Two-parameter homogenization for a GL problem in a perforated domain

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Abstract. Let Ω ⊂ R^2 be a simply connected domain, let ω be a simply connected subdomain of Ω, and set A = Ω \ ω. Suppose that J is the class of complex-valued maps on the annular domain A with degree 1 both on ∂Ω and on ∂ω. We consider the variational problem for the Ginzburg-Landau energy E_λ among all maps in J. Because only the degree of the map is prescribed on the boundary, the set J is not necessarily closed under a weak H^1-convergence.

We show that the attainability of the minimum of E_λ over J is determined by the value of cap(A) — the H^1-capacity of the domain A. In contrast, it is known, that the existence of minimizers of E_λ among the maps with a prescribed Dirichlet boundary data does not depend on this geometric characteristic.

When cap(A) ≥ π (A is either subcritical or critical), we show that the global minimizers of E_λ exist for each λ > 0 and they are vortexless when λ is large. Assuming that λ → ∞, we demonstrate that the minimizers of E_λ converge in H^1(A) to an S^1-valued harmonic map which we explicitly identify.

When cap(A) < π (A is supercritical), we prove that either (i) there is a critical value λ_0 such that the global minimizers exist when λ < λ_0 and they do not exist when λ > λ_0, or (ii) the global minimizers exist for each λ > 0. We conjecture that the second case never occurs.

Further, for large λ, we establish that the minimizing sequences/minimizers in supercritical domains develop exactly two vortices—a vortex of degree 1 near ∂Ω and a vortex of degree −1 near ∂ω.

1 Introduction

Let Ω_o and Ω_i be two smooth bounded simply connected domains in R^2 such that Ω_i ⊂ Ω_o. Consider an annular type domain A = Ω_o \ Ω_i and introduce the notations: Γ_o = ∂Ω_o, Γ_i = ∂Ω_i, so that ∂A = Γ_o \ Γ_i.

We next define a perforated domain A_δ obtained by ”punching” holes of size δ in A. To this end we first introduce a unit periodicity cell V. Let a and b be two linearly independent vectors
in \( \mathbb{R}^2 \) and set \( P = \{sa + tb; s, t \in (0, 1)\} \). Let \( U \) be a smooth simply connected domain such that \( \overline{U} \subset P \). Set \( \Gamma = \partial U \) then the unit cell is defined as \( V = P \setminus \overline{U} \).

Consider, for a small number \( \delta > 0 \), a point \( x_\delta \in \mathbb{R}^2 \) and define \( Z_\delta = \{ \delta m + x_\delta + \delta P \subset A \} \). Then the perforated domain is defined as follows:

\[
A_\delta = A \setminus \bigcup_{m \in Z_\delta} (\delta m + x_\delta + \delta \overline{U}),
\]  

with the boundary

\[
\partial A_\delta = \Gamma_o \cup \Gamma_i \cup \bigcup_{m \in Z_\delta} (\delta m + x_\delta + \delta \Gamma),
\]

(1.2)

Our goal is to study asymptotic behavior as \( \delta \to 0 \) and \( \lambda \to \infty \) of solutions (minimizers) of the following minimization problem:

\[
m_\delta := \operatorname{Inf}\left\{ E_\lambda(u) = \frac{1}{2} \int_{A_\delta} |\nabla u|^2 + \frac{\lambda}{4} \int_{A_\delta} (1 - |u|^2)^2 ; u \in \mathcal{J} \right\}.
\]  

(1.3)

Here, \( \lambda = \lambda(\delta) \to \infty \) is a GL parameter (the exact relation between \( \lambda \) and \( \delta \) will be specified later), \( E_\lambda \) is a Ginzburg-Landau (GL) energy functional. The class \( \mathcal{J} \) of testing maps is

\[
\mathcal{J} = \left\{ u \in H^1(A_\delta; \mathbb{C}), \operatorname{tr} u \in H^{1/2}(\partial A_\delta; S^1), \deg(u, \partial \Gamma_o) = \right. \\
= \deg(u, \partial \Gamma_i) = 1, \deg(u, \delta m + x_\delta + \delta \Gamma) = 0, \text{for any } m \in Z_\delta \right\}
\]  

(1.4)

The degrees here are computed with respect to the direct orientation of the connected components of \( \partial A_\delta \).

To begin with, we note that the definition of \( \mathcal{J} \) is meaningful. Indeed, if \( \gamma \) is a simple closed curve and \( u \in H^{1/2}(\gamma; S^1) \), then \( u \) has a degree on \( \gamma \) (since \( u \in \operatorname{VMO}(\gamma; S^1) \) and such maps have a degree [20]. If \( u \) is a minimizer of (1.3)-(1.4), then actually \( u \in C^\infty(A_\delta) \) [?], thus for a minimizer, the degree is a classical winding number.

Recall that the minimizers of the problem (1.3)-(1.4) exists (subcritical domain) or may not exist (supercritical domain) depending on the \( H^1 \)-capacity of the domain \( A_\delta \) [?],[?].

It turns out that asymptotic behavior of minimizers of (1.3)-(1.4) for subcritical domains can be understood by establishing asymptotic equivalence (see Theorem ?? below) between these minimizers and the minimizers of the following problem
The infimum is always attained in (1.5) [9], Chapter 1. Note that while the variational problem (1.3)-(1.4) is nonlinear, the latter problem (1.5) has an underlying linear problem for the phase of the corresponding harmonic maps and therefore it is much easier to analyze both asymptotically and numerically. Indeed homogenization for the problem (1.5) as \( \delta \to 0 \) has been established in [?], and therefore the above mentioned asymptotic equivalence provides the homogenization result for (1.3)-(1.4).

While the main effort in this work is on establishing the asymptotic equivalence in the subcritical domains, we also find the asymptotic behavior of the minimizing sequences for supercritical domains Theorems 1.2-2.2.

We conclude the introduction by formulating our main result on asymptotic equivalence.

We begin with the following

**Proposition 1.** Let \( M_\delta \) be the minimizer in (1.5). Then there exists \( \lim_{\delta \to 0} M_\delta = M \in (0, +\infty) \).

**Proof of Proposition 1:** Recall the following "torsion" problem:

\[
\inf \left\{ \frac{1}{2} \int_A |\nabla \phi|^2; \; \phi \in H^1(A_\delta ; \mathbb{R}^2), \; \phi|_{\Gamma_o} = 1, \; \phi|_{\Gamma_i} = 0, \; \phi|_{\delta m + x_\delta + \delta \Gamma} \text{ unknown constant} \right\} \tag{1.6}
\]

Let \( N_\delta \) be an infimum (actually minimum) in (1.6). Recall ([?]) that there exists

\[ \lim_{\delta \to 0} N_\delta = N \in (0, +\infty) \]

Problems (1.5) and (1.6) are related as follows: We have \( M_\delta = \frac{1}{2} \int_A |\nabla \Phi|^2 \), where \( \Phi \) solves

\[
\begin{align*}
\Delta \Phi & = 0, & \text{in } A_\delta \\
\Phi & = 0 & \text{on } \Gamma_i \\
\Phi & = \text{unknown constant} & \text{on } \Gamma_o \\
\Phi & = \text{unknown constant} & \text{on each } \delta m + x_\delta + \delta \Gamma \\
\int_{\Gamma_o} \frac{\partial \phi}{\partial \nu} & = 2\pi \\
\int_{\delta m + x_\delta + \delta \Gamma} \frac{\partial \phi}{\partial \nu} & = 0, & \forall m \in Z_\delta
\end{align*}
\]
(see, e.g, Chapter 1 in [9] where it is shown that if $\psi$ is the phase in the local representation of a harmonic function $u = \exp(i\psi)$, then $\Phi$ the harmonic conjugate of $\psi$ and $\Phi$ is globally defined unlike $\psi$. On the other hand, if $\phi$ is a minimizer of (1.6), then it is easy to see that $\phi$ satisfies:

$$\begin{align*}
\Delta \phi &= 0, & \text{in } A_\delta \\
\phi &= 0 & \text{on } \Gamma_i \\
\phi &= 1 & \text{on } \Gamma_o \\
\phi &= \text{unknown constant} & \text{on each } \delta m + x_\delta + \delta \Gamma \\
\int_{\delta m + x_\delta + \delta \Gamma} \frac{\partial \phi}{\partial \nu} = 0, & \forall m \in Z_\delta.
\end{align*}$$

(1.8)

Thus $\phi = C\Phi$. Since

$$N_\delta = \frac{1}{2} \int_{A_\delta} |\nabla \phi|^2 = \frac{1}{2} \int_{\Gamma_o} \frac{\partial \phi}{\partial \nu} = \frac{1}{2} \int_{\Gamma_o} C \frac{\partial \Phi}{\partial \nu} = \frac{1}{2} C^2 2\pi,$$

we find that $\Phi = \frac{\pi}{N_\delta} \Phi$. Thus $M_\delta = \frac{\pi^2}{N_\delta}$, and the conclusion of the proposition follows.

We now show that the existence of minimizers of (1.3) is governed by the value of $M$.

**Proposition 2.** Assume that $M < 2\pi$. Then, for sufficiently small $\delta > 0$, the infimum $m_\delta$ in (1.3) is attained.

**Proof of Proposition 2.** For small $\delta$, we have $M_\delta < 2\pi$. Since test functions for (1.5) are also test functions for (1.3), we find that $m_\delta \leq M_\delta < 2\pi$. It remains to recall that Proposition 5 in [?] states that if $m_\delta < 2\pi$, then $m_\delta$ is attained.

**Theorem 1.** Assume that $M < 2\pi$ and that

$$\lim_{\delta \to 0} \lambda(\delta)\delta^2 = \delta = \infty.$$

Then:

a) for sufficiently small $\delta > 0$, minimizers $u_\delta$ of (1.3) are **unique** up to a rotation, i.e., if $u'_\delta$ and $u_\delta$ are minimizers, then $u'_\delta = \alpha u_\delta$ for some $\alpha \in S^1$. 

b) if $u^\delta$ is a minimizer of (1.5) (a harmonic map), then there is some $\alpha_\delta \in S^1$ such that, as $\delta \to 0$, $u_\delta - \alpha_\delta u^\delta \to 0$ both in $H^1(A_\delta)$ and uniformly in $A_\delta$.

In particular, $|u_\delta| \to 1$ uniformly in $A_\delta$, so that $u_\delta$ is **vortexless** for sufficiently small $\delta$.

**Corollary 1.** The map $\alpha_\delta u_\delta/\alpha^\delta$, initially defined in perforated domain $A_\delta$, has an extension $w_\delta$ defined in $A$ such that $w_\delta \to 1$ in $H^1(A)$.

**Remark 1.** Clearly if $u$ minimizes either (1.3) or (1.5) so does $\alpha u$, $\forall \alpha \in S^1$, and therefore one can not hope to ”get rid” of the constant $\alpha_\delta$ in the above statements.
Recall that the $H^1$-capacity is given by

$$\text{cap}(A) = \text{Min}\left\{ \int_A |\nabla u|^2 : u \in H^1(A), u|_{\partial\Omega} = 0, u|_{\partial\omega} = 1 \right\}.$$  

For example, if $A = \{x : r < |x| < R\}$, then $\text{cap}(A) = \frac{2\pi}{\ln(R/r)}$. In general, $\text{cap}(A)$ is a measure of the "thickness" of $A$.

Formula (??) and the relationship (??) between $\text{cap}(A)$ and $I_0$ suggest that there are three different types of domains:

a) "subcritical" or "thin", when $\text{cap}(A) > \pi$ (or, equivalently, $I_0 < 2\pi$);

b) "critical", when $\text{cap}(A) = \pi$ (or, equivalently, $I_0 = 2\pi$);

c) "supercritical" or "thick", when $\text{cap}(A) < \pi$ (or, equivalently, $I_0 > 2\pi$).

We now return to the question of existence of minimizers. The main tool in proving the existence is the following

**Proposition 3.** Assume that $m_\lambda < 2\pi$. Then $m_\lambda$ is attained.

The results of this type were first established for the Yamabe problem by Th. Aubin in [5] and subsequently proved to be extremely useful in minimization problems with possible lack of compactness of minimizing sequences; see [19], [16], [17], [13] and the more recent papers [18], [23] and [29]. The proof of Proposition 3 relies on the following

**Lemma 1.** (Price lemma) Let $(u_n)$ be a bounded sequence in $\mathcal{J}$ such that $u_n \rightharpoonup u$ in $H^1(A)$. Then:

$$\liminf_n \frac{1}{2} \int_A |\nabla u_n|^2 \geq \frac{1}{2} \int_A |\nabla u|^2 + \pi(|1 - \deg(u, \partial\Omega)| + |1 - \deg(u, \partial\omega)|).$$  

(1.9)

In addition,

$$\frac{1}{2} \int_A |\nabla u|^2 \geq \pi|\deg(u, \partial\Omega) - \deg(u, \partial\omega)|.$$  

(1.10)

The argument we use here works for arbitrary fixed degrees [7]. The general form of the estimate (1.9) shows [7] that the minimal energy needed to jump from the degree $d$ (for the maps $u_n$) to the degree $\delta$ (for the map $u$) on a given connected component of $\partial A$ is $\pi|d - \delta|$.

As an immediate consequence of Proposition 3 and the upper bound (??), we obtain the following

**Theorem 2.** Assume that $A$ is subcritical or critical. Then $m_\lambda$ is attained for each $\lambda \geq 0$.  

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In the subcritical and critical case, we further address the following natural 

**Question 2.** What is the behavior of minimizers \( u_\lambda \) of (1.1)-(1.2) as \( \lambda \to \infty \)?

The answer is given by

**Theorem 3.** Let \( \text{cap}(A) \geq \pi \), i.e. \( A \) is subcritical or critical. Let \( u_\lambda \) be a minimizer of (1.1)-(1.2). Then \( |u_\lambda| \to 1 \) uniformly in \( \overline{A} \) as \( \lambda \to \infty \). In addition, up to a subsequence, \( u_\lambda \to u_\infty \) in \( H^1(A) \), where \( u_\infty \) is a minimizer of (??)-(??).

Theorem 2 combined with the method developed in [9] yield the stronger convergence \( u_\lambda \to u_\infty \in C^{1,\alpha}(A) \), \( 0 < \alpha < 1 \); see [7]. We also prove in [7] that, for large \( \lambda \), minimizers are unique modulo multiplication by a constant in \( S^1 \) and are symmetric if the domain itself is symmetric. Whenever minimizers \( u_\lambda \) exist, they are smooth [7]. This is not a standard regularity result, because the boundary conditions satisfied by the \( u_\lambda \)'s are of mixed type—Dirichlet for the modulus \( |u_\lambda| \) and Neumann for the phase \( \arg u_\lambda \).

We now turn to the supercritical case \( \text{cap}(A) < \pi \). Concerning existence of minimizers, we prove that there are exactly two possibilities (see Fig. 1).

**Theorem 4.** Let \( \text{cap}(A) < \pi \), that is let \( A \) be supercritical. Then either 

a) \( m_\lambda \) is attained for all \( \lambda \) (\( m_\lambda < 2\pi \));

or

b) there exists a critical value \( \lambda_1 \in (0, \infty) \) such that \( m_\lambda \) is attained when \( \lambda < \lambda_1 \) while it is not attained when \( \lambda > \lambda_1 \).

In contrast with supercritical/critical case, we prove that minimizing sequences (or minimizers, if they exist) must develop vortices (zeros of non-zero degree). If \( A \) is a circular annulus, the presence of vortices indicates the “breaking of symmetry” of the minimizer—similar phenomenon in problems for harmonic maps was studied in [1] and [11]. The inheritance of symmetry of the domain by minimizers of the GL functional was considered in [28].

**Theorem 5. (Rise of vortices)** Let \( A \) be supercritical.

In case (a) of Theorem 4, let \( u_\lambda \) be a minimizer of (1.1)-(1.2). Then, for large \( \lambda \), the map \( u_\lambda \) has exactly two simple zeros \( \zeta_\lambda \) and \( \xi_\lambda \) of degrees 1 and −1, respectively, such that \( \zeta_\lambda \to \partial \Omega \) and \( \xi_\lambda \to \partial \omega \) as \( \lambda \to \infty \).

In case (b) of Theorem 4, fix \( \lambda > \lambda_1 \) and let \( (u^k_\lambda) \) be a minimizing sequence for (1.1)-(1.2). Then \( u^k_\lambda = v^k_\lambda + w^k_\lambda \), where \( w^k_\lambda \to 0 \) in \( H^1(A) \) as \( k \to \infty \) and \( v^k_\lambda \) is smooth. Further, the map \( v^k_\lambda \) satisfies the GL equation and has exactly two simple zeros, \( \zeta_k \) and \( \xi_k \), of degrees 1 and −1, respectively, such that \( \zeta_k \to \partial \Omega \) and \( \xi_k \to \partial \omega \) as \( k \to \infty \).
We introduce the decomposition $u^k_{\lambda} = v^k_{\lambda} + w^k_{\lambda}$ because $u^k_{\lambda}$ belongs merely to $H^1$ and thus need not be continuous. Although there is no natural notion of zeros of $u^k_{\lambda}$, it is meaningful to consider zeros of $v^k_{\lambda}$, because it is a smooth map. An intuitive interpretation of this statement is that $u^k_{\lambda}$ essentially has two zeros for large $k$.

Further, in case (b), we prove (Step 5 in the proof of Theorem 4 in Section 4) that, near $\zeta_k$, the map $u^k_{\lambda}$ essentially behaves as a conformal map $\Phi_k$ from $\Omega$ into $\mathbb{D}_1$ with $\Phi_k(\zeta_k) = 0$ and, near $\xi_k$, as an anti-conformal map $\Psi_k$ from $\mathbb{C} \setminus \overline{\mathbb{D}}_1$ into $D_1$ with $\Psi_k(\xi_k) = 0$. A similar conclusion is valid in case (a) as well.

We believe that case (a) never occurs, hence we propose the following

**Conjecture.** In the supercritical case, there exists a finite value $\lambda_1 > 0$ such that $m_{\lambda}$ is never attained when $\lambda > \lambda_1$.

A formal argument in support of this conjecture is as follows. Assume that case (a) holds. For a large $\lambda$, let $d = \text{dist}(\{\zeta_\lambda, \xi_\lambda\}, \partial A)$ (cf. Theorem 4). It is easy to verify that

$$\lambda/4 \int_A (1 - |u_\lambda|^2)^2 \geq C_1 d^2.$$ 

On the other hand, various examples suggest that

$$1/2 \int_A |\nabla u_\lambda|^2 \geq 2\pi - C_2 d^2,$$

where $C_1$ and $C_2$ do not depend on $\lambda$ or $d$. If it can be proved that the second inequality does indeed hold, then the upper bound contradicts the existence of minimizers for large $\lambda$.

Finally, we discuss specific features of the critical case. It is known that, for variational problems with lack of compactness, the critical case could inherit the properties of either the supercritical or the subcritical case ([20], [15], [11], [18]). In our problem, the results are the same in critical and subcritical case, the supercritical case being qualitatively different. However, while the proof of the existence is the same in both subcritical and critical cases, the proof of $H^1$-convergence of the minimizers $u_\lambda$ as $\lambda \to \infty$ for the subcritical case cannot be extended to the critical case and a more subtle argument is required.

We conclude the introduction with a brief review of existing work on minimization of GL functionals related to the problem considered in this paper.

The GL functionals have been extensively studied for general domains. The asymptotics as $\lambda \to \infty$ of global minimizers for the GL functional and their vortex structure for the Dirichlet boundary data (for which the degree is fixed) was considered by Bethuel, Brezis, and Hélein in [9] and [10]. The existence and the qualitative behavior of minimizers in [9] and [10] do not depend on the size (capacity) of the domain.

A minimization problem for the GL functional with the magnetic field in simply connected domains for classes of functions with no prescribed boundary conditions was studied by Serfaty [32]-[33] and by Sandier and Serfaty [31]. In this case, the qualitative changes in the behavior of minimizers are described in terms of a parameter defined by the external magnetic field. In
particular, the existence of a threshold value for this parameter corresponding to a transition from vortex-less minimizers to minimizers with vortices was proved in [32]. For multiply connected domains, a similar result for Bose-Einstein condensate was established by Aftalion, Alama and Bronsard [2]. More recently Alama and Bronsard [4] considered energy minimizers for the GL functional with magnetic field in multiply connected domains, in the London limit as the GL parameter tends to infinity. Note that in the present work we establish an existence result for an arbitrary value of the GL parameter in the absence of the magnetic field.

The existence of local minimizers for the GL functional with the magnetic field over three-dimensional tori was considered by Rubinstein and Sternberg [30]. Their approach relies on the fact that, when the GL parameter \( \lambda \) is large, the boundedness of the nonlinear term in the GL energy forces the minimizing maps to be, in some sense, “close” to \( S^1 \)-valued maps. The first step in their proof consists of finding, for \( \lambda = \infty \), local minimizers for the GL functional in different homotopy classes of \( S^1 \)-valued maps (existence of these homotopy classes is due to White [34]). This step is reminiscent of the method used by Brezis and Coron in [17] for harmonic maps. The next step consists of proving, for \( \lambda \) large, the existence of local minimizers close to the ones obtained for \( \lambda = \infty \). These existence results are not influenced by the domain size (capacity). Note that [30] generalizes the earlier results of Jimbo and Morita [26] obtained for solids of revolution with a convex cross-section.

If adapted to our case, the methods of [30] yield, for large \( \lambda \), the existence of local minimizers in \( J \) that are \( H^1 \)-close to the minimizers of \( E_\lambda \) in \( K \). If \( A \) is subcritical or supercritical, it can be proved that these critical points are, for large \( \lambda \), the genuine minimizers [7]. However, they are not minimizers when \( \lambda \) is large and \( A \) is supercritical.

The minimization problem for GL functional with the degree boundary conditions in a special case of a narrow circular annulus was studied by Golovaty and Berlyand [25]. The techniques developed there rely on the radial symmetry and cannot be applied to general domains.

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2 Existence of minimizers

The following simple fact will be repeatedly used in the sequel. Let \((u_n)\) be a bounded sequence in \(H^1(A)\) such that for every \(n\) we have \(|u_n| = 1\) a.e. on \(\partial A\). If \(u_n \rightharpoonup u\) in \(H^1\), clearly \(|u| = 1\) a.e. on \(\partial A\) and both \(\deg(u, \partial \Omega)\) and \(\deg(u, \partial \omega)\) are well-defined.
Proof of the Price lemma: Set \( v_n = u_n - u \). We have

\[
\int_A |\nabla u_n|^2 = \int_A |\nabla u|^2 + \int_A |\nabla v_n|^2 + o(1),
\]

as \( n \to \infty \). Given an arbitrary \( f \in C^\infty(\overline{A} ; [-1, 1]) \), we integrate by parts the pointwise inequality \( |\nabla v_n|^2 \geq 2 f \text{Jac } v_n \) to obtain

\[
\int_A |\nabla v_n|^2 \geq 2 \int_A f v_n \times v_n,\tau + \int_A (f_v v_n \times v_n - f_y v_n, x \times v_n),
\]

where \( \partial A \) has the counterclockwise orientation. The above equality follows from the identity

\[
2 \text{Jac } v_n = (v_n \times v_n, y) + (v_n, x \times v_n),
\]

when \( v_n \) is smooth. The same inequality for \( v_n \in H^1 \) follows by approximation. Since \( v_n \rightharpoonup 0 \) in \( H^1 \), (2.1) and (2.2) yield

\[
\int_A |\nabla u_n|^2 \geq \int_A |\nabla u|^2 + \int_{\partial A} f v_n \times v_n,\tau + o(1),
\]

via an embedding argument. On the other hand, if \( \Gamma \) is any connected component of \( \partial A \), then

\[
\int_\Gamma v_n \times v_n,\tau = \int_\Gamma u_n \times u_n,\tau - \int_\Gamma u \times u,\tau + o(1).
\]

Indeed, if \( g_n \rightharpoonup g \) in \( H^{1/2}(\Gamma) \) and \( h \in H^{1/2}(\Gamma) \), then

\[
\int_\Gamma g_n h,\tau = \int_\Gamma g \times h,\tau + o(1) \quad \text{and} \quad \int_\Gamma h \times g_n,\tau = \int_\Gamma h \times g,\tau + o(1),
\]

where the integrals are understood in the sense of an \( H^{1/2} - H^{-1/2} \) duality. Then (2.4) follows easily from (2.5) and the fact that \( u_n|_{\Gamma^c} \to u|_{\Gamma^c} \) in \( H^{1/2}(\Gamma) \).

Now choose \( f \) such that \( f = \text{sgn}(1 - \text{deg}(u, \partial \Omega)) \) on \( \partial \Omega \), \( f = -\text{sgn}(1 - \text{deg}(u, \partial \omega)) \) on \( \partial \omega \), and \( -1 \leq f \leq 1 \) in \( A \). By combining (2.1), (2.3), (2.4), and the degree formula (??), we obtain (1.9).

Equation (1.10) relies on the pointwise inequality \( |\nabla u|^2 \geq 2 |\text{Jac } u| \), which yields

\[
\int_A |\nabla u|^2 \geq 2 \int_A |\text{Jac } u| \geq 2 \int_A \text{Jac } u \geq \int_{\partial A} u \times u,\tau = 2\pi |\text{deg}(u, \partial \Omega) - \text{deg}(u, \partial \omega)|,
\]

following an integration by parts and taking into account (??).
Proof of Proposition 3: Let \((u_n)\) be a minimizing sequence for \(E_\lambda\) in \(J\). Up to a subsequence, we can assume that \(u_n \to u\) for some \(u \in H^1(A)\). Set \(D = \text{deg}(u, \partial \Omega)\) and \(d = \text{deg}(u, \partial \omega)\). If \(d = D = 1\), then \(u \in J\) and \(u\) is a minimizer of (1.1)-(1.2). If both \(D \neq 1\) and \(d \neq 1\) then (1.9) implies that

\[
2\pi > m_\lambda = \lim_n E_\lambda(u_n) \geq \liminf_n \frac{1}{2} \int_A |\nabla u_n|^2 \geq \pi(|1 - d| + |1 - D|) \geq 2\pi,
\]

(2.7)

which is a contradiction. Finally, if only one of two integers \(d\) and \(D\) is equal to 1, then \(|d - D| \geq 1\) and \(|1 - d| + |1 - D| \geq 1\). By combining (1.9) and (1.10) we obtain \(m_\lambda \geq 2\pi\) once again—this is impossible.

Proof of Proposition 4: Let \(u\) be a minimizer of (1.1)-(1.2) and set \(g = u_{|\partial A}\). If \(v\) minimizes \(E_\lambda\) among all the maps \(w \in H^1(A)\) such that \(w_{|\partial A} = g\), then \(v \in J\) and \(m_\lambda \leq E_\lambda(v) \leq E_\lambda(u) = I_0\). We claim that the last inequality is strict. Arguing by contradiction, assume that \(E_\lambda(v) = E_\lambda(u)\). Then \(u\) minimizes \(E_\lambda\) with respect to its own boundary conditions; in particular, \(u\) satisfies the GL equation \(-\Delta u = \lambda u(1 - |u|^2)\). Since \(|u| = 1\) a.e., we find that \(u\) is harmonic and has the modulus 1. Thus \(u\) has to be a constant, which contradicts the fact that \(u \in K\).

Proof of Theorem 4: The mapping \(\lambda \mapsto m_\lambda\) is clearly both non-decreasing and continuous. In view of the upper bound (2.1), there is some \(\lambda_1 \in [0, \infty)\) such that \(m_\lambda < 2\pi\) if \(\lambda < \lambda_1\), and \(m_\lambda = 2\pi\) if \(\lambda \geq \lambda_1\). We first claim that \(m_\lambda\) is not attained if \(\lambda > \lambda_1\). Arguing by contradiction, we assume that there are some \(\lambda > \lambda_1\) and \(u \in J\) such that \(E_\lambda(u) = m_\lambda = 2\pi\). As in the proof of Proposition 2, we cannot have \(|u| = 1\) a.e. Thus \(\int_A (1 - |u|^2)^2 > 0\) and, therefore, \(E_{\lambda'}(u) < E_\lambda(u)\) if \(\lambda' < \lambda\).

For any \(\lambda'\) such that \(\lambda_1 < \lambda' < \lambda\), this implies that \(m_{\lambda'} \leq E_{\lambda'}(u) < 2\pi\), which is a contradiction.

In view of Proposition 1, \(m_\lambda\) is attained for all \(\lambda < \lambda_1\). Case (a) in Theorem 4 corresponds to \(\lambda_1 \in (0, \infty)\) and case (b) to \(\lambda_1 = \infty\). Therefore, in order to complete the proof of Theorem 4, it remains to rule out the possibility that \(\lambda_1 = 0\). This amounts to proving the following

Lemma 2. We have \(m_0 < 2\pi\).

Proof of Lemma 2: We start by considering a circular annulus, \(A = \{z \in \mathbb{R}^2 : r < |z| < R\}\). Set \(u(z) = \frac{z}{R + r} + \frac{rR}{(R + r)^2}\). It is easy to check that \(u(z) = \frac{z}{|z|}\) on \(\partial A\), so that \(u \in J\). On the other hand, it is also straightforward to verify that \(E_0(u) = 2\pi \frac{R - r}{R + r} < 2\pi\), hence \(m_0 < 2\pi\).

Consider now the case of a general \(A\). Recall that there is a conformal representation \(\Phi\) of \(A\) into some circular annulus \(\bar{A}\); moreover, \(\Phi\) extends to a \(C^1\)-diffeomorphism of \(\bar{A}\) into \(\bar{A}\) and...
we may choose $\Phi$ in order to preserve the orientation of curves [3]. Let $F : H^1(\mathcal{A}) \to H^1(\mathcal{A})$, $F(u) = u \circ \Phi$. If $\mathcal{J}(\mathcal{A})$ and $\mathcal{J}(\mathcal{A})$ stand for the corresponding classes of testing maps, we claim that $F$ is a bijection between $\mathcal{J}(\mathcal{A})$ and $\mathcal{J}(\mathcal{A})$. In order to prove this statement, we have to show that the degrees on the connected components of the boundary are preserved by $\Phi$. Indeed, let $\Gamma$ be a connected component of $\partial A$ and let $\gamma = \Phi(\Gamma)$. Since $\Phi$ is orientation-preserving, we have

$$\deg(g, \gamma) = \deg(g \circ \Phi, \Gamma)$$

(2.8) for $g \in C^\infty(\gamma; S^1)$. Using the density of $C^\infty(\gamma; S^1)$ in $H^1/2(\gamma; S^1)$ and the continuity of the map $g \mapsto g \circ \Phi$ from $H^1/2(\gamma; S^1)$ into $H^1/2(\Gamma; S^1)$, we find that (2.8) is still valid for $g \in H^1/2(\gamma; S^1)$. Thus $F$ maps $\mathcal{J}(\mathcal{A})$ into $\mathcal{J}(\mathcal{A})$. Similarly, $F^{-1}$ maps $\mathcal{J}(\mathcal{A})$ into $\mathcal{J}(\mathcal{A})$. So $F$ is a bijection between $\mathcal{J}(\mathcal{A})$ and $\mathcal{J}(\mathcal{A})$.

Using the conformal invariance of the Dirichlet integral, we find that $m_0$ has the same value for both $A$ and $\mathcal{A}$. Since $m_0 < 2\pi$ for circular annuli, the proof of Lemma 2 is complete.

3 Proof of Theorem 3

Let $u_\lambda$ be a minimizer of (1.1)-(1.2) for a given $\lambda \geq 0$. We start by observing that a sequence $(u_\lambda)$ is bounded in $H^1(\mathcal{A})$. Indeed, the upper bound (??) implies that $(\nabla u_\lambda)$ is bounded in $L^2(\mathcal{A})$.

Thus, by a Poincaré-type inequality, $(u_\lambda - a_\lambda)$ is bounded in $H^1(\mathcal{A})$, where $a_\lambda = \frac{1}{|\partial \Omega|} \int_{\partial \Omega} u_\lambda$. Since $|u_\lambda| = 1$ a.e. on $\partial \Omega$, $a_\lambda$ is bounded, so that $u_\lambda$ is bounded in $H^1(\mathcal{A})$.

Let $u_\infty \in H^1(\mathcal{A})$ be such that, up to some subsequence, $u_{\lambda_n} \rightharpoonup u_\infty$ in $H^1(\mathcal{A})$. In view of (??), we have $\int_A (1 - |u_\lambda|^2)^2 \to 0$, and thus $u_\infty \in H^1(A; S^1)$.

In the subcritical case, we identify $u_\infty$ using the Price lemma and the following simple

Lemma 3. Let $u \in H^1(A; S^1)$. Then $\deg(u, \partial \Omega) = \deg(u, \partial \omega)$.

Proof of Lemma 3: Differentiating the equality $|u|^2 = 1$ a.e. we find that $u \cdot u_x = u \cdot u_y = 0$ a.e., so that $\text{Jac } u = 0$ a.e. On the other hand, an integration by parts used in conjunction with the degree formula (??) yields

$$0 = \int_A \text{Jac } u = \frac{1}{2} \int_{\partial A} u \times u_r = \pi(\deg(u, \partial \Omega) - \deg(u, \partial \omega)).$$

(3.1)

For the convenience, we divide the remainder of the proof of Theorem 3 into five steps.
Step 1. Identification of \( u_\infty \) and strong-\( H^1(A) \) convergence in the subcritical case.

By combining the Price Lemma, Proposition ??, Lemma 3, and the upper bound (??), we have

\[
2\pi > I_0 \geq \liminf_n m_{\lambda_n} \geq \liminf_n \frac{1}{2} \int_A |\nabla u_{\lambda_n}|^2 \geq \frac{1}{2} \int_A |\nabla u_\infty|^2 + 2\pi |1 - \deg(u_\infty, \partial \Omega)|, \tag{3.2}
\]

in the subcritical case \( I_0 < 2\pi \). It follows from (3.2) that \( \deg(u_\infty, \partial \Omega) = \deg(u_\infty, \partial \omega) = 1 \), that is \( u_\infty \in \mathcal{K} \), and \( I_0 \geq \frac{1}{2} \int_A |\nabla u_\infty|^2 \). Recalling the definition of \( I_0 \), we find that \( u_\infty \) minimizes (??)-(??). Then it follows from (3.2) that

\[
I_0 \geq \liminf_n \frac{1}{2} \int_A |\nabla u_{\lambda_n}|^2 \geq \frac{1}{2} \int_A |\nabla u_\infty|^2 = I_0, \tag{3.3}
\]

which implies that \( u_{\lambda_n} \to u_\infty \) in \( H^1(A) \).

Step 2. An improved upper bound for \( m_\lambda \)

The following result is a slight improvement of the upper bound (??).

Lemma 4. There exist constants \( C > 0 \) and \( \lambda_0 > 0 \), such that \( m_\lambda \leq I_0 - \frac{C}{\lambda} \) for \( \lambda > \lambda_0 \).

Proof of Lemma 4: Let \( u \) minimize (??)-(??), then \( u \in C^\infty(\overline{A}) \) [10]. Consider an arbitrary \( f \in C^\infty_0(A; \mathbb{R}) \) and set \( v_\lambda = (1 - f/\lambda)u \). The map \( v_\lambda \) coincides with \( u \) on \( \partial A \) and thus belongs to \( J \). It is easy to observe that \( |\nabla v_\lambda|^2 = (1 - f/\lambda)^2 |\nabla u|^2 + |\nabla f|^2 / \lambda^2 \), since \( u \) is \( S^1 \)-valued. Thus

\[
m_\lambda \leq E_\lambda(v_\lambda) = \frac{1}{2} \int_A |\nabla u|^2 - \frac{1}{\lambda} \int_A f(|\nabla u|^2 - f) + O\left(\frac{1}{\lambda^2}\right). \tag{3.4}
\]

The conclusion of Lemma 4 follows from (3.4) by choosing \( f \) such that \( 0 \leq f \leq |\nabla u|^2 \) in \( A \) and \( 0 < f < |\nabla u|^2 \) in some nonempty open subset of \( A \).

Step 3. Candidates for \( u_\infty \) in the critical case

Lemma 5. Assume that \( A \) is critical. Then either \( u_\infty \) minimizes (??)-(??), or \( u_\infty \) is identically equal to a constant of modulus 1.
Proof of Lemma 5: We rely on the Price Lemma, Lemma 3, and the upper bound \( (??) \). As in (3.2), we have

\[
2\pi = I_0 \geq \liminf_n m_{\lambda_n} \geq \frac{1}{2} \int_A |\nabla u_\infty|^2 + 2\pi |1 - \text{deg}(u_\infty, \partial \Omega)|. \tag{3.5}
\]

If \( \text{deg}(u_\infty, \partial \Omega) = \text{deg}(u_\infty, \partial \omega) = 1 \) then, as in Step 1, we find that \( u_\infty \) minimizes \( (??)-(??) \). On the other hand, if \( \text{deg}(u_\infty, \partial \Omega) = \text{deg}(u_\infty, \partial \omega) \neq 1 \), then (3.5) implies that \( u_\infty \) must be identically equal to a constant. Since \( |u_\infty| = 1 \) a.e. on \( \partial A \), this constant is of modulus 1.

Step 4. Identification of \( u_\infty \) and strong-\( H^1(A) \) convergence in the critical case

We rely on the following

Lemma 6. \[27\] Let \( (v_\lambda) \) be a family of solutions of the GL equation \( -\Delta v_\lambda = \lambda v_\lambda (1 - |v_\lambda|^2) \) in \( A \). Assume that \( |v_\lambda| \leq 1 \) and \( E_\lambda(v_\lambda) \leq C \) uniformly in \( \lambda \). Then \( (v_\lambda) \) is bounded in \( C^\infty_{\text{loc}}(A) \). In addition, the following pointwise estimates hold:

\[
1 - |v_\lambda(z)|^2 \leq \frac{D}{\lambda d^2(z)}, \quad z \in A \tag{3.6}
\]

and

\[
|D^k v_\lambda(z)| \leq \frac{D_k}{d^k(z)}, \quad z \in A, k \in \mathbb{N}; \tag{3.7}
\]

here, \( d(z) = \text{dist}(z, \partial A) \) and the constants \( D, D_k \) depend only on \( C \).

In order to identify \( u_\infty \), we rule out the possibility that \( u_\infty \) is a constant. We argue by contradiction. Let \( \Gamma \) be a simple curve in \( A \) enclosing \( \partial \omega \). Let \( U \) be the domain enclosed by \( \partial \Omega \) and \( \Gamma \) and set \( V = A \setminus U \). Integrating the pointwise inequality \( |\nabla u_\lambda|^2 \geq 2 \text{Jac} u_\lambda \) over \( U \) and using the degree formula \( (??) \), we find that

\[
\frac{1}{2} \int_U |\nabla u_\lambda|^2 \geq \pi - \frac{1}{2} \int_{\Gamma} u_\lambda \times u_{\lambda, \tau}, \tag{3.8}
\]

where \( \Gamma \) is counterclockwise oriented. Similarly, the inequality \( |\nabla u_\lambda|^2 \geq -2 \text{Jac} u_\lambda \) yields

\[
\frac{1}{2} \int_V |\nabla u_\lambda|^2 \geq \pi - \frac{1}{2} \int_{\Gamma} u_\lambda \times u_{\lambda, \tau}. \tag{3.9}
\]

Thus

\[
m_{\lambda} \geq \frac{1}{2} \int_A |\nabla u_\lambda|^2 \geq 2\pi - \int_{\Gamma} u_\lambda \times u_{\lambda, \tau}. \tag{3.10}
\]
Next we observe that \( u_\lambda \) satisfies the assumption of the Lemma 6 for every \( \lambda \). Indeed, any minimizer of (1.1)-(1.2) satisfies the GL equation. Since \( |u_\lambda| = 1 \) a.e. on \( \partial A \), we have \( |u_\lambda| \leq 1 \) in \( A \), by the maximum principle [9]. Finally, we have \( E_\lambda(u_\lambda) \leq 2\pi \) for each \( \lambda \).

Since \( u_\infty \) is a constant, in view of Lemma 6, we have for large \( \lambda \) that \( 1/2 \leq |u_\lambda| \leq 1 \) on \( \Gamma \) and \( \deg(u_\lambda, \Gamma) = 0 \). Thus \( u_\lambda \) admits the representation \( u_\lambda = \rho_\lambda e^{i\varphi_\lambda} \) on \( \Gamma \) for large \( \lambda \). Here \( 1/2 \leq \rho_\lambda \leq 1 \) and \( \varphi_\lambda \) is \textbf{single-valued}. Therefore, we have

\[
\int_{\Gamma} u_\lambda \times u_{\lambda,\tau} = \int_{\Gamma} \rho_\lambda^2 \varphi_{\lambda,\tau} = \int_{\Gamma} (\rho_\lambda^2 - 1)\varphi_{\lambda,\tau}. \tag{3.11}
\]

On the other hand, the assumption that \( u_\infty \) is a constant and Lemma 6 imply that \( \nabla \varphi_\lambda \to 0 \) uniformly on \( \Gamma \), as \( \lambda \to \infty \). This fact, in conjunction with (3.11) and the estimate (3.6), yield

\[
\int_{\Gamma} u_\lambda \times u_{\lambda,\tau} = o\left(\frac{1}{\lambda}\right). \tag{3.12}
\]

The equation (3.12) along with (3.10) imply that

\[
m_\lambda \geq 2\pi - o\left(\frac{1}{\lambda}\right) \quad \text{as} \quad \lambda \to \infty. \tag{3.13}
\]

Since the inequality (3.13) contradicts the conclusion of Lemma 4, for large \( \lambda \), it follows that \( u_\infty \) is not a constant. In view of Step 3, the map \( u_\infty \) minimizes (??)-(??), hence \( u_{\lambda_n} \to u_\infty \) strongly in \( H^1 \) (cf. Step 1).

**Step 5.** Show that \( |u_\lambda| \to 1 \) uniformly in \( A \) as \( \lambda \to \infty \)

As we have, the sequence \((u_\lambda)\) is bounded in \( H^1(A) \). Moreover, if \( u_{\lambda_n} \to u_\infty \) weakly in \( H^1 \), it follows from Step 1 and Step 4 that \( u_{\lambda_n} \to u_\infty \) strongly in \( H^1 \) and \( u_\infty \) minimizes (??)-(??) . For such a sequence \((u_{\lambda_n})\), it remains to prove that \( |u_{\lambda_n}| \to 1 \) uniformly in \( \overline{A} \) as \( n \to \infty \).

Fix some \( a \in (0,1) \). We have to establish the inequality

\[
|u_{\lambda_n}(z)| \geq a \quad \text{in} \quad A \quad \text{for large} \ n. \tag{3.14}
\]

Recall the following

**Lemma 7.** [20] Let \( g_n, g \in \text{VMO}(\partial A; S^1) \) be such that \( g_n \to g \) in \( \text{VMO} \). Let \( \tilde{g}_n, \tilde{g} \) be the corresponding harmonic extensions to \( A \). Then, for each \( \varepsilon > 0 \), there is some \( \delta = \delta(\varepsilon) > 0 \) (independent of \( n \)) such that

\[
|\tilde{g}_n(z)| \geq 1 - \varepsilon \quad \text{if} \quad d(z) \leq \delta. \tag{3.15}
\]
Lemma 8. [9] Let \( v \in H^1_0(A) \) be such that \( \Delta v \in L^\infty \). Then, for some \( C \) depending only on \( A \), we have
\[
\|\nabla v\|_{L^\infty} \leq C\|v\|_{L^\infty}^{1/2}\|\Delta v\|_{L^\infty}^{1/2}. \tag{3.16}
\]

Set \( g_n = u_{\lambda_n}|_{\partial A}, \ g = u_\infty|_{\partial A} \). Since \( H^{1/2}(\partial A) \subset \text{VMO}(\partial A) \) and \( u_{\lambda_n} \to u_\infty \) in \( H^1(A) \), we find that \( g_n \to g \) in VMO. We consider a decomposition \( u_{\lambda_n} = \tilde{g}_n + v_{\lambda_n} \), where \( v_{\lambda_n} \in H^1_0(A) \) is the solution of \( -\Delta v_{\lambda_n} = \lambda_n u_{\lambda_n}(1 - |u_{\lambda_n}|^2) \). Observe that
\[
|v_{\lambda_n}| \leq |\tilde{g}_n| + |u_{\lambda_n}| \leq 2. \tag{3.17}
\]
Here we rely on the inequality \( |u_{\lambda_n}| \leq 1 \) and on the fact that, \( \tilde{g}_n \) being the harmonic extension of a map of modulus 1, itself has the modulus that does not exceed 1. Using Lemma 8 in conjunction with (3.17) and the definition of \( v_{\lambda_n} \), we find that
\[
|\nabla v_{\lambda_n}| \leq C\sqrt{2\lambda_n}. \tag{3.18}
\]
Then
\[
|v_{\lambda_n}(z)| \leq C_1\sqrt{\lambda_n}d(z) \tag{3.19}
\]
for some \( C_1 \) independent of \( n \), since \( v_{\lambda_n} = 0 \) on \( \partial A \). Combining (3.19) with Lemma 7 we obtain that there exist constants \( C_2 = C_2(a) \) and \( n_0 = n_0(a) \), such that
\[
|u_{\lambda_n}(z)| \geq a \quad \text{if} \quad d(z) \leq \frac{C_2}{\sqrt{\lambda_n}} \quad \text{and} \quad n \geq n_0. \tag{3.20}
\]
Returning to the proof of (3.14), we proceed as in [9]. We argue by contradiction. Suppose that (up to a subsequence) there are points \( z_n \in A \) such that \( |u_{\lambda_n}(z_n)| \leq a \). In view of (3.20), we have
\[
d(z_n) \geq \frac{C_2}{\sqrt{\lambda_n}}, \tag{3.21}
\]
for large \( n \).

By (3.7), given an arbitrary \( C_3 \in (0, C_2) \), there exists a constant \( C_4 > 0 \) independent of \( n \) and such that \( |\nabla u_{\lambda_n}(z)| \leq C_4\sqrt{\lambda_n} \) when \( |z - z_n| \leq \frac{C_3}{\sqrt{\lambda_n}} \). Since \( |u_{\lambda_n}(z_n)| \leq a \), we thus have
\[
|u_{\lambda_n}(z)| \leq \frac{1 + a}{2} \quad \text{if} \quad |z - z_n| \leq \frac{C_3}{\sqrt{\lambda_n}} \quad \text{and} \quad n \text{ is large}, \tag{3.22}
\]
provided we choose \( C_3 \) sufficiently small. For such \( C_3 \) and for a sufficiently large \( n \), we have
\[
\lambda_n \int_A (1 - |u_{\lambda_n}|^2)^2 \geq \lambda_n \int_{\Pi_n} (1 - |u_{\lambda_n}|^2)^2 \geq C_5, \tag{3.23}
\]
where \( \Pi_n := \{ z : |z - z_n| \leq C_3 / \sqrt{\lambda_n} \} \) and \( C_5 \) is independent of \( n \).

On the other hand, the upper bound (??), the strong-\( H^1 \) convergence \( u_{\lambda n} \to u_\infty \), together with the fact that \( u_\infty \) minimizes (??)-(??) yield
\[
I_0 \geq \lim_n \left( \frac{1}{2} \int_A |\nabla u_{\lambda n}|^2 + \frac{\lambda_n}{4} \int_A (1 - |u_{\lambda n}|^2)^2 \right) = I_0 + \lim_n \frac{\lambda_n}{4} \int_A (1 - |u_{\lambda n}|^2)^2.
\] (3.24)

Thus we must have
\[
\lim_n \frac{\lambda_n}{4} \int_A (1 - |u_{\lambda n}|^2)^2 = 0.
\] (3.25)

For large \( n \), the equations (3.23) and (3.25) contradict each other. Therefore, (3.14) holds and the proof of Theorem 2 is complete.

4 Rise of vortices

Except when otherwise noted, we assume that the domain \( A \) is supercritical throughout this section. First suppose that case (a) in Theorem 4 holds. As noted at the beginning of the proof of Theorem 3, the family \( (u_\lambda) \) is bounded in \( H^1(A) \). Thus, up to a subsequence, \( u_{\lambda n} \rightharpoonup u_\infty \), where \( u_\infty \in H^1(A; S^1) \). Next suppose that case (b) in Theorem 4 holds. Consider a minimizing sequence \( (u_k) \) for a fixed \( \lambda > \lambda_1 \). By the same argument as above, \( (u_k) \) is bounded in \( H^1(A) \) and, up to a subsequence, \( u_{k n} \rightharpoonup u_\infty \), where \( u_\infty \in H^1(A; \mathbb{C}) \). We begin by identifying \( u_\infty \).

**Lemma 9.** In both cases in Theorem 4 the map \( u_\infty \) is identically equal to a constant of modulus 1.

**Proof of Lemma 9:** Assume first case (a). By combining the Price Lemma, the upper bound (??), and Lemma 3, we find that
\[
2\pi \geq \liminf \frac{1}{2} \int_A |\nabla u_{\lambda n}|^2 \geq \frac{1}{2} \int_A |\nabla u_\infty|^2 + 2\pi |1 - \deg(u_\infty, \partial \Omega)|.
\] (4.1)

If \( \deg(u_\infty, \partial \Omega) \neq 1 \), then \( \nabla u_\infty = 0 \) a.e. and \( u_\infty \) has to be a constant. This constant is of modulus 1, since \( |u_\infty| = 1 \) a.e. on \( \partial A \). On the other hand, if \( \deg(u_\infty, \partial \Omega) = 1 \), then \( u_\infty \in K \) and (4.1) yields
\[
2\pi \geq \frac{1}{2} \int_A |\nabla u_\infty|^2 \geq I_0,
\] (4.2)

but in the supercritical case \( I_0 > 2\pi \). Thus \( u_\infty \) is a constant of modulus 1.
Next assume case (b). As the proof of Theorem 4 shows, \( m_\lambda = 2\pi \) for \( \lambda > \lambda_1 \). The Price Lemma implies that
\[
2\pi = m_\lambda = \lim_n E_\lambda(u^{k_n}) \geq E_\lambda(u_\infty) + \pi(|1 - \deg(u_\infty, \partial\Omega)| + |1 - \deg(u_\infty, \partial\omega)|). \tag{4.3}
\]
If \( \deg(u_\infty, \partial\Omega) = \deg(u_\infty, \partial\omega) = 1 \), then \( u_\infty \in \mathcal{J} \) and \( u_\infty \) minimizes (1.1)-(1.2) by (4.3). This, however, is impossible, since \( m_\lambda \) is not attained for \( \lambda > \lambda_1 \). If \( \deg(u_\infty, \partial\Omega) \neq 1 \) and \( \deg(u_\infty, \partial\omega) \neq 1 \), then (4.3) implies that \( u_\infty \) has to be equal to a constant (of modulus 1). Finally, if exactly one among \( \deg(u_\infty, \partial\Omega) \) and \( \deg(u_\infty, \partial\omega) \) equals 1, then (4.3) combined with (1.10) yields
\[
2\pi \geq 2\pi + \frac{\lambda}{4} \int_A (1 - |u_\infty|^2)^2. \tag{4.4}
\]
Therefore, \( u_\infty \) is a constant of modulus 1, which is in contradiction with the assumption on the degrees of \( u_\infty \). We conclude that \( u_\infty \) is a constant of modulus 1.

As a byproduct of the above lemma, it is easy to establish Proposition ??.

**Proof of Proposition ??:** Since \( m_\lambda \) is not decreasing, for each sequence \( \lambda_n \to \infty \) we have
\[
\lim_{\lambda \to \infty} m_\lambda = \lim_n m_{\lambda_n}.
\]
Assume first that \( A \) is subcritical or critical. Consider a sequence \( (\lambda_n) \) such that \( u_{\lambda_n} \to u_\infty \) strongly in \( H^1(A) \), where \( u_\infty \) minimizes (??)-(??). By combining the upper bound (??) with the definition of \( I_0 \), we find that
\[
I_0 \geq \lim_{\lambda \to \infty} m_\lambda = \lim_n E_{\lambda_n}(u_{\lambda_n}) \geq \frac{1}{2} \int_A |\nabla u_\infty|^2 = I_0. \tag{4.5}
\]
Thus \( \lim_{\lambda \to \infty} m_\lambda = I_0 \), as claimed.

Assume next that \( A \) is supercritical. In case (b), we have \( m_\lambda = 2\pi \) for large \( \lambda \) and, thus, (??) holds. In case (a), consider a sequence \( (\lambda_n) \) such that \( u_{\lambda_n} \rightharpoonup u_\infty \) weakly in \( H^1(A) \), where \( u_\infty \) is a constant of modulus 1. Using the Price lemma and the upper bound (??), we obtain
\[
2\pi \geq \lim_{\lambda \to \infty} m_\lambda = \lim_n E_{\lambda_n}(u_{\lambda_n}) \geq 2\pi, \tag{4.6}
\]
which yields \( \lim_{\lambda \to \infty} m_\lambda = 2\pi \) and (??) follows.

**Proof of Theorem 5. Case (b):** For \( \lambda > \lambda_1 \), we consider the behavior of a minimizing sequence \( (u^k) \). For the convenience of the reader, we divide the proof into six steps.

**Step 1. Decomposition of \( u^k \)**
Suppose that $v^k$ minimizes the GL energy $E_\lambda$ among all maps $v \in H^1(A)$ such that $v = u^k$ on $\partial A$. Clearly, (i) $v^k$ satisfies the GL equation $-\Delta v^k = \lambda u^k(1 - |v^k|^2)$, (ii) $|v^k| \leq 1$ (by the maximum principle), (iii) $v^k \in \mathcal{J}$, and (iv) the sequence $(v^k)$ is still a minimizing sequence for $E_\lambda$ in $\mathcal{J}$, since $E_\lambda(v^k) \leq E_\lambda(u^k)$. Set $w^k = u^k - v^k \in H^1_0(A)$.

**Lemma 10.** We have $w^k \to 0$ in $H^1(A)$ as $k \to \infty$.

**Proof of Lemma 10:** In view of Lemma 9, we may assume that, up to a subsequence, $u^k_n \rightharpoonup u$ and $v^k_n \rightharpoonup v$ weakly in $H^1(A)$, where $u$, $v$ are constants of modulus 1. Since $u^k = v^k$ on $\partial A$ we have $u = v$ and, hence, $w^k_n \to 0$. In fact, since this conclusion holds for every subsequence of the original sequence, it follows that $w^k \rightharpoonup 0$ weakly in $H^1(A)$.

Inserting the equality $u^k = v^k + w^k$ into the expression for $E_\lambda(u^k)$ and using the fact that $w^k \rightharpoonup 0$, we obtain

$$E_\lambda(u^k) = E_\lambda(v^k) + \frac{1}{2} \int_A |\nabla w^k|^2 + \int_A \nabla v^k \cdot \nabla w^k + o(1).$$

(4.7)

Furthermore,

$$\frac{1}{2} \int_A |\nabla w^k|^2 + \int_A \nabla v^k \cdot \nabla w^k \to 0 \quad \text{as } k \to \infty,$$

(4.8)

since both $(u^k)$ and $(v^k)$ are minimizing sequences. On the other hand, if we multiply by $w^k$ the GL equation satisfied by $v^k$ and integrate, we find that

$$\left| \int_A \nabla v^k \cdot \nabla w^k \right| = \left| \int_A \lambda v^k \cdot w^k(1 - |v^k|^2) \right| \leq \lambda \int_A |w^k| \to 0 \quad \text{as } k \to \infty,$$

(4.9)

by an embedding argument. Equation (4.8) when used in conjunction with (4.9) yields

$$\lim_k \int_A |\nabla w^k|^2 = 0.$$

Since $w^k = 0$ on $\partial A$, we find that $w^k \to 0$ in $H^1(A)$ by the Poincaré’s inequality and Lemma 10 follows.

In conclusion, modulo a small remainder $w_k$ in $H^1(A)$, we may replace a minimizing sequence $(u^k)$ by the minimizing sequence $(v^k)$, having two additional properties (i) and (ii). In the rest of the proof, we will study the behavior of the sequence $(v^k)$.

**Step 2.** Concentration of the energy near $\partial A$

We fix two simple curves $\gamma$ and $\Gamma$ in $A$, such that $\gamma$ encloses $\partial \omega$ and $\Gamma$ encloses $\gamma$. Let $U$ be the domain enclosed by $\partial \Omega$ and $\Gamma$, $V$ be the domain enclosed by $\gamma$ and $\partial \omega$ and set $W = A \setminus (\overline{U} \cup \overline{V})$. 18
Lemma 11. When $k \to \infty$, we have

$$
\int_A \left(1 - |v^k|^2 \right)^2 \to 0, 
$$

(4.10)

$$
\| \nabla v^k \|_{L^\infty(W)} \to 0, 
$$

(4.11)

$$
\| \partial_z v^k \|_{L^2(U)} \to 0 \quad \text{and} \quad \| \partial_{\bar{z}} v^k \|_{L^2(V)} \to 0, 
$$

(4.12)

$$
\frac{1}{2} \int_U |\nabla v^k|^2 \to \pi \quad \text{and} \quad \int_U \text{Jac} v^k \to \pi, 
$$

(4.13)

$$
\frac{1}{2} \int_V |\nabla v^k|^2 \to \pi \quad \text{and} \quad \int_V \text{Jac} v^k \to -\pi. 
$$

(4.14)

**Proof of Lemma 11:** We integrate the identities

$$
\frac{1}{2} |\nabla v^k|^2 = \text{Jac} v^k + 2 |\partial_z v^k|^2 
$$

and

$$
\frac{1}{2} |\nabla v^k|^2 = -\text{Jac} v^k + 2 |\partial_{\bar{z}} v^k|^2 
$$

over $U$ and $V$, respectively. We find that

$$
E_\lambda(v^k) = \int_U \text{Jac} v^k - \int_V \text{Jac} v^k + 2 \int_U |\partial_z v^k|^2
$$

$$
+ 2 \int_V |\partial_{\bar{z}} v^k|^2 + 2 \int_W |\nabla v^k|^2 + \frac{\lambda}{4} \int_A (1 - |v^k|^2)^2.
$$

(4.15)

An integration by parts combined with the degree formula (?) yields,

$$
\int_U \text{Jac} v^k = \pi - \frac{1}{2} \int_{\gamma} v^k \times v^k_{\tau} \quad \text{and} \quad - \int_U \text{Jac} v^k = \pi - \frac{1}{2} \int_{\gamma} v^k \times v^k_{\tau}
$$

(4.16)

for the counterclockwise orientation on $\gamma$ and $\Gamma$.

We claim that,

$$
\nabla v^k \to 0 \quad \text{in} \ C^0_\text{loc}(A),
$$

(4.17)
as $k \to \infty$. Then the conclusions of Lemma 11 can be obtained as follows. Using (4.17) we pass to the limit in (4.16) and, in turn, in (4.15). Here we take into account the facts that $|v^k| \leq 1$ and $\lim_{k} E_{\lambda}(v^k) = 2\pi$.

It remains to establish (4.17). Since $|v^k| \leq 1$, we have that $|\Delta v^k| \leq \lambda$. Since the sequence $(v^k)$ is bounded in $H^1$, it follows from standard elliptic estimates [24] that $(v^k)$ is bounded in $W^{2,p}_{\text{loc}}(A)$ for every $1 < p < \infty$. Furthermore, $(v^k)$ is relatively compact in $C^1_{\text{loc}}(A)$ due to the Sobolev embeddings. In view of Lemma 9, each subsequence of $(v^k)$ contains a further subsequence converging weakly in $H^1$ to a constant map of modulus 1. It is easy to see that this property, along with the fact that $(v^k)$ is relatively compact in $C^1_{\text{loc}}(A)$, implies (4.17). Note for further use, that the same argument implies that $|v^k| \to 1$ in $C^1_{\text{loc}}(A)$.

**Step 3. Existence of zeros**

**Lemma 12.** There is some $k_0 \in \mathbb{N}$ such that, for $k \geq k_0$, the map $v^k$ has at least one zero $\zeta_k$ in $U$, at least one zero $\xi_k$ in $V$, and no zeros in $\overline{W}$. In addition, for every zero $\zeta_k$ in $U$ and $\xi_k$ in $V$, we have $\text{dist}(\zeta_k, \partial \Omega) \to 0$ and $\text{dist}(\xi_k, \partial \omega) \to 0$, respectively, as $k \to \infty$.

**Proof of Lemma 12:** Non-existence of zeros in $\overline{W}$ for large $\lambda$ and the last property follow from the fact that $|v^k| \to 1$ in $C^1_{\text{loc}}(A)$. It remains to establish existence of zeros in $U$ and in $V$ for large $\lambda$.

We argue by contradiction. Assume, for example, that, up to a subsequence, $v^k \neq 0$ in $U$. Then we claim that, for every $k$, there exists $C_k > 0$ such that $C_k \leq |v^k| \leq 1$ in $U$. Since $|v^k| \to 1$ in $C^1_{\text{loc}}(U)$, it remains to show that $v^k$ is bounded away from zero near $\partial \Omega$. Indeed, Lemma 7 applied to $g = v^k|_{\partial A}$, $g_n \equiv g$, implies that there is some $\delta_1 > 0$ such that $\overline{g}(z) \geq 3/4$ if $d(z) < \delta_1$. On the other hand, if we set $w^k = v^k - \overline{g}(z) \in H^1_0(A)$, then $\Delta w^k \in L^\infty(A)$ and $w^k \in C^1_0(A)$. Therefore, there is some $\delta_2 > 0$ such that $|w^k(z)| \leq 1/4$ if $d(z) < \delta_2$. We conclude that $|v^k(z)| \geq 1/2$ if $d(z) < \text{Min}(\delta_1, \delta_2)$ and the claim follows.

Set $y_k = v^k/|v^k|$. This map belongs to $H^1(U; S^1)$, since $C_k \leq |v^k| \leq 1$ in $U$. Due to Lemma 3 we have $\text{deg}(y_k, \Gamma) = \text{deg}(y_k, \partial \Omega)$, hence $\text{deg}(y_k, \Gamma) = 1$ since $y_k = v^k$ on $\partial \Omega$. Therefore, $\text{deg}(v^k, \Gamma) = \text{deg}(y_k, \Gamma) = 1$. This is impossible since, up to a subsequence, $v^k \to v$ in $C^1(\Gamma)$, and $v$ is a constant of modulus 1. The proof of Lemma 12 is complete.

**Step 4. Rescaling of $v^k$**

Recall that $\nabla v^k \to 0$ and $|v^k| \to 1$ in $C^1(\Gamma)$. Thus, we can extend $v^k|_\Omega$ to $\Omega$ so that the extension $u^k_1$ satisfies $\|\nabla u^k_1\|_{L^\infty(\Omega; U)} \to 0$ and $1/2 \leq |u^k_1| \leq 1$ in $\Omega \setminus U$ for large $k$. Similarly, $v^k|_V$ has an extension $u^k_2$ to $\mathbb{C} \setminus \overline{\omega}$ satisfying $\|\nabla u^k_2\|_{L^\infty(\mathbb{C} \setminus V)} \to 0$ and $1/2 \leq |u^k_2| \leq 1$ in $\mathbb{C} \setminus V$ for large $k$.

Let $\Phi$ be a fixed conformal representation of $\Omega$ into $\mathbb{D}_1$. It is well-known that conformal representations $\Phi_k$ of $\Omega$ into $\mathbb{D}_1$ satisfying the property $\Phi_k(\zeta_k) = 0$ are given by $\Phi_k(z) = \alpha \frac{\Phi(z) - \Phi(\zeta_k)}{1 - \Phi(\zeta_k)\Phi(z)}$. 20
Similarly, we may construct a conformal representation \( \Psi \) on \( S^1 \) of \( \gamma \). Moreover, the trace of \( \gamma \) on \( S^1 \) has modulus 1 and degree 1 (since \( \Phi_k \) preserves the orientation of curves). It is easy to see that, for an appropriate choice of \( \alpha \), we may assume that \( \partial_\gamma y_k(0) \geq 0 \). Similarly, we may construct a conformal representation \( \Psi_k \) of \( \mathbb{C} \setminus \gamma \) onto \( \mathbb{D}_1 \) vanishing at \( \xi_k \) and such that \( \zeta_k = \nu_2 \circ \Psi_k^{-1} \) has the same properties as \( y_k \).

In the remaining part of the proof, we study the asymptotic properties of \( y_k \) and \( z_k \) and relate these properties to the asymptotic behavior of \( v^k \). The reason we prefer to deal with \( y_k \) and \( z_k \) instead of \( v^k \) is a lack of strong-\( H^1 \)-convergence: as we have already seen, up to a subsequence, \( v^{k_n} \rightharpoonup v \), where \( v \) is some constant of modulus 1. In particular, \( (v^{k_n}) \) is not strongly convergent in \( H^1 \), since the degrees change in the limit. However, as we will establish below, \( y_k \) and \( z_k \) do strongly converge in \( H^1(\mathbb{D}_1) \). We focus on the behavior of \( y_k \); the analysis for \( z_k \) is the same.

Recall some elementary properties of the \( \Phi_k \).

**Lemma 13.** [7] For every \( r \in (0,1) \), there are constants \( C_j = C_j(r) \) independent of \( k \) and such that:

(i) \( \Phi_k^{-1}(\mathbb{D}_r) \subset \{ z \in \Omega ; |z - \zeta_k| \leq C_1 d(\zeta_k, \partial \Omega) \text{ and } d(z, \partial \Omega) \geq C_2 d(\zeta_k, \partial \Omega) \}; \)

(ii) \( |\nabla \Phi_k^{-1}| \leq C_3 d(\zeta_k, \partial \Omega) \) in \( \mathbb{D}_r \).

For each \( R_1, R_2 > 0 \), there is an \( r \in (0,1) \) independent of \( k \) such that

(iii) \( \Phi_k(\{ z \in \Omega ; |z - \zeta_k| \leq R_1 d(\zeta_k, \partial \Omega) \text{ and } d(z, \partial \Omega) \geq R_2 d(\zeta_k, \partial \Omega) \}) \subset \mathbb{D}_r \).

**Lemma 14.** We have \( y_k \to \text{id} \) and \( z_k \to \text{id} \) strongly in \( H^1(\mathbb{D}_1) \) and in \( C^1_{\text{loc}}(\mathbb{D}_1) \).

**Proof of Lemma 14:** Since the Dirichlet integral is conformally invariant, using Lemma 11 we have

\[
\int_{\mathbb{D}_1} |\nabla y_k|^2 = \int_{\Omega} |\nabla v^k_1|^2 = \int_{\Omega} |\nabla v^k|^2 + \int_{\Omega \setminus U} |\nabla v^k|^2 = 2\pi + o(1),
\]

as \( k \to \infty \). Similarly

\[
\int_{\mathbb{D}_1} (|\nabla y_k|^2 - 2 \text{Jac} y_k) = o(1),
\]

as \( k \to \infty \).

The fact that \( |y_k| \leq 1 \), combined with (4.18) implies that \( (y_k) \) is bounded in \( H^1(\mathbb{D}_1) \). Let \( y \in H^1(\mathbb{D}_1) \) be such that, up to a subsequence, \( y_{k_n} \rightharpoonup y \). Then \( |y| = 1 \) a.e. on \( S^1 \).

Since the map \( u \mapsto \int_{\mathbb{D}_1} (|\nabla u|^2 - 2 \text{Jac} u) \) is convex and continuous for \( u \in H^1(\mathbb{D}_1) \) (and, thus, weakly l.s.c.), equation (4.19) and the fact that \( y_{k_n} \rightharpoonup y \) imply

\[
\int_{\mathbb{D}_1} (|\nabla y|^2 - 2 \text{Jac} y) = 4 \int_{\mathbb{D}_1} |\partial_\gamma y|^2 \leq 0.
\]

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Thus $\partial_\gamma y = 0$ a.e. in $\mathbb{D}_1$, that is $y$ is holomorphic in $\mathbb{D}_1$. Set $g = y_{|S^1} \in H^{1/2}(S^1; S^1)$, whose Fourier expansion is of the form $g = \sum_{l=0}^{\infty} a_l e^{it\theta}$. Then $\deg g = \sum_{l=0}^{\infty} l|a_l|^2$ (when $g$ is smooth, this equation is equivalent to the degree formula (4.3); the same equality still holds for a general $g \in H^{1/2}(S^1; S^1)$ [14]). On the other hand, since $y$ is holomorphic, it is the harmonic extension of $g$, hence

$$
\int_\mathbb{D} |\nabla y|^2 = 2\pi \sum_{l=0}^{\infty} l|a_l|^2 = 2\pi \deg g \leq 2\pi,
$$

(4.21)

where the last inequality follows from (4.18). Therefore, either $\deg g = 0$ and $y$ is a constant of modulus 1 or $\deg g = 1$.

First, we rule out the possibility that $y$ is a constant. For a large $k$, the set

$$M_k := \{ z \in \Omega ; \, |z - \zeta_k| \leq C_1 d(\zeta_k, \partial \Omega) \text{ and } d(z, \partial \Omega) \geq C_2 d(\zeta_k, \partial \Omega)\}
$$

is contained in $U$ and thus $|\Delta v_k^1| = \lambda |v_k^1(1 - |v_k|^2)| \leq \lambda$ in $M_k$. Using Lemma 13 (ii) and Lemma 12, we find that

$$|\Delta y_k| = \frac{1}{2} |\nabla \Phi_k^{-1}| (\Delta v_k^1) \circ \Phi_k^{-1} \to 0 \quad \text{uniformly in } \mathbb{D}_r \text{ as } k \to \infty.
$$

(4.22)

Since $(y_k)$ is bounded in $H^1$, it follows from standard elliptic estimates that $(y_k)$ is relatively compact in $C^1_{\text{loc}}(\mathbb{D}_1)$. In particular, $y_{k_n} \to y$ uniformly in $\mathbb{D}_{1/2}$. Recalling that $y_k(0) = 0$, we find that $y(0) = 0$, that is, $y$ cannot be a constant of modulus 1.

Next, we identify $y$. Lemma 7 applied to $g_n \equiv g$ implies that $|y(z)| \to 1$ uniformly as $|z| \to 1$. We recall that a holomorphic map $y$ in $\mathbb{D}$ satisfying $|y(z)| \to 1$ uniformly as $|z| \to 1$ is a Blaschke product, i.e., $y(z) = \alpha \prod_{j=1}^{d} \frac{z - a_j}{1 - \bar{a}_j z}$ for some $\alpha \in S^1$ and $a_1, \ldots, a_d \in \mathbb{D}$ [21]. Here $d$ is the degree of $y_{|S^1}$. In our case $d = 1$ and $y(0) = 0$, thus $y = \alpha \text{id}$ with $\alpha \in S^1$. Since $\partial_\gamma y_k(0) \geq 0$, we have $\alpha = \partial_\gamma y(0) \geq 0$, hence $\alpha = 1$ and $y = \text{id}$.

The uniqueness of the weak limit implies that $y_k \to \text{id}$ in $H^1$. Formula (4.18) combined with the fact that $\int_\mathbb{D} |\nabla \text{id}|^2 = 2\pi$ yields $y_k \to \text{id}$ in $H^1$. Further, since the sequence $(y_k)$ is relatively compact in $C^1_{\text{loc}}(\mathbb{D})$, it follows that $y_k \to \text{id}$ in $C^1_{\text{loc}}(\mathbb{D})$.

**Step 5.** Holomorphic (anti-holomorphic) behavior of $v_k$ near $\partial \Omega$ ($\partial \omega$).

As an immediate consequence of Lemma 14, we obtain the following

**Lemma 15.** We have $v_k - \Phi_k \to 0$ in $L^2_{\text{loc}}(\overline{A} \setminus \partial \omega)$ and $v_k - \overline{\Phi}_k \to 0$ in $L^2_{\text{loc}}(\overline{A} \setminus \partial \Omega)$. 

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Proof of Lemma 15: We prove the first assertion. Fix a compact $K \subset \overline{A \setminus \partial \omega}$. Since the curves $\gamma$ and $\Gamma$ introduced in Step 2 are arbitrary, we have, thanks to Lemma 11,

$$\int_{K \setminus U} |\nabla v^k|^2 \to 0 \quad \text{as } k \to \infty. \quad (4.23)$$

On the other hand, Lemma 13 (i) and the fact that $d(\zeta_k, \partial \Omega) \to 0$ imply that $\Phi_k(K \setminus U) \subset \overline{D \setminus D_{r_k}}$ for some sequence $r_k \to 1$. The conformal invariance of the Dirichlet integral yields

$$\int_{K \setminus U} |\nabla \Phi|^2 = \int_{\Phi_k(K \setminus U)} |\nabla \text{id}|^2 \leq \int_{\overline{D \setminus D_{r_k}}} |\nabla \text{id}|^2 \to 0 \quad \text{as } k \to \infty. \quad (4.24)$$

Finally,

$$\int_{K \cap U} |\nabla \Phi - \nabla v^k|^2 \leq \int_{\Phi_k(U)} |\nabla \Phi - \nabla y_k|^2 \to 0 \quad \text{as } k \to \infty, \quad (4.25)$$

by Lemma 14 and the conformal invariance. The conclusion of Lemma 15 follows by combining the estimates (4.23)-(4.25).

Step 6. Uniqueness of zeros of $v^k$ and their degrees of for large $k$.

We argue by contradiction and assume that, possibly up to a subsequence, $v^k$ has two distinct zeros $\zeta_k$ and $\tilde{\zeta}_k$ in $U$. Without loss of generality, we may further assume that

$$d(\zeta_k, \partial \Omega) \geq d(\tilde{\zeta}_k, \partial \Omega). \quad (4.26)$$

Let $\Phi_k$ and $\tilde{\Phi}_k$ be the corresponding conformal representations. Given any $r \in (0, 1)$, we claim that $\Phi^{-1}_k(D_r) \cap \tilde{\Phi}^{-1}_k(D_r) = \emptyset$ for a sufficiently large $k$. Indeed, suppose that $z \in \Phi^{-1}_k(D_r) \cap \tilde{\Phi}^{-1}_k(D_r)$ and let $C_1$ be as defined in Lemma 13. We have

$$|z - \zeta_k| \leq C_1 d(\zeta_k, \partial \Omega), \quad |z - \tilde{\zeta}_k| \leq C_1 d(\tilde{\zeta}_k, \partial \Omega), \quad (4.27)$$

by Lemma 13 (i) and, therefore

$$|\tilde{\zeta}_k - \zeta_k| \leq 2C_1 d(\zeta_k, \partial \Omega). \quad (4.28)$$

Equations (4.26) and (4.28), along with Lemma 13 (iii) imply the existence of some fixed $\rho \in (0, 1)$ such that $\Phi_k(\tilde{\zeta}_k) \in \overline{D_\rho}$ for every $k \in \mathbb{N}$. However, this is impossible for large $k$, since on the one hand $y_k = v^k \circ \Phi^{-1}_k \to \text{id}$ in $C^1(\overline{D_\rho})$ (and thus, for large $k$, $y_k|_{\overline{D}_r}$ is into), while on the other hand $y_k(\Phi_k(\zeta_k)) = y_k(\Phi_k(\tilde{\zeta}_k)) = 0$ for each $k$. The claim is proved.
Now fix \( r \in (1/\sqrt{2}, 1) \) so that \( \int_{\mathbb{D}_r} |\nabla \text{id}|^2 = 2\pi r^2 > \pi \). Setting \( \tilde{y}_k = v^k \circ \Phi_k^{-1} \), we obtain from Lemma 14 that
\[
\frac{1}{2} \int_U |\nabla v^k|^2 \geq \frac{1}{2} \int_{\Phi_k^{-1}(\mathbb{D}_r) \cup \Phi_k^{-1}(\mathbb{D}_r)} |\nabla y_k|^2 + \frac{1}{2} \int_{\mathbb{D}_r} |\nabla \tilde{y}_k|^2 \to 2\pi r^2,
\]
(4.29)
as \( k \to \infty \). Given our choice of \( r \), equation (4.29) contradicts equation (4.13) thus proving the uniqueness of \( \zeta_k \).

Next, we determine, for large \( k \), the degree of \( v^k \) around \( \zeta_k \). Since \( y_k \to \text{id} \) strongly in \( C^1_{\text{loc}} \) and \( y_k(0) = 0 \), it follows for large \( k \) that \( y_k \) has a zero of degree 1 at the origin. Since the diffeomorphism \( \Phi_k \) is orientation preserving, we find that \( v^k \) has a zero of degree 1 at \( \zeta_k \) for large \( k \). Similarly, \( v^k \) has a zero of degree \(-1\) at \( \xi_k \) for large \( k \).

**Proof of Theorem 5. Case (a):** Our purpose is to describe the behavior of a family \( (u_\lambda) \) of minimizers of (1.1)-(1.2) as \( \lambda \to \infty \). The proof follows essentially the same lines as the one in case (b). We point out the changes that have to be made.

Step 1 is not needed here, since the minimizers already satisfy the GL equation and the property \( |u_\lambda| \leq 1 \). The equations
\[
\lambda \int_A (1 - |u_\lambda|^2)^2 \to 0, \quad (4.30)
\]
\[
\|\nabla u_\lambda\|_{L^\infty(W)} \to 0, \quad (4.31)
\]
\[
\|\partial_x u_\lambda\|_{L^2(U)} \to 0 \quad \text{and} \quad \|\partial_z u_\lambda\|_{L^2(V)} \to 0, \quad (4.32)
\]
\[
\frac{1}{2} \int_U |\nabla u_\lambda|^2 \to \pi \quad \text{and} \quad \int_U \text{Jac} u_\lambda \to \pi, \quad (4.33)
\]
\[
\frac{1}{2} \int_V |\nabla u_\lambda|^2 \to \pi \quad \text{and} \quad \int_V \text{Jac} u_\lambda \to -\pi. \quad (4.34)
\]
correspond to (4.10)-(4.14) in Step 2. However, while (4.10)-(4.14) were obtained via (4.17), the estimate (3.12) has to be used in case (a). Note that, although we established (3.12) in the critical case, the only assumption that needed there was that all possible weak-\( H^1 \) limits of sequences \( (u_{\lambda_n}) \) are constants. Hence (3.12) is still valid in the present context.

Using the same proof as in Step 3 in case (b), we find for large \( \lambda \) that \( u_\lambda \) has zeros \( \zeta_\lambda \) and \( \xi_\lambda \) in \( U \) and in \( V \), respectively. Moreover,
Lemma 16. We have that $\lambda^{1/2}d(\zeta_\lambda, \partial \Omega) \to 0$ and $\lambda^{1/2}d(\xi_\lambda, \partial \omega) \to 0$ as $\lambda \to \infty$.

Proof of Lemma 16: We establish the first assertion. By (3.7), we have for some constant $C$ independent of large $\lambda$ that

$$|\nabla u_\lambda(z)| \leq \frac{C}{d(\zeta_\lambda, \partial \Omega)} \quad \text{if} \quad |z - \zeta_\lambda| \leq \frac{1}{2}d(\zeta_\lambda, \partial \Omega). \tag{4.35}$$

Thus, choosing $c_\lambda = \frac{1}{2} \min \{1, 1/C\} d(\zeta_\lambda, \partial \Omega)$, we have $D_{c_\lambda}(\zeta_\lambda) \subset A$ and $|u_\lambda| \leq 1/2$ in $D_{c_\lambda}(\zeta_\lambda)$. Therefore,

$$\lambda \int_A (1 - |u_\lambda|^2)^2 \geq \lambda \int_{D_{c_\lambda}(\zeta_\lambda)} (1 - |u_\lambda|^2)^2 \geq \frac{9\pi \lambda c_\lambda^2}{16}. \tag{4.36}$$

The conclusion of Lemma 16 follows by combining (4.30) with (4.36).

Next, we consider the rescaled maps $y_\lambda = u_\lambda \circ \Phi^{-1}_\lambda$ and $z_\lambda = u_\lambda \circ \Psi^{-1}_\lambda$, where $\Phi_\lambda$ and $\Psi_\lambda$ are suitable conformal representations vanishing at $\zeta_\lambda$ and $\xi_\lambda$, respectively. Step 4 works using the same proof as before except when establishing the analog of (4.22), which is

$$|\Delta y_\lambda| \to 0 \quad \text{in} \quad C^{0}_{\text{loc}}(\mathbb{D}). \tag{4.37}$$

The argument that leads to (4.37) is as follows. Let $r \in (0, 1)$ be given. By combining Lemma 13 (i) and (ii) with Lemma 16, we have

$$\|\Delta y_\lambda\|_{L^\infty(D_r)} = \frac{1}{2} \|\nabla \Phi^{-1}_\lambda \|_{L^\infty(D_r)} \leq C_3 \lambda d^2(\zeta_\lambda, \partial \Omega) \to 0, \tag{4.38}$$

as $\lambda \to \infty$.

Finally, Steps 5 and 6 are the same, and no changes are needed in the proof.

References


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