty} \sin (\kappa' V't) \, d\kappa' \]  
(12)
where \( V = c \sin \left( \frac{\tau}{2} \right) \) and \( V' = c \cosh \left( \frac{\tau}{2} \right) \).

In (12) each integral is equal numerically to \( \frac{1}{2} J_0(2t/\tau) \); hence \( y_0 \) is zero for \( t \) negative, \( J_0(2t/\tau) \) for \( t \) positive, and \( \frac{1}{2} \) for \( t \) zero.

Now we may write down a similar expression \( y_r \) which is zero for \( t \) negative and equal to \( J_2(2t/\tau) \) for \( t \) positive.

We have
\[ y_r = \frac{1}{\pi} \int_0^{\pi} \cos \left( \frac{t}{\tau} \sin \phi - 2r\phi \right) \, d\phi + \frac{1}{\pi} \int_1^{\infty} \sin \left\{ \left( \frac{u+1}{u} \right) t - r\pi \right\} \frac{du}{u^{2r+1}} \]
\[ = \frac{a}{2\pi} \int_0^{\pi} \cos (\kappa(x - Vt)) \, d\kappa + \left( \frac{a}{2\pi} \right) \frac{e^{-\frac{x}{2}}}{} \int_0^{\infty} \sin (\kappa' V't) \, d\kappa' \]  
(13)
with the same notation as in (12).

Comparing (12) and (13) we see how the central particle may be regarded as a source of displacement which is suddenly created at a certain instant. Physically, the idea is simpler if \( y \) is a velocity suddenly created by an impulse, but the analytical expressions are less direct. The first integrals in (12) and (13) correspond to periods (> \( \pi \tau \)) for which simple wave-motion is possible, the greatest wave-velocity being finite; the second integrals arise from component periods less than \( \pi \tau \). Of the value of \( y_r \) at any time one half is associated with each type of integral.

In the ordinary analysis, (12) and (13) would be replaced by twice the first integral in each case, with \( 0 \) and \( \infty \) as the limits, thus assuming the possibility of representing the effect of a suddenly created source of disturbance by simple wave-trains of all possible wave-lengths; the above illustration suggests that the existence of two types of integral may be necessary in a dispersive medium and that the apparent instantaneous propagation is connected with this fact. In this illustration the infinite velocity enters dynamically by the neglect of the inertia of the string. If we supposed this to be of uniform density we should have something analogous to two interacting media, the particles being connected by

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stretches of non-dispersive medium along which a disturbance could travel with an abrupt front.

3. Another illustration which may be treated in the same way is taken from Lord Rayleigh’s paper, already quoted, on propagation of waves along connected systems of similar bodies. Let each mass be connected to its immediate neighbours on the two sides by an elastic rod capable of bending but without inertia. The potential energy function is of the form

\[ V = \ldots + \frac{1}{2} b(2y_{r-1} - y_{r-2} - y_r)^2 + \frac{1}{2} b(2y_r - y_{r-1} - y_{r+1})^2 \]

\[ + \frac{1}{2} b(2y_{r+1} - y_r - y_{r+2})^2 + \ldots \ldots \ldots \] \hspace{1cm} (14)

With the same initial conditions, the coordinate \( y_0 \) unity, we have for the subsequent motion—writing \( \tau^2 = a^2 / c^2 = \mu / 4b \),

\[
\begin{align*}
4\tau^2 y_r &= -y_{r-2} + 4y_{r-1} - 6y_r + 4y_{r+1} - y_{r+2}, \quad r > 1, \\
4\tau^2 y_1 &= 4y_0 - 7y_1 + 4y_2 - y_3, \\
4\tau^2 y_0 &= -6y_0 + 8y_1 - 2y_2. \\
\end{align*}
\]

Forming a series for \( y_r \) in terms of \( t \) we find thus can be expressed for all values of \( r \) in the form

\[ y_r = J_r \left( \frac{t}{\tau} \right) \cos \left( r\frac{\pi}{2} - \frac{t}{\tau} \right). \] \hspace{1cm} (16)

Writing this in an integral form, with \( x \) for \( ra \), we have

\[
\begin{align*}
y &= J_{x/a} \left( \frac{ct}{a} \right) \cos \left( \frac{x\pi}{2a} - \frac{ct}{a} \right) \\
&= \frac{1}{\tau} \int_0^\pi \cos \left\{ \frac{2}{a} (x\phi - ct \sin^2 \phi) \right\} d\phi \\
&= \frac{a}{\pi} \int_0^{\pi/a} \cos \left\{ \kappa (x - \nabla t) \right\} d\kappa, \hspace{1cm} (17)
\end{align*}
\]

with

\[ V = c \frac{\sin^2 \left( \frac{\pi}{2} \kappa a \right)}{\frac{1}{2} \kappa a}. \]

Further, we have the group velocity \( U = c \sin (\kappa a) \). In (17) the disturbance is expressed in simple trains of wavelengths ranging from \( 2a \) to \( \infty \); in this range both \( V \) and \( U \) are positive and their variation with the wave-length is shown in fig. 4. The hyperbolic curves in the same diagram show the limiting forms of \( U \) and \( V \) if we wish to pass to the case of a continuous beam by making \( a \) small; the ordinary theory gives \( V = (\text{const.}) \kappa \), hence \( c \) must become infinitely large so that in the limit the product \( ac \) is finite.
Fig. 5 is obtained by plotting the positions of the particles at the instants $t/\tau = 0, 2, 6, 11$; we can trace the progress of the disturbance outwards from the central particle. Fig. 5
may be compared with fig. 2 of the previous case; here, owing to smaller wave-lengths being associated with larger velocities, we may observe the front part of the disturbance alternating between advancing as a crest and as a trough.

If we wish to regard the origin as the seat of a suddenly created disturbance we must analyse \( y_0 \) and \( y_r \) into periods and we find as before two types of integrals.

Using certain known integrals involving Bessel functions *, we may reduce the expressions to a form suitable for our purpose: we find in this way

\[
y_0 = \frac{a}{2\pi} \int_0^\pi \cos (\kappa V t) d\kappa + \frac{a}{2\pi} \int_0^\infty \sin (\kappa' V' t) d\kappa', \quad \quad \quad \quad (18)
\]

\[
y_r = \frac{a}{2\pi} \int_0^\pi \cos \{\kappa(x - V t)\} d\kappa + \frac{a}{2\pi} \int_0^\infty e^{-\kappa' x} \sin (\kappa' V' t) d\kappa', \quad (19)
\]

where \( V = c \sin^2(\frac{1}{2}\kappa a) / \frac{1}{2}\kappa a; \quad V' = c \sinh^2(\frac{1}{2}\kappa' a) / \frac{1}{2}\kappa' a. \)

Comparing the expressions in (17), (18), and (19) we see that they allow of the same interpretation as the similar forms in the previous illustration.

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**XIV. The Principles of Dynamics.** By Norman Campbell, Fellow of Trinity College, Cambridge †.

1. Introduction.
2. Definition of the problems discussed.
3. The relation of mathematics and physics.
4. An example of this relation.
5. The relation "A."
6, 7, 8. The fundamental conceptions of dynamics.
9. The application of dynamics to experiment.
10. The assumptions that must be made to make such application possible.
11. "Absolute and relative motion."
12. "The velocity of the sun in space."
13. "Absolute translation and absolute rotation."*

§ 1. It is doubtless rash nowadays to attempt to offer any remarks on a subject which has been so much discussed as the basis of dynamics with the hope that they shall be at the same time novel and useful. Ever since the publication of the brilliant treatise of Prof. Mach on the history of Mechanics the questions which he raised have been canvassed eagerly by inquirers of every nation, and it might

† Communicated by the Author.