HOMOGENIZATION OF HARMONIC MAPS AND SUPERCONDUCTING COMPOSITES

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Abstract. We consider the Neumann boundary value problem for harmonic maps in an annular domain with a large number of small holes. On the boundary of the domain the degree condition is prescribed. We obtain an effective (homogenized) anisotropic nonlinear problem. We also discuss applications of these results to homogenized description of superconducting composites and superfluidity.

Key words. Homogenization, Ginzburg-Landau energy, Harmonic map, Superconductivity, Superfluidity.

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1. Introduction. We consider a homogenization problem for harmonic maps which describes an ideal superconductor reinforced by a large number of very thin insulating rods.

Let $\Omega$ and $G$ be bounded simply connected domains in $\mathbb{R}^2$ with smooth boundaries such that $\bar{G} \subset \Omega$. For simplicity of presentation we assume for a moment that $\Omega$ and $G$ are concentric disks. We further consider a periodic set $\bar{B}_\epsilon = \bigcup_i B_i^\epsilon$ in $\mathbb{R}^2$ which consists of identical inclusions $B_i^\epsilon$ of size $\epsilon > 0$ centered at the sites of a plane lattice of periods $\varepsilon h_1$ and $\varepsilon h_2 (h_2 > h_1 > 1)$ in the $x_1$ and $x_2$ directions respectively. Again for the sake of simplicity we assume for a moment that the inclusions are discs of diameter $\varepsilon$ and denote by $B_\varepsilon = \bigcup_i B_i^\varepsilon$ the set of inclusions inside the annulus $Q = \Omega \setminus G$. Then $Q_\varepsilon = \Omega \setminus (\bar{G} \cup \bigcup_i B_i^\varepsilon)$ is a non simply connected bounded domain in $\mathbb{R}^2$ (i.e. $Q_\varepsilon$ is a domain with "holes" $B_i^\varepsilon$, $\varepsilon > 0$ is a sufficiently small positive number).

Consider the following boundary value problem:

\[ \Delta u_\varepsilon + u_\varepsilon |\nabla u_\varepsilon|^2 = 0, \text{ in } Q_\varepsilon \quad (0.1) \]

\[ |u_\varepsilon| = 1 \quad (0.2) \]

\[ \frac{\partial u_\varepsilon}{\partial \nu} = 0, \text{ on } \partial Q_\varepsilon \quad (0.3) \]

\[ \text{deg} (u_\varepsilon, \partial \Omega) = d, \text{deg} (u_\varepsilon, \partial G) = d, \quad (0.4) \]

\[ \text{deg} (u_\varepsilon, \partial B_i^\varepsilon) = 0, i = 1, 2, \ldots N_\varepsilon \quad (0.5) \]

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where \( u(x_1, x_2) \) is a complex-valued function \( Q_\varepsilon \to S^1 \), \( d \) is an integer, \( \deg(v, \Gamma) \) is the Brouwer degree or winding number of a function \( v \) considered as a map from \( \Gamma \) into the unit sphere \( S^1 \) (for example, the function \( e^{i\theta} \) where \( \theta \) is the polar angle satisfies (0.4) and also explains our choice of orientation).

Our main goal is to describe asymptotic behavior of the solution \( u_\varepsilon \) as \( \varepsilon \to 0 \). It is shown below that the solutions \( u_\varepsilon \) converge in a certain sense (see Theorem 1 for precise formulation) to the solution \( u \) of the following homogenized problem (\( u \) maps \( Q = \Omega \setminus G \) into \( S^1 \)):

\[
\sum_{k,l} a_{kl} \frac{\partial^2 u}{\partial x_k \partial x_l} + u \sum_{k,l} a_{kl} \frac{\partial u}{\partial x_k} \frac{\partial \bar{u}}{\partial x_l} = 0 \quad \text{in} \quad Q, \tag{0.6}
\]

\[
\frac{\partial u}{\partial n} \equiv \sum_{k,l} a_{kl} \frac{\partial u}{\partial x_k} \cos(\nu, x_k) = 0 \quad \text{on} \quad \partial Q. \tag{0.7}
\]

Furthermore, assuming that the flux density is set (extended) to be zero on \( \bigcup B_i \), i.e.

\[
j_\varepsilon(x) = \begin{cases} \text{Im} [\bar{u}_\varepsilon \nabla u_\varepsilon], & x \in Q_\varepsilon \\ 0, & x \in \bigcup B_i \end{cases} \tag{0.8}
\]

it converges weakly in \( L_2(Q) \) to the vector-function (homogenized flux):

\[
j(x) = \text{Im} \left[ \bar{u} \sum_{k,l} a_{kl} \frac{\partial u}{\partial x_l} e^k \right]. \tag{0.9}
\]

The constant tensor \( a_{ik} \) is determined by solutions of some linear problems on the unit cell of periodicity (the so-called cell problem).

We remark here that the most distinctive feature of the problem (0.1)-(0.5) is the degree condition (0.4), which is both quite physical (unlike the Dirichlet condition which is hard to control in actual experiments) and mathematically challenging. To illustrate mathematical difficulties caused by the degree condition, we recall from [3] that the minimizers of the Ginzburg-Landau type energy

\[
\frac{1}{2} \int_Q \left\{ |\nabla u|^2 + \frac{1}{\delta^2} (|u|^2 - 1)^2 \right\}, \tag{0.10}
\]

with Dirichlet boundary data \( g \) of a given degree converge to the solution of a corresponding harmonic map problem with the same Dirichlet data in star-shaped domains. However replacing the Dirichlet boundary data by the degree condition (and the natural boundary condition) changes the picture drastically and the analogous convergence results are no longer true. For simply connected domains it can be shown by using relatively simple techniques and for nonsimply connected domains the analogous analysis becomes much more delicate ([18], [19]).

Furthermore the main idea behind the proof of the convergence result for harmonic maps is the reduction of the original nonlinear problem to a linear one for the phase \( \Psi_\varepsilon \). However since \( \Psi_\varepsilon \) is not a single-valued function (in fact it becomes a single-valued function on a Riemannian surface which looks like a spiral staircase), it is much more
convenient to consider the limit $\varepsilon \to 0$ for its conjugate $\Phi_\varepsilon$. It is single-valued and the main difficulties are caused by the degree condition, which prevents us from using standard results on homogenization for linear problems in domains with holes (in fact holes for the phase $\Psi_\varepsilon$ become some sort of dual holes for $\Phi_\varepsilon$). We also remark that we learned how to introduce an auxiliary problem for the conjugate of the phase from [3]. In our work this auxiliary problem is used for homogenization purposes. We would like to acknowledge here that the formulation of this problem was inspired by reading the book [3] from which we have learned many ideas and methods about the theory of harmonic maps into $S^1$. Note that some physical results describing homogenized properties of a superconducting composite and superfluid with rods are also presented in [17].

We shall now explain the motivation of our work and describe physical problems which lead to the boundary value problem (0.1)-(0.5).

Recall (see for example [1]) that the free energy of a superconductor $Q$ is determined by the following expression (up to an additive constant)

$$F(u, A) = \int_Q \left\{ |\nabla u - iAu|^2 + \lambda^2 |\text{curl} A|^2 + \frac{1}{\xi^2} (|u|^2 - 1)^2 \right\}, \quad (0.11)$$

which is called the Ginzburg-Landau functional [2] (see also [15] and references therein). Here $u = u(x)$ is a complex-valued function called the wave function (order parameter) of superconducting charge-carriers (Cooper pairs), $A$ is the magnetic (vector) potential (i.e. $\text{curl} A$ is the magnetic field), $\xi$ and $\lambda$ are numerical parameters which have the dimensionality of length and depend on the material properties of the superconductor and its temperature ($\xi$ is the (Ginzburg-Landau) coherence length and the parameter $\lambda$ is the London penetration depth (depth of the magnetic field penetration [1])). Electromagnetic processes in the superconductor are described by the pair $(u, A)$ which minimizes the functional (1) under suitable boundary conditions. The wave function can be normalized so that $|u|^2$ is proportional to the density of the charge carriers and the supercurrent is determined by the formula

$$j_S = \text{Im}(\bar{u}\nabla u) - A|u|^2. \quad (0.12)$$

We consider a composite superconductor $Q$ reinforced by dielectric (nonconducting) rods. Namely, we assume that the sample is confined between two long coaxial vertical cylinders $\Omega$ and $G$, and that the superconducting matrix (medium) $Q = \Omega \setminus G$ is reinforced by a large number, $N_\varepsilon$, of thin rods $B^\varepsilon_i$ ($i = 1, \ldots, N_\varepsilon$), which are aligned in the same direction as the cylinders (i.e. the rods are also vertical). If we put such a sample in a weak magnetic field (which does not destroy its superconductivity) and then decrease the temperature below critical, a current appears in the superconductor and the magnetic field is expelled from the superconductor. After the applied magnetic field is terminated, a part of this field is trapped by the space $G$ and, perhaps, by rods $B^\varepsilon_i$. According to the theory of superconductivity [1] the field is expelled from the superconductor $Q_\varepsilon = Q \setminus \bigcup B^\varepsilon_i$ almost completely and the space $G$ traps an integer number $d$ of quanta of the magnetic field. We assume that the field passes through the interior of the cylinder $G$ (cavity) but not through the rods. Due to the quantization this assumption is reasonable for very thin rods. Then the vector-potential $A$ satisfies the equalities

$$\int_{\partial G} (A \cdot \tau) = \int_{\partial \Omega} (A \cdot \tau) = 2\pi d, \quad \int_{\partial B^\varepsilon_i} (A \cdot \tau) = 0, \quad (i = 1, \ldots, N_\varepsilon), \quad (0.13)$$

$$\int_{\partial G} (A \cdot \tau) = \int_{\partial \Omega} (A \cdot \tau) = 2\pi d, \quad \int_{\partial B^\varepsilon_i} (A \cdot \tau) = 0, \quad (i = 1, \ldots, N_\varepsilon), \quad (0.13)$$

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$$\int_{\partial G} (A \cdot \tau) = \int_{\partial \Omega} (A \cdot \tau) = 2\pi d, \quad \int_{\partial B^\varepsilon_i} (A \cdot \tau) = 0, \quad (i = 1, \ldots, N_\varepsilon), \quad (0.13)$$
where $\tau$ is the unit tangent vector to the boundaries $\partial \Omega, \partial G, \partial B_i^\varepsilon$, which together with the unit outward normal $\nu$ form the properly oriented frame $(\nu, \tau)$. The absence of supercurrent flow through boundaries can be written as follows:

$$j_{S\nu} = (j_S \cdot \nu) = 0, \text{ on } \partial \Omega, \partial G, \partial B_i^\varepsilon. \quad (0.14)$$

For simplicity we denote by $G, \Omega, B_i^\varepsilon$ the two dimensional cross-sections of the corresponding cylinders by a (horizontal) plane orthogonal to their axes, and by $\partial \Omega, \partial G, \partial B_i^\varepsilon$ the corresponding one dimensional boundaries (see fig. 1).

**Fig. 1.1. Cross-section of the sample**

Furthermore, we assume that the cylinders are infinitely long and that the magnetic field does not depend on the vertical coordinate $x_3$. Hence the problem becomes planar and the wave function $u_\varepsilon$ and the vector-potential $A_\varepsilon$ satisfy the equalities

$$|u_\varepsilon| = 1, \text{ i.e. } u_\varepsilon = e^{i\Psi_\varepsilon} \text{ in } Q_\varepsilon, \quad (0.15)$$

$$A_\varepsilon = \text{grad} \Psi_\varepsilon \text{ in } Q_\varepsilon, \quad (0.16)$$

$$\int_{\partial G} \frac{\partial \Psi_\varepsilon}{\partial \tau} = \int_{\partial \Omega} \frac{\partial \Psi_\varepsilon}{\partial \tau} = 2\pi d, \quad \int_{\partial B_i^\varepsilon} \frac{\partial \Psi_\varepsilon}{\partial \tau} = 0, \quad (0.17)$$

$$\frac{\partial \Psi_\varepsilon}{\partial \nu} - A_{\varepsilon \nu} = 0 \text{ on } \partial Q_\varepsilon. \quad (0.18)$$
Here (0.15) and (0.16) follow from (0.11), (0.17) follows from (0.13) and (0.16); (0.18) is a consequence of (0.14).

Notice, that the pair \((u_\varepsilon, A_\varepsilon)\) is determined up to the gauge transformation

\[ u \to ue^{i\varphi}, \quad A \to A + \text{grad}\varphi, \]

where \(\varphi\) is a single-valued function in \(Q_\varepsilon\). Consequently, the vector-potential \(A_\varepsilon\) can be chosen so that the conditions

\[ \text{div}A_\varepsilon = 0, \quad \text{in } Q_\varepsilon, \]

\[ A_{\varepsilon\nu} = 0 \quad \text{on } \partial Q_\varepsilon \tag{0.19} \]

hold. Then, from (0.16) and (0.18) we obtain, that locally the phase \(\Psi_\varepsilon\) satisfies the equation

\[ \Delta \Psi_\varepsilon = 0 \quad \text{in } Q_\varepsilon, \tag{0.20} \]

and the boundary condition

\[ \frac{\partial \Psi_\varepsilon}{\partial \nu} = 0 \quad \text{on } \partial Q_\varepsilon. \tag{0.21} \]

In view of (0.15) we can identify the function \(u_\varepsilon(x)\) with the mapping of \(Q_\varepsilon\) into the unit sphere \(S^1\). Then, taking into account (0.15), (0.17), (0.19), (0.20) and analogously to [3] we conclude that \(u_\varepsilon\) minimizes the functional

\[ F(u_\varepsilon) = \int_{Q_\varepsilon} |\nabla u_\varepsilon|^2 \tag{0.22} \]

in the class of functions from \(H^1(Q_\varepsilon, S^1)\) satisfying the conditions

\[ \deg(u_\varepsilon, \partial G) = \deg(u_\varepsilon, \partial \Omega) = d, \quad \deg(u_\varepsilon, \partial B_i^\varepsilon) = 0, \tag{0.23} \]

where \(\deg(u, \Gamma)\) denotes the degree (i.e. the winding number of \(u\) considered as a map of a one-dimensional curve \(\Gamma\) into the unit sphere \(S^1\), as in [3]). Thus, \(u_\varepsilon(x)\) is a harmonic map of \(Q_\varepsilon\) into \(S^1\) with prescribed degree (of a map) on the boundaries \(\partial G, \partial \Omega, \partial B_i^\varepsilon\), i.e it satisfies (0.1)-(0.5) (and is determined up to a multiplicative constant with absolute value one [3]).

Notice, that (0.12), (0.15) and (0.16) imply that the current in \(Q_\varepsilon\) is zero. The magnetic field is zero in \(Q_\varepsilon\) due to (0.16).

The minimization problem of functional (0.22) in the class \(H^1(Q_\varepsilon, S^1)\) with the condition (0.23) on the boundary also describes the flow of a superfluid liquid in the infinite ring \(\Omega \setminus G\) with a large number of stiff rods. In this case the function \(u_\varepsilon(x)\) is the wave function of superfluid particles [5] and the condition (0.23) corresponds to the quantum character of rotation of the superfluid liquid in the ring \(\Omega \setminus G\).

In case when the number of domains \(B_i^\varepsilon \subset \Omega \setminus G\) is large and their diameters are small, the local behavior of \(u_\varepsilon(x)\) in the domain \(Q_\varepsilon\) is rather complicated. Therefore, the problem of describing the asymptotic behavior of \(u_\varepsilon(x)\), as \(\varepsilon \to 0\), when the number of domains \(B_i^\varepsilon\) increases indefinitely and their diameters tend to zero, naturally
arises. This leads us to the homogenization problem for the functional (0.22) under conditions (0.23), i.e. to the description of its $\Gamma$-limit [4], which is the main goal of this work.

Finally we explain the physical meaning of our homogenization result. Since it is quite difficult to measure the wave function itself we focus on the convergence of the fluxes (0.8), (0.9). For simplicity we consider the superfluid problem in a cylindrical domain with rods. Then in physical variables the flux density of the condensate particles has the form (see, for example, [1])

$$j_\varepsilon(x) = \frac{h}{m} \Im [\bar{u}_\varepsilon \nabla u_\varepsilon],$$

(0.24)

where $m$ is the mass of a condensate particle and $h$ is Planck’s constant. The presence of the rods can be described by introducing a new effective (homogenized) mass. If $a_{ik} = \hat{a}I$ is a scalar tensor ($h_1 = h_2$) then the effective mass is increased to $\hat{m} = \frac{m}{\hat{a}}$, $0 < \hat{a} < 1$. In general the presence of the rods in an isotropic superfluid can be described by a virtual anisotropic superfluid in which the effective mass of the condensate particles depends on the direction (similarly to virtual mass in hydrodynamics). For the superconducting problem in the absence of a magnetic field the analogous explanation applies to a change in the coupling constant $\frac{\hbar q m}{\hat{m}}$, where $q$ and $m$ are the charge and the mass of the Cooper pairs (see for example [1]).

The present article is organized as follows. In Section 1 we give a rigorous formulation of the problem and statement of the main result. In Section 2 we obtain a suitable representation for the solution of the minimization problem for the functional (0.22) with the condition (0.23). Here the ideas of the book [3] are essentially used. The proofs of the main results are given in Sections 3, 4 and 5. Finally, in Section 6 we derive the formulas connecting the homogenized conductivity tensor (effective conductivity of a conducting medium with dielectric inclusions) and the polarization tensor (effective conductivity of a medium with absolutely conducting inclusions), which is essential for the formulation of the main result.

2. Formulation of the problem and statement of main result. We consider domains in the space $\mathbb{R}^2$, where $(e_1, e_2)$ denotes unit vectors of rectangular coordinate system and $(x_1, x_2)$ denotes corresponding coordinates of the point $x \in \mathbb{R}^2$. Let $\Omega, G, (\bar{G} \subset \Omega)$ and $B$ be smooth bounded simply-connected domains in $\mathbb{R}^2$ containing the origin 0 and let $\Pi$ be a rectangle containing $\bar{B}$:

$$\Pi = \left\{ x \in \mathbb{R}^2 : -\frac{h_1}{2} < x_1 < \frac{h_1}{2}, -\frac{h_2}{2} < x_2 < \frac{h_2}{2} \right\}.$$

$\Pi$ should be thought of as a unit periodicity cell, which contains an inclusion $B$. We further rescale the periodicity cell to make it size of order of $\varepsilon$ where $\varepsilon > 0$ is a small real number. Denote by $B_\varepsilon$ the set of $\Omega \setminus \bar{G}$ such that

$$B_\varepsilon = \varepsilon \bigcup_{m,n} (B + mh_1e^1 + nh_2e^2),$$

where union is taken over such integers $m, n$ for which the cell $\varepsilon (\Pi \setminus B + mh_1e^1 + nh_2e^2)$ belongs to $\Omega \setminus \bar{G}$ for sufficiently small positive $\varepsilon$. We will denote $B_\varepsilon^i$ ($i = 1, 2, \ldots N_\varepsilon \sim \frac{\text{vol}(\Omega \setminus \bar{G})}{\varepsilon^2h_1h_2}$)
connected components of this set (inclusions or cross-sections of the rods reinforcing the superconducting medium).

Let us consider the domain

\[ Q_\varepsilon = \Omega \setminus \left( \overline{G} \cup \bigcup_i \overline{B}_\varepsilon^i \right) \]

In a particular case when \( \Omega \) and \( G \) are both discs centered at the origin and \( B \) is also a disc, the domain \( Q_\varepsilon \) is an annulus \( \Omega \setminus G \) from which the inclusions or rods \( B_\varepsilon \) are deleted.

Denote by \( \mathcal{E}_\varepsilon^d \) the class of maps from \( Q_\varepsilon \) into the unit sphere \( S^1 \) which satisfies the following conditions:

\[ \mathcal{E}_\varepsilon^d = \{ v \in H^1 (Q_\varepsilon, S^1) : \deg (v, \partial \Omega) = d, \deg (v, \partial G) = d, \]

\[ \deg (v, \partial B_i^\varepsilon) = 0, i = 1, 2, \ldots N \} . \]

Here \( d \in \mathbb{Z} \) is an integer, \( \deg (v, \Gamma) \) stands for the Brouwer degree of a map of a closed curve \( \Gamma \subset \mathbb{R}^2 \) into \( S^1 \) (the winding number or index of a vector field on \( \Gamma \)). For smooth vector fields \( v(x) \) the following formula

\[ \deg(v, \Gamma) = \frac{1}{2i\pi} \int_\Gamma v \times \frac{\partial v}{\partial \tau} \]

holds. Here \( \tau \) is a unit tangent vector to \( \Gamma \) which forms a properly oriented frame \((\nu, \tau)\) together with the unit outward normal \( \nu \) to \( \Gamma \); \( \times \) stands for the vector product. This formula also holds for \( v \in H^1 (Q_\varepsilon, S^1) \) and \( \Gamma = \partial \Omega, \partial G, \partial B_i^\varepsilon [6] \).

We identify vector fields from \( H^1 (Q_\varepsilon, S^1) \) with complex-valued functions on \( Q_\varepsilon \) with absolute value one.

Let us consider the minimization problem

\[ \min_{u_\varepsilon \in \mathcal{E}_\varepsilon^d} \int_{Q_\varepsilon} |\nabla u_\varepsilon|^2. \tag{1.2} \]

It will be shown below that there exists a unique (up to a multiplicative constant) solution \( u_\varepsilon(x) \) of this problem and this solution satisfies in \( Q_\varepsilon \) the following Euler-Lagrange equation:

\[ \Delta u_\varepsilon + u_\varepsilon |\nabla u_\varepsilon|^2 = 0, \quad |u_\varepsilon| = 1 \tag{1.3} \]

and Neumann boundary condition on \( \partial Q_\varepsilon \):

\[ \frac{\partial u_\varepsilon}{\partial \nu} = 0. \tag{1.4} \]

The main goal of this paper is to study the asymptotic behavior of \( u_\varepsilon(x) \) as \( \varepsilon \to 0 \) (i.e. large number of very thin reinforcing rods). Let us introduce a quantitative characteristic of the effect of the rods. Consider the following boundary problems.
(k = 1, 2) in the domain Π \ B (a unit periodicity cell):

\[
\begin{align*}
\Delta w_k &= 0, & \text{in } \Pi \setminus \bar{B} \\
\frac{\partial w_k}{\partial \nu} &= 0, & \text{on } \partial B \\
w_k - x_k \text{ and } D(w_k - x_k) \text{ are equal on the opposite sides of the rectangle } \Pi
\end{align*}
\]

(1.5)

Here \( D = \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2} \) is the normal derivative on \( \partial B \), \( k = 1, 2 \).

There exist a unique (up to an additive constant) solution of this problem. Set

\[ a_{kl} = \frac{1}{|\Pi|} \int_{\Pi \setminus \bar{B}} (\nabla w_k, \nabla w_l), \quad (1.6) \]

where \( |\Pi| \) is the area of the rectangle \( \Pi \). The collection of numbers \( \{a_{kl}, k, l = 1, 2\} \) form a symmetric positive definite tensor in \( \mathbb{R}^2 \) (conductivity tensor [9],[12],[16]). We further introduce the following class of mappings of the domain \( Q = \Omega \setminus G \) into \( S^1 \)

\[ \mathcal{E}^d = \{ v \in H^1(Q, S^1) : \deg(v, \partial \Omega) = \deg(v, \partial G) = d \}. \]

Now we can formulate our main result.

**Theorem 1.** Let \( u_\varepsilon \in \mathcal{E}^d \) be a solution of the problem (1.3) - (1.4). Then for any \( \varepsilon > 0 \) there exists a real number \( \theta_\varepsilon \) such that \( e^{i \theta_\varepsilon} u_\varepsilon \) converges in norm in \( L^2(Q_\varepsilon) \) to the solution \( u \) of the following minimization problem

\[ \min_{u \in \mathcal{E}^d} \int_Q \sum_{k,l} a_{kl} \frac{\partial u}{\partial x_k} \frac{\partial \bar{u}}{\partial x_l}. \quad (1.7) \]

The solution \( u \in \mathcal{E}^d \) satisfies the Euler equation in the domain \( Q \):

\[ \sum_{k,l} a_{kl} \frac{\partial^2 u}{\partial x_k \partial x_l} + u \sum_{k,l} a_{kl} \frac{\partial u}{\partial x_k} \frac{\partial \bar{u}}{\partial x_l} = 0. \quad (1.8) \]

and the Neumann boundary condition on \( \partial \Omega \) and \( \partial G \):

\[ \frac{\partial u}{\partial \nu} \equiv \sum_{k,l} a_{kl} \frac{\partial u}{\partial x_l} \cos(\nu, x_k) = 0. \quad (1.9) \]

Assuming that the flux density is set (extended) to be zero on \( \bigcup B^i_\varepsilon \), i.e.

\[
j_\varepsilon(x) = \begin{cases} 
\text{Im} [\bar{u}_\varepsilon \nabla u_\varepsilon], & x \in Q_\varepsilon \\
0, & x \in \bigcup B^i_\varepsilon
\end{cases}
\]

(1.10)

it converges weakly in \( L^2(Q) \) to the vector-function (homogenized flux):

\[ j(x) = \text{Im} \left[ \bar{u} \sum_{k,l} a_{kl} \frac{\partial u}{\partial x_l} e^k \right]. \quad (1.11) \]

The proof of this theorem is presented in Sections 3 - 6.
3. Representation of the solution of the problem (1.2).

Let us consider the boundary value problem

\[
\begin{aligned}
\Delta \Phi_{\varepsilon} &= 0, \quad \text{in } \Omega_{\varepsilon} \\
\Phi_{\varepsilon} &= 0, \quad \text{on } \partial \Omega \\
\Phi_{\varepsilon} &= C_{\varepsilon}, \quad \text{on } \partial G \\
\Phi_{\varepsilon} &= C_{\varepsilon i}, \quad \text{on } \partial B_{\varepsilon}^i \\
\int_{\partial G} \frac{\partial \Phi_{\varepsilon}}{\partial \nu} &= 2\pi d, \\
\int_{\partial B_{\varepsilon}^i} \frac{\partial \Phi_{\varepsilon}}{\partial \nu} &= 0,
\end{aligned}
\]  

(2.1)

where \(\nu\) is a unit outward normal to the contours \(\partial G, \partial B_{\varepsilon}^i (i = 1, 2, \ldots, N_{\varepsilon})\), constants \(C_{\varepsilon}, C_{\varepsilon i} (i = 1, 2, \ldots, N_{\varepsilon})\) are not given and determined as a part of the solution \(\{\Phi_{\varepsilon}(x), C_{\varepsilon}, C_{\varepsilon i}, i = 1, 2, \ldots, N_{\varepsilon}\}\) of the problem.

There exists a unique solution of the problem (2.1) and it minimizes the functional

\[
F(\varphi_{\varepsilon}) = \frac{1}{2} \int_{Q_{\varepsilon}} |\nabla \varphi_{\varepsilon}|^2 + 2\pi d \varphi_{\varepsilon}|_{\partial G}
\]

in the class of functions

\[
V = \{ \varphi_{\varepsilon} \in H^1(Q_{\varepsilon}, R) : \varphi_{\varepsilon} = 0 \text{ on } \partial \Omega, \varphi_{\varepsilon} = \text{const} = \varphi_{\varepsilon}|_{\partial G} \text{ on } \partial G, \varphi_{\varepsilon} = \text{const} = \varphi_{\varepsilon}|_{\partial B_{\varepsilon}^i} \text{ on } \partial B_{\varepsilon}^i \}.
\]

We cut the domain \(Q_{\varepsilon}\) along a horizontal line \(L_{\varepsilon}\) joining \(\partial G\) and \(\partial \Omega\). We also assume that \(L_{\varepsilon}\) is at a distance greater than \(C_{\varepsilon} (\varepsilon > 0)\) from the set \(\bigcup B_{\varepsilon}^i\) (see fig. 1). We then denote the obtained cut domain by \(Q_{\varepsilon}^C\). Note that \(Q_{\varepsilon}^C\) has no closed contours containing \(G\).

We next consider the system of the Cauchy-Riemann equations in the domain \(Q_{\varepsilon}^C\)

\[
\begin{aligned}
\frac{\partial \Psi_{\varepsilon}}{\partial x_1} &= \frac{\partial \Phi_{\varepsilon}}{\partial x_2} & \text{in } Q_{\varepsilon}^C \\
\frac{\partial \Psi_{\varepsilon}}{\partial x_2} &= -\frac{\partial \Phi_{\varepsilon}}{\partial x_1}
\end{aligned}
\]  

(2.2)

where \(\Phi_{\varepsilon}(x)\) is a solution of the problem (2.1). These equations are compatible since \(\Delta \Phi_{\varepsilon} = 0\) in \(Q_{\varepsilon}^C\) and therefore locally solvable. We can construct the solution of (2.2) by integrating along paths inside the domain \(Q_{\varepsilon}^C\). Due to the properties of function \(\Phi_{\varepsilon}(x)\) we obtain a single-valued function \(\Psi_{\varepsilon}(x)\) defined in \(Q_{\varepsilon}^C\) which satisfies the following conditions:

\[
\begin{aligned}
\Delta \Psi_{\varepsilon} &= 0 \quad \text{in } Q_{\varepsilon}^C \\
\frac{\partial \Psi_{\varepsilon}}{\partial \nu} &= 0 \quad \text{on } \partial \Omega, \partial G \text{ and } \partial B_{\varepsilon}^i \\
\frac{\partial \Psi_{\varepsilon}}{\partial x_2} &= 0 \quad \text{on } L_{\varepsilon} \\
[\Psi_{\varepsilon}] &= -2\pi d \quad \text{on } L_{\varepsilon}
\end{aligned}
\]  

(2.3)
where square brackets mean the difference between the limit values (from above and below) across the line $L_{\varepsilon}$, i.e. $[v] = v_+ - v_-$. The solution of the problem (2.3) is unique up to an additive constant. We introduce then the normalization condition

$$
\int_{Q_{\varepsilon}^C} \Psi_{\varepsilon} \equiv \hat{\Psi}_{\varepsilon} = 0. \quad \text{(2.4)}
$$

and consider the following class of functions defined in $Q_{\varepsilon}^C$

$$
E_{\varepsilon}^d = \left\{ \Psi_{\varepsilon} \in H^1(Q_{\varepsilon}^C, \mathbb{R}); \left[ \Psi_{\varepsilon} \right] = -2\pi d \text{ on } L_{\varepsilon}; \hat{\Psi}_{\varepsilon} = 0 \right\}.
$$

It follows from (2.3), (2.4) that the function $\Psi_{\varepsilon}(x)$ is the solution of the following minimization problem

$$
\min_{\Psi_{\varepsilon} \in E_{\varepsilon}^d} \int_{Q_{\varepsilon}^C} |\nabla \Psi_{\varepsilon}|^2. \quad \text{(2.5)}
$$

Introduce $u_{\varepsilon} = e^{i\Psi_{\varepsilon}}$. Then $v_{\varepsilon} \in \mathcal{E}_{\varepsilon}^d$ and

$$
\int_{Q_{\varepsilon}} |\nabla u_{\varepsilon}|^2 = \int_{Q_{\varepsilon}^C} |\nabla \Psi_{\varepsilon}|^2.
$$

Thus the function $u_{\varepsilon} = e^{i\Psi_{\varepsilon}}$ is a solution of the minimization problem (1.2) and is determined up to a multiplicative constant with absolute value one. It follows from (2.3) that the function $\Psi_{\varepsilon}(x)$ admits a continuation into a multi-valued function on $Q_{\varepsilon}$ whose values change by $2\pi d$ $(d \in \mathbb{Z})$ when going around the domain $G$ (in fact it becomes a single-valued function on a Riemannian surface which looks like a spiral staircase).

We will use the obtained representation $u_{\varepsilon} = e^{i\Psi_{\varepsilon}}$ in the proof of Theorem 1. The proof consists of three major steps. First, in Section 3 we study the asymptotic behavior of the solution $\Phi_{\varepsilon}(x)$ of the problem (2.1), as $\varepsilon \to 0$. Then in Section 4 we use (2.2), the result of Section 3 and compensated compactness theorem [7],[8] to obtain the asymptotics of the solution $\Psi_{\varepsilon}(x)$ of the problem (2.3)- (2.4). Finally in Section 5 we use the representation $u_{\varepsilon} = e^{i\Psi_{\varepsilon}}$ to prove the claim of Theorem 1 about the solution of the problem (1.2).

Notice that the problem (2.1) is known as an electrostatic boundary problem with absolutely conducting inclusions $B_{\varepsilon}$, and the problem (2.3) - (2.4) locally equivalent to the Neumann boundary problem for the Laplace equation (absolutely nonconducting inclusions).

The asymptotics of the solutions of such types of problems (i.e. homogenization of the Neumann boundary problem and the electrostatic problem) were studied by several authors (see, e.g. [9] - [12] and more recent work [19] where the notion of $H$-zero convergence has been introduced) and their results suggest a local structure of homogenized equations. But we can not use these results directly because of the unusual domains and most important because of the boundary conditions which include prescribing the degree of the solution on the boundary. That is why we suggest a new approach for homogenization of these problems, based on passing to the limit as $\varepsilon \to 0$ in integral identities, that determine a weak solution of these problems in $L_2$ (We call it the weakest solutions. As oppose to the usual weak solutions the weakest solution of the second order PDE is obtained by doubled integration by parts ).
4. Asymptotic behavior of solutions of problems (2.1) and (0.27). Consider the following boundary value problem in the domain $\Pi \setminus \bar{B}$:

$$\begin{cases}
\Delta v_k = 0 & \text{in } \Pi \setminus \bar{B} \\
v_k = \text{const} = C_k & \text{on } \partial B \\
\int_{\partial B} \frac{\partial v_k}{\partial p} = 0 \\
v_k - x_k \text{ and } D(v_k - x_k) \text{ are equal on the opposite sides} & \text{of the rectangle } \Pi
\end{cases}$$

(3.1)

Here the constants $C_k$ ($k = 1, 2$) are unknown and determined in the course of solving the problem. There exists a unique solution $\{v_k, C_k = v_k|_{\partial B}\}$ of the problem (up to an arbitrary constant). Put

$$b_{kl} = \frac{1}{|\Pi|} \int_{\Pi \setminus B} (\nabla v_k, \nabla v_l).$$

The collection of numbers $\{b_{kl}, k, l = 1, 2\}$ defines a symmetric positive definite tensor in $\mathbb{R}^2$ called the polarization tensor. This tensor characterizes the asymptotic behavior of the solution $\{\Phi_{\varepsilon}(x), C_{\varepsilon}, C_{\varepsilon i}, i = 1, \ldots, N_{\varepsilon}\}$ of the problem (2.1). Now we extend the function $\Phi_{\varepsilon}(x)$ into $B_{\varepsilon}^1$ by putting $\Phi_{\varepsilon} = C_{\varepsilon i}$ ($i = 1, \ldots, N_{\varepsilon}$) in $B_{\varepsilon}^1$ to obtain the function $\Phi_{\varepsilon}(x) \in H^1(\Omega)$, for which the same notation is kept.

**Theorem 2.** The extended function $\Phi_{\varepsilon}(x)$ converges in $L^2(\Omega)$ as $\varepsilon \to 0$ to the function $\Phi(x)$ the solution (more precisely, the functional part of the solution $(\Phi, C)$) of the following homogenized problem:

$$\begin{cases}
\sum_{k,l} b_{kl} \frac{\partial^2 \Phi}{\partial x_k \partial x_l} = 0 & \text{in } Q \\
\Phi = 0 & \text{on } \partial \Omega \\
\Phi = \text{const} = C & \text{on } \partial G \\
\int_{\partial G} \frac{\partial \Phi}{\partial n} = 2\pi d
\end{cases}$$

(3.2)

where

$$\frac{\partial}{\partial n} = \sum_{k,l} b_{kl} \cos(\nu, x_k) \frac{\partial}{\partial x_l}$$

is the derivative with respect to the outward conormal to the contour $\partial G$.

**Proof.** Taking into account (2.1) and using Green’s theorem we obtain

$$\int_{Q} |\nabla \Phi_{\varepsilon}|^2 = 2\pi d C_{\varepsilon}$$

(3.3)

and

$$C_{\varepsilon} = \frac{1}{|\partial G|} \int_{\partial G} \Phi_{\varepsilon},$$

(3.4)
where $|\partial G|$ is the length of the contour $\partial G$.

Since $\Phi_\varepsilon = 0$ on $\partial \Omega$, the inequalities
\[
\int_{\partial G} \Phi_\varepsilon^2 \leq A_1 \int_Q |\nabla \Phi_\varepsilon|^2, \quad \int_{\partial G} \Phi_\varepsilon^2 \leq A_2 \int_Q |\nabla \Phi_\varepsilon|^2
\]
hold. Consequently, from (3.3), (3.4) we have
\[
\|\Phi_\varepsilon\|_{H^1(Q)} \leq A,
\]
where the constants $A_1$, $A_2$ and $A$ do not depend on $\varepsilon$.

Hence the family of functions $\{\Phi_\varepsilon(x), \varepsilon \to 0\}$ is weakly compact in $H^1(Q)$, so we can extract a subsequence $\{\Phi_{\varepsilon_\nu}(x), \varepsilon_\nu \to 0, \nu = 1, 2, \ldots\}$ weakly converging in $H^1(Q)$ to some function $\Phi(x) \in H^1(Q)$. It follows from the embedding theorem that this subsequence converges in the norm to $\Phi(x)$ in $L_2(Q)$, $L_2(\partial \Omega)$ and $L_2(\partial G)$. Taking into account that $\Phi_{\varepsilon_\nu}(x) = 0$ on $\partial \Omega$ and $\Phi_{\varepsilon_\nu}(x) = C_{\varepsilon_\nu}$ on $\partial G$, we conclude that $\Phi(x)$ equals zero on $\partial \Omega$ and is equal to a constant $C$ on $\partial G$, with $\lim_{\nu \to 0} C_{\varepsilon_\nu} = C$.

We will show that $(\Phi(x), C)$ is a solution of the problem (3.2).

Let us introduce the class of functions
\[
F_\varepsilon(Q_\varepsilon) = \left\{ \zeta_\varepsilon(x) \in C^2(Q_\varepsilon); \quad \zeta_\varepsilon(x) = \text{const} = a \text{ on } \partial G, \quad \int_{\partial G} \frac{\partial \zeta_\varepsilon}{\partial \nu} = 0, \quad \zeta_\varepsilon = a_{\varepsilon_i} \text{ on } \partial B_{\varepsilon_i}^i, \quad \int_{\partial B_{\varepsilon_i}^i} \frac{\partial \zeta_\varepsilon}{\partial \nu} = 0, \zeta_\varepsilon(x) = 0 \text{ on } \partial \Omega \right\},
\]
where $a$, $a_{\varepsilon_i}$ ($i = 1, \ldots, N_{\varepsilon}$) are arbitrary constants. It follows from (2.1) that $\Phi_\varepsilon(x)$ satisfies following integral equality
\[
\int_{Q_\varepsilon} \Phi_\varepsilon \Delta \zeta_\varepsilon = 2\pi da,
\]
which holds for any $\zeta_\varepsilon \in F_\varepsilon(Q_\varepsilon)$. We will choose the function $\zeta_\varepsilon \in F_\varepsilon(Q_\varepsilon)$ in a special way and pass to the limit, as $\varepsilon = \varepsilon_\nu \to 0$ in this equality. We will construct this function starting with an arbitrary function $\zeta(x)$ from the following class:
\[
F(Q) = \{ \zeta(x) \in C^3(Q); \quad \zeta(x) \equiv 0 \text{ in some one-sided neighborhood of } \partial \Omega \text{ in } Q; \quad \zeta(x) \equiv \text{const} = a \text{ in a one-sided neighborhood of } \partial G \text{ in } Q \}.
\]

Here the constant $a$ and widths of neighborhoods $\partial \Omega$ and $\partial G$ in $Q$ (strips) are arbitrary. As a result of passing to the limit in (3.5) we obtain the equality
\[
\int_Q \Phi \sum b_{ki} \frac{\partial^2 \zeta}{\partial x_k \partial x_l} = 2\pi da
\]
Taking into account that $\Phi = 0$ on $\partial \Omega$ and $\Phi = \text{const} = C$ on $\partial G$ we conclude from (3.6) that $\Phi$ is a solution of the problem (3.2). Indeed choosing first $\zeta(x)$ with $a = 0$ in (3.6) we have a homogeneous identity for $\Phi(x)$ with any finite $\zeta(x)$. Hence due to the theory of weak solutions of elliptic equations [13] $\Phi(x)$ is an infinitely
differentiable function in $Q$ and satisfies the first equation in (3.2). As it was proven above $\Phi(x) \in H^3(Q)$ and takes constant values on the smooth boundaries $\partial \Omega$ and $\partial G$. Therefore it follows from the theory of elliptic boundary value problems [13],[14] that $\Phi(x)$ is a smooth function up to the boundary. Take this into account and choose $\zeta(x)$ with an arbitrary $a \neq 0$ in (3.6). Then integration by parts provides the desired integral equation on $\partial G$. Thus we have shown that $\Phi(x)$ is a solution of the problem (3.2).

We next construct the function $\zeta_\varepsilon(x)$ in the following way. Let us cover the space $\mathbb{R}^2$ by rectangles $\Pi^i_{\varepsilon+}$ centered at the points $x^i = \varepsilon(h_1 m_i + h_2 n_i)$, $m_i, n_i \in \mathbb{Z}$ with sides $\varepsilon h_1 + \varepsilon \gamma$, $\varepsilon h_2 + \varepsilon \gamma$, where $\gamma > 1$ will be chosen later. The rectangles $\Pi^i_{\varepsilon-}$ centered at the same points $x^i$ with sides $\varepsilon h_1$, $\varepsilon h_2$ and $\varepsilon h_1 - \varepsilon \gamma$, $\varepsilon h_2 - \varepsilon \gamma$ respectively are located inside $\Pi^i_{\varepsilon+}$. Let $\{\varphi_{\varepsilon i}(x), x \in \mathbb{R}^2\}$ be a partition of unity corresponding to above covering, i.e. a collection of functions satisfying the following conditions:

$$\varphi_{\varepsilon i}(x) \in C^2, 0 \leq \varphi_{\varepsilon i}(x) \leq 1, \varphi_{\varepsilon i}(x) = 0 \text{ outside } \Pi^i_{\varepsilon+} \text{ and } \varphi_{\varepsilon i}(x) = 1 \text{ inside } \Pi^i_{\varepsilon-},$$

$${\sum \varphi_{\varepsilon i}(x) = 1 \text{ and } D^m \varphi_{\varepsilon i}(x) = O(\varepsilon^{-m \gamma})}$$

Let $v_{\varepsilon k}(x)$ be a solution of cell problem (3.1) extended by the constant $C_k$ on $B \subset \Omega$. Consider the function $\varepsilon v_{\varepsilon k}(x) - x_k$ in the rectangle $\Pi^i_{\varepsilon+}$ centered at 0. It follows from (3.1) that this function can be extended on $\mathbb{R}^2$ by setting $v_{\varepsilon k}(x) = \varepsilon v_{\varepsilon k}(x/x - x^i) - (x_k - x^i) = 0$ for $x \in \Pi^i_{\varepsilon-}$. The obtained function $v_{\varepsilon k}(x)$, satisfies the equation $\Delta v_{\varepsilon k}(x) = 0$ everywhere outside the periodic in $\mathbb{R}^2$ set $\bigcup B^i_\varepsilon$, where $B^i_\varepsilon = \varepsilon B + x^i$. Set

$$v^i_{\varepsilon k}(x) = v_{\varepsilon k}(x) + (x_k - x^i), x \in \Pi^i_{\varepsilon+}. \quad (3.7)$$

Then

$$\begin{cases}
  \Delta v^i_{\varepsilon k}(x) = 0 & \text{in } \Pi^i_{\varepsilon+} \setminus B^i_\varepsilon \\
  v^i_{\varepsilon k} = \varepsilon C_k & \text{in } B^i_\varepsilon \\
  \int \frac{\partial v^i_{\varepsilon k}}{\partial \nu} = 0
\end{cases} \quad (3.8)$$

Let $\zeta(x)$ be an arbitrary function from the class $F(Q)$ which is equal to $a$ on $\partial G$. Extending it by 0 outside of $\Omega$ and by constant $a$ inside $G$, we obtain a function $\zeta(x) \in C^3(\mathbb{R}^2)$.

We define the function $\zeta_\varepsilon(x) \in F_\varepsilon(Q_\varepsilon)$ as a restriction on $Q_\varepsilon$ of the function $\tilde{\zeta}_\varepsilon(x) \mathbb{R}^2 \to \mathbb{R}$ defined by

$$\tilde{\zeta}_\varepsilon(x) = \sum_k \left[ \zeta(x^i) + \sum_k v^i_{\varepsilon k}(x) \frac{\partial \zeta}{\partial x_k}(x^i) + \frac{1}{2} \sum_{k,l} v^i_{\varepsilon k}(x) v^j_{\varepsilon j}(x) \frac{\partial^2 \zeta}{\partial x_k \partial x_l}(x^i) \right] \varphi_{\varepsilon i}(x) \quad (3.9)$$

It follows from (3.8) and properties of the functions $\varphi_{\varepsilon i}(x)$ and $\zeta(x)$, that $\zeta_\varepsilon = \tilde{\zeta}_\varepsilon|_{Q_\varepsilon} \in F_\varepsilon(Q_\varepsilon)$. Let us consider the function $f_\varepsilon(x): Q \to \mathbb{R}$

$$f_\varepsilon(x) = \left\{ \begin{array}{ll}
  \Delta \zeta_\varepsilon(x) & x \in \Omega_\varepsilon \\
  0 & x \in \bigcup B^i_\varepsilon
\end{array} \right. \quad (3.9)$$

**Lemma 1.** $f_\varepsilon(x)$ converges weakly in $L_2(Q)$, as $\varepsilon = \varepsilon_\nu \to 0$, to the function

$$f(x) = \sum b_{kl} \frac{\partial^2 \zeta}{\partial x_k \partial x_l}$$
where $b_{kl}$ is the polarization tensor.

**Proof.** We first show that the functions $\{ f_\varepsilon(x) \}$ are uniformly bounded in the norm of $f_\varepsilon(x)$ in $L^2(Q)$ in $\varepsilon$. Taking into account (3.7) and properties of the partition of unity we represent $\zeta_\varepsilon(x)$ as

$$
\zeta_\varepsilon(x) = \sum_i \left\{ \zeta^i + \sum_k \zeta^i_k (x_k - x^i_k) + \frac{1}{2} \sum_{k,l} \zeta^i_{kl} (x_k - x^i_k) (x_l - x^i_l) - \zeta(x) \right\} \varphi_{\varepsilon i}(x) +
$$

$$
+ \sum_i \left\{ \sum_k \left[ \zeta^i_k - \zeta^i_{kl} (x_l - x^i_l) - \zeta_k(x) \right] \varphi_{\varepsilon k}(x) \right\} \varphi_{\varepsilon i}(x) +
$$

$$
+ \frac{1}{2} \sum_i \left\{ \sum_{k,l} (\zeta^i_{kl} - \zeta^i_{kl}(x)) \varphi_{\varepsilon k}(x) \varphi_{\varepsilon l}(x) \right\} \varphi_{\varepsilon i}(x) + \zeta(x) +
$$

$$
+ \sum_k \zeta_k(x) \varphi_{\varepsilon k}(x) + \frac{1}{2} \sum_{k,l} \zeta_{kl}(x) \varphi_{\varepsilon k}(x) \varphi_{\varepsilon l}(x).
$$

(3.10)

Here we use the notations:

$$
\zeta_k(x) = \frac{\partial \zeta}{\partial x_k}(x), \quad \zeta_{kl}(x) = \frac{\partial^2 \zeta}{\partial x_k \partial x_l}(x).
$$

Superscript $i$ denotes the values of $\zeta$, $\zeta_k$ and $\zeta_{kl}$ evaluated at the point $x^i$. Note that the derivatives of order $j$ of the expression in the first brace in (3.10) have the order $O(\varepsilon^{3-j})$. Due to the definition of $v_{\varepsilon k}(x)$ we also have $D^m v_{\varepsilon k} = O(\varepsilon^{1-m})$ and therefore the same estimates hold for the second and the third braces in (3.10). Finally from the definition of the partition of unity we get $D^m \varphi_{\varepsilon k} = O(\varepsilon^{-m})$. Taking into account these estimates and choosing $\gamma$ such that $1 < \gamma < 5/3$, and making use of (3.9) we obtain

$$
\int_Q f_\varepsilon^2 = \int_{Q_\varepsilon} |\Delta \zeta_\varepsilon|^2 < C,
$$

(3.11)

where $C$ does not depend on $\varepsilon$. Then only three last terms in (3.10) give contributions, which is do not disappear as $\varepsilon \to 0$.

For any function $\Psi(x) \in C^1(\overline{Q})$ we can write

$$
\int_Q f_\varepsilon \Psi = \int_{Q_\varepsilon} \Delta \zeta_\varepsilon \Psi = \sum_i \int_{\Pi_{\varepsilon} \setminus B^i_{\varepsilon}} \Delta \zeta_\varepsilon \Psi + \sum_i \int_{\Pi^i_{\varepsilon} \setminus \Pi^i_{\varepsilon-}} \Delta \zeta_\varepsilon \Psi.
$$

(3.12)

Due to (3.9)

$$
\Delta \zeta_\varepsilon(x) = \sum_{k,l} \frac{\partial^2 \zeta}{\partial x_k \partial x_l}(x^i) (\nabla v^i_{\varepsilon k}(x), \nabla v^i_{\varepsilon l}(x)), \quad x \in \Pi^i_{\varepsilon} \setminus B^i_{\varepsilon}
$$
and hence
\[ \int_{\Pi_i^+ \setminus B_i^{\varepsilon}} \Delta \zeta_{\varepsilon} \Psi = \sum_{k,l} \frac{\partial^2 \zeta_{\varepsilon}}{\partial x_k \partial x_l} (x^i) \Psi(x^i) \int_{\Pi_i^+ \setminus B_i^{\varepsilon}} (\nabla v_{\varepsilon k}^i(x), \nabla v_{\varepsilon l}^j(x))(1 + O(\varepsilon^{\gamma - 1})). \tag{3.13} \]
as $\varepsilon \to 0$. According to the definition $v_{\varepsilon k}^i(x) = \varepsilon v_k(x/\varepsilon - x^i)$ and therefore
\[ \int_{\Pi_i^+ \setminus B_i^{\varepsilon}} (\nabla v_{\varepsilon k}^i(x), \nabla v_{\varepsilon l}^j(x)) = \varepsilon^2 \int_{\Omega \setminus B} (\nabla v_k, \nabla v_l)(1 + O(\varepsilon^{\gamma - 1})) = \varepsilon^2 |\Pi| b_{kl}(1 + O(\varepsilon^{\gamma - 1})). \tag{3.14} \]
It follows from (3.11), (3.14) that the first sum over $i$ in (3.12) converges, as $\varepsilon \to 0$, to the integral
\[ \int \sum_{k,l} b_{kl} \frac{\partial^2 \zeta_{\varepsilon}}{\partial x_k \partial x_l} \Psi. \]
The second sum in (3.12) converges to zero due to (3.11) and the estimate
\[ \text{meas} \left( \bigcup_i \left( \Pi_i^+ \setminus \Pi_i^{-} \right) \right) = O(\varepsilon^{\gamma - 1}). \tag{3.15} \]
Thus
\[ \lim_{\varepsilon \to 0} \int_Q f_{\varepsilon} \Psi = \int_Q \sum_{k,l} b_{kl} \frac{\partial^2 \zeta_{\varepsilon}}{\partial x_k \partial x_l} \Psi, \]
We combine this with (3.11) and the Lemma follows.

It remains to pass to the limit when $\varepsilon = \varepsilon_\nu \to 0$ in the equality (3.5), where $\{ a_{\varepsilon} = a, \; \zeta_{\varepsilon} = \tilde{\zeta}_{\varepsilon} \mid \Omega \}$ and $\tilde{\zeta}_{\varepsilon}$ is defined by the formula (3.9). Since
\[ \int_{Q_\varepsilon} \Phi_{\varepsilon} \Delta \zeta_{\varepsilon} = \int_Q f_{\varepsilon} \Phi_{\varepsilon}, \]
where $\Phi_{\varepsilon}$ is extended on $B_i^{\varepsilon}$ by constants $z_{\varepsilon i}$, using Lemma 1 and strong convergence of $\Phi_{\varepsilon}(x)$ to $\Phi(x)$ as $\varepsilon = \varepsilon_\nu \to 0$, we obtain the equality (3.6).

Thus, the extended solutions $\Phi_{\varepsilon}(x)$ of the problem (2.1) converges in $L_2(Q)$ to the solution $\Phi(x)$ of the problem (2.4) as $\varepsilon = \varepsilon_\nu \to 0$. It remains to notice, that the problem (3.2) has a unique solution and therefore $\Phi_{\varepsilon}(x)$ converges to $\Phi(x)$, as $\varepsilon \to 0$. Theorem 2 is proven.

**Corollary 1.** The derivatives $\frac{\partial \Phi}{\partial x_k}$ ($k = 1, 2$) converge weakly in $L_2(Q)$ to $\frac{\partial \Phi}{\partial x_k}$. 
5. Asymptotic behavior of the solution of the problem (2.3) - (2.4).

Recall that the solution $\Psi_\varepsilon$ of the problem (2.3) was normalized by the condition (2.4). Assume that the cuts $L_\varepsilon$ (see the definition of the domain $Q^\varepsilon_C$) approach a fixed horizontal segment $L$, joining $\partial G$ and $\partial \Omega$ (or coincide with it), as $\varepsilon \to 0$.

Let us denote by $Q^C$ the domain $\Omega \setminus \bar{G}$ with a cut along $L$, i.e.

$$Q^C = (\Omega \setminus \bar{G}) \setminus L.$$  

**Theorem 3.** The solution $\Psi_\varepsilon(x)$ of the problem (2.3)-(2.4) converges strongly in $L_2(Q_\varepsilon)$ as $\varepsilon \to 0$ to the function $\Psi(x)$ which is the solution of the following problem

\[
\begin{align*}
\sum_{k,l} a_{kl} \frac{\partial^2 \Psi}{\partial x_k \partial x_l} &= 0, \quad \text{in } Q^C \\
\frac{\partial \Psi}{\partial \nu_a} &= 0, \quad \text{on } \partial \Omega \text{ and } \partial G \\
\left[\Psi\right] &= -2\pi d \quad \text{on } L \\
\left[\frac{\partial \Psi}{\partial \nu_a}\right] &= 0, \quad \text{on } L \\
\oint_{\Omega^C} \Psi &= 0
\end{align*}
\]

(4.1)

where $a_{kl}$ is the conductivity tensor defined in (1.6),

$$\frac{\partial}{\partial \nu_a} = \sum_{k,l} a_{kl} \cos(\nu, x_k) \frac{\partial}{\partial x_l}$$

is the conormal derivative and the square brackets denote the difference of limiting values across $L$. The problem (4.1) has a unique solution which minimizes the functional

$$F(\Psi) = \int_{\Omega} \sum_{k,l} a_{kl} \frac{\partial \Psi}{\partial x_k} \frac{\partial \Psi}{\partial x_l}$$

in the class

$$E^d = \left\{ \psi \in H^1(Q^C, \mathbb{R}); [\psi] = -2\pi d, \text{ on } L; \hat{\psi} = 0 \right\}.$$  

**Proof.** For the sake of simplicity we assume that the cuts $L_\varepsilon$ and $L$ coincide for each $\varepsilon$. Then $Q^C = Q^\varepsilon_C \cup \bigcup \{B_\varepsilon \}$. As it was noted above, the solution $\Psi_\varepsilon$ of the problem (2.3) minimizes the Dirichlet integral (2.4) in the class $E^d_\varepsilon$. Taking into account specificity of the domain $Q^\varepsilon_C$, we construct a function $g_\varepsilon(x) \in E^d_\varepsilon$ such that its Dirichlet integral over $Q^\varepsilon_C$ is bounded by a constant $C > 0$, which does not depend on $\varepsilon$. Hence

$$\int_{Q^\varepsilon} |\nabla \Psi_\varepsilon|^2 \leq \int_{Q^\varepsilon} |\nabla g_\varepsilon|^2 < C.$$  

(4.3)
Since subdomains $B_i^\varepsilon \subset \Omega^C$ are distance $\geq const \varepsilon$ from each other and from the cut $L$, the solution $\Psi_\varepsilon(x)$ can be extended into $B_i^\varepsilon$ such that the extended function $\tilde{\Psi}_\varepsilon(x) \in H^1(Q^C)$ satisfies the following inequalities (see [9], [10])

$$\int_{\Pi_i^\varepsilon} |\nabla \tilde{\Psi}_\varepsilon|^2 \leq C_1 \int_{\Pi_i^\varepsilon \setminus B_i^\varepsilon} |\nabla \Psi_\varepsilon|^2$$  \hspace{1cm} (4.4)

and

$$\int_{\Pi_i^\varepsilon} |\tilde{\Psi}_\varepsilon - \Psi_i^\varepsilon|^2 \leq C_2 \varepsilon^2 \int_{\Pi_i^\varepsilon \setminus B_i^\varepsilon} |\nabla \Psi_\varepsilon|^2,$$  \hspace{1cm} (4.5)

where the constants $C_1, C_2$ do not depend on $\varepsilon$ and $i$, and $\Psi_i^\varepsilon$ is the (integral) mean value of the function $\Psi_\varepsilon(x)$ in $\Pi_i^\varepsilon \setminus B_i^\varepsilon$. It follows from (4.3) that

$$\int_{Q^C} |\nabla \tilde{\Psi}_\varepsilon|^2 \leq C_1 \int_{Q^C} |\nabla \Psi_\varepsilon|^2$$  \hspace{1cm} (4.6)

Then using the equalities

$$\int_{Q^C} \tilde{\Psi}_\varepsilon = \sum_i \Psi_i^\varepsilon |\Pi_i^\varepsilon| + \sum_i \int_{\Pi_i^\varepsilon} (\tilde{\Psi}_\varepsilon - \Psi_i^\varepsilon)$$

and

$$\sum_i \Psi_i^\varepsilon |\Pi_i^\varepsilon| = \frac{|\Pi|}{|\Pi \setminus B|} \tilde{\Psi} = 0,$$

with the help of (4.5) we obtain

$$\left( \int_{Q^C} \tilde{\Psi}_\varepsilon \right)^2 < C_3 \varepsilon^2 \int_{Q^C} |\nabla \Psi_\varepsilon|^2,$$  \hspace{1cm} (4.7)

where $C_3 = C_2 \text{meas} Q^C$ does not depend on $\varepsilon$.

Finally applying the Poincare inequality

$$\int_{Q^C} |\tilde{\Psi}|^2 \leq C_4 \left( \int_{Q^C} \tilde{\Psi}_\varepsilon \right)^2 + C_5 \int_{Q^C} |\nabla \tilde{\Psi}_\varepsilon|^2$$

and taking into account (4.3), (4.4) we get the inequality

$$\int_{Q^C} |\tilde{\Psi}|^2 \leq C \int_{Q^C} |\nabla \tilde{\Psi}_\varepsilon|^2,$$  \hspace{1cm} (4.8)

with the constant $C > 0$ independent on $\varepsilon$.

It follows from (4.6), (4.8) and (4.3) that the family of functions $\{\tilde{\Psi}_\varepsilon, \varepsilon \to 0\}$ is bounded in $H^1(Q^C)$, and therefore we can extract a subsequence $\{\tilde{\Psi}_\varepsilon, \varepsilon = \varepsilon_\nu \to$
which converges weakly in $H^1(Q^C)$ to a function $\Psi(x) \in H^1(Q^C)$. The embedding theorems imply the convergence in $L_2(Q^C)$ and in $L_2(L^\pm)$, where $L^\pm$ stands for the upper and lower sides of the cut $L$. Combining this with (4.7) we conclude that $\Psi_\varepsilon(x)$ converges to $\Psi(x)$ strongly in $L_2(Q^C)$, as $\varepsilon \to 0$, and $\Psi(x)$ belongs to the class $E^\varepsilon$. We now show that $\Psi(x)$ is smooth enough in the domain $Q^C$ and satisfies in this domain the equation

$$\sum_{k,l} a_{kl} \frac{\partial^2 \Psi}{\partial x_k \partial x_l} = 0. \quad (4.9)$$

It can be done in the same manner as in the proof of Theorem 2. Namely, let $\zeta(x) \in C^3_0(Q^C)$ be an arbitrary function from the class $C^3$ with compact support in $Q^C$. Starting from $\zeta(x)$ we construct the function (analogous (3.9)):

$$\zeta_\varepsilon(x) = \sum_i \left[ \zeta(x^i) + \sum_k u_{i,k}^{\varepsilon}(x) \frac{\partial \zeta}{\partial x_k}(x^i) + \frac{1}{2} \sum_{k,l} u_{i,k}^{\varepsilon}(x) u_{i,l}^{\varepsilon}(x) \frac{\partial^2 \zeta}{\partial x_k \partial x_l}(x^i) \right] \varphi_\varepsilon(x),$$

where $u_{i,k}^{\varepsilon}(x) = u_{x,k}(x) + (x - x^i)$ for $x \in \Pi^\varepsilon \setminus B^\varepsilon$ and $\tilde{u}_{x,k}(x)$ is a periodic extension of the function $\varepsilon u_k(x/\varepsilon) - x$ which is defined in the cell $\Pi^\varepsilon \setminus B^\varepsilon$ centered at the origin with the help of the solution $u_k(x)$ of the cell problem (1.5). It is easy to see that $\zeta_\varepsilon(x) \in C^2(Q^C_\varepsilon)$. Moreover its support is at a fixed positive distance from $L$, $\partial \Omega$, $\partial G$ and $\partial \zeta_\varepsilon \frac{\partial \nu}{\partial \nu} = 0$ on $\partial B^\varepsilon_i$.

These properties of $\zeta_\varepsilon(x)$ together with (2.3) imply that $\Psi_\varepsilon$ satisfies the equality

$$\int_{Q^C_\varepsilon} \Psi_\varepsilon \Delta \zeta_\varepsilon = 0.$$  

By letting

$$f_\varepsilon(x) = \begin{cases} 
\Delta \zeta_\varepsilon(x), & \text{when } x \in Q^C_\varepsilon \\
0, & \text{when } x \in \bigcup_i B^\varepsilon_i,
\end{cases}$$

we may rewrite this equality as follows

$$\int_{Q^C} \Psi_\varepsilon f_\varepsilon = 0. \quad (4.10)$$

**Lemma 2.** For $\varepsilon = \varepsilon_\nu \to 0$ $f_\varepsilon(x)$ converges weakly in $L_2(Q^C)$ to the function

$$f(x) = \sum_{k,l} a_{kl} \frac{\partial^2 \zeta}{\partial x_k \partial x_l},$$

where $a_{kl}$ is the conductivity tensor.
The proof of this Lemma is similar to the proof of Lemma 1. In this case it must be taken into account that the function \( u_{\varepsilon k}(x) = u'_{\varepsilon k}(x) - (x_k - x'_k) \) is periodic and satisfies the equation \( \Delta u_{\varepsilon k} = 0 \) in \( \mathbb{R}^2 \setminus \bigcup_i B^i_{\varepsilon} \).

Now let us pass to the limit in these equalities, as \( \varepsilon \rightarrow 0 \). Hence due to the theory of weak solutions of elliptic equations \([13]\), we have that \( \Psi \) is uniformly bounded in \( \Omega \).

Now let us pass to the limit in the equality (4.10). Then, keeping in mind the strong convergence of \( \tilde{\Psi}_{\varepsilon} \) to \( \Psi \) and Lemma 1, we conclude that \( \Psi \) satisfies the following integral equality

\[
\int_{Q^C} \Psi \frac{\partial^2 \zeta}{\partial x_k \partial x_l} = 0 \tag{4.11}
\]

for any \( \zeta \in C^2_0(Q^C) \).

It remains to show that the conormal derivative of \( \Psi \) equals to zero on \( \partial \Omega \) and \( \partial G \) and has no jump across the cut \( L \). In order to do this we should choose the function \( \zeta(x) \) in (4.11) from a wider class (not necessarily with a compact support in \( Q^C \)). However this would create additional difficulties the in the construction of the function \( \zeta(x) \) with the needed properties. That is why we choose a round about way. We use the system (2.2). Set

\[
P_{\varepsilon k}(x) = \begin{cases} 
\frac{\partial \Psi_{\varepsilon}}{\partial x_k}(x), & \text{when } x \in Q^C_{\varepsilon} (k = 1, 2) \\
0, & \text{when } x \in \bigcup B^i_{\varepsilon} \subset Q^C
\end{cases} \tag{4.12}
\]

Since \( \Phi_{\varepsilon}(x) \) is extended by a constant \( C_{\varepsilon i} \) on \( B^i_{\varepsilon} \subset Q^C \) and \( \frac{\partial \Phi_{\varepsilon}}{\partial x_k} = 0 \) on \( B^i_{\varepsilon} \), so the system (2.2) can be rewritten in the domain \( Q^C \) as follows

\[
\begin{align*}
P_{\varepsilon 1}(x) &= \frac{\partial \Phi_{\varepsilon}}{\partial x_2}(x) \\
P_{\varepsilon 2}(x) &= -\frac{\partial \Phi_{\varepsilon}}{\partial x_1}(x) 
\end{align*} \quad \text{in } Q^C. \tag{4.13}
\]

Let us pass to the limit in these equalities, as \( \varepsilon = \varepsilon_{\nu} \rightarrow 0 \), where the subsequence \( \varepsilon_{\nu} \) was chosen in the proof of Theorem 3. In order to do this we need to find the asymptotics of \( P_{\varepsilon k}(x) \), as \( \varepsilon = \varepsilon_{\nu} \rightarrow 0 \).

Consider the vector-function \( P_{\varepsilon}(x) = \{P_{\varepsilon 1}(x), P_{\varepsilon 2}(x)\} \).

**Lemma 3.** \( P_{\varepsilon}(x) \) converges weakly in \( (L^2(Q^C))^2 \), as \( \varepsilon = \varepsilon_{\nu} \rightarrow 0 \), to the vector-function

\[
\sum_{k, l} a_{kl} \frac{\partial \Psi}{\partial x_l}(x)e^k.
\]

**Proof.** It follows from the above consideration that the norm of \( P_{\varepsilon}(x) \) in \( (L^2(Q^C))^2 \) is uniformly bounded in \( \varepsilon \). Therefore there exists a subsequence \( \{\varepsilon_{\nu} \rightarrow 0\} \) such that the corresponding subsequence of the vector-functions \( P_{\varepsilon}(x) \) converges weakly in \( (L^2(Q^C))^2 \). This subsequence can be extracted from the sequence \( \{\varepsilon_{\nu} \rightarrow 0, \nu = \)
Let \( u_k(x) \) be a solution of the cell problem (1.5). Let us extend it to the domain \( B \) so that the extended function \( \tilde{u}_k(x) \) belongs to the class \( C^2(\Pi) \). Introduce the following functions

\[
\tilde{u}_{ck}(x) = \varepsilon \tilde{u}_k \left( \frac{x}{\varepsilon} - x^i \right) - (x_k - x^i_k)
\]

and

\[
\tilde{u}^i_{ck}(x) = \tilde{u}_{ck}(x) + (x_k - x^i_k) \quad x \in \Pi^i_\varepsilon +.
\]

Clearly \( \tilde{u}_k(x) \) is a periodic function in \( \mathbb{R}^2 \) and

\[
\begin{align*}
\Delta \tilde{u}_{ck}(x) &= 0 \quad \text{in } \Pi^i_\varepsilon + \setminus B^i_\varepsilon, \\
\frac{\partial \tilde{u}^i_{ck}}{\partial \nu} &= 0 \quad \text{on } \partial B^i_\varepsilon.
\end{align*}
\]

Let \( V(x) = \{ V_1(x), V_2(x) \} \) be an arbitrary vector-function with compact support in \( Q^C \) from the class \( C^2 \). Using this function we construct a vector-function \( V_\varepsilon = \{ V_\varepsilon^1, V_\varepsilon^2 \} \) defined as follows

\[
V_\varepsilon^i(x) = \sum_i \varphi_{ci}(x) \left[ \sum_l V_l(x^i) \frac{\partial \tilde{u}^i_{ck}}{\partial x^l}(x) + \sum_{k,l} \frac{\partial V_l}{\partial x^i}(x^i) \frac{\partial \tilde{u}^i_{ck}}{\partial x^l}(x) \tilde{u}^i_{ck}(x) \right],
\]

where \( \{ \varphi_{ci} \} \) is the above mentioned partition of unity.

It is clear that \( V_\varepsilon(x) \in C_0^2(Q^C) \) and according to (4.15) the normal component satisfies

\[
V_{\varepsilon\nu} = 0 \text{ on } \partial B^i_\varepsilon
\]

and

\[
\text{div} V_\varepsilon(x) = \sum_{k,l} (\nabla u^i_{ck}, \nabla u^i_{cl}) \frac{\partial V_l}{\partial x_k}(x^i) \text{ in } \Pi^i_{\varepsilon-} \setminus B^i_\varepsilon.
\]

It is not difficult to check that

\[
|\text{curl } V_\varepsilon(x)| = \left| \sum_{k,l} \frac{\partial V_l}{\partial x_k}(x^i) \left( \frac{\partial \tilde{u}^i_{ck}}{\partial x_1} \frac{\partial \tilde{u}^i_{ck}}{\partial x_2} - \frac{\partial \tilde{u}^i_{ck}}{\partial x_2} \frac{\partial \tilde{u}^i_{ck}}{\partial x_1} \right) \right| \text{ in } \Pi^i_{\varepsilon-}.
\]

To estimate \( \text{div } V_\varepsilon \) and \( \text{curl } V_\varepsilon \) in \( \Pi^i_{\varepsilon+} \setminus \Pi^i_{\varepsilon-} \) we represent \( V_{\varepsilon j}(x) \) in the form

\[
V_{\varepsilon j}(x) = \sum_i \varphi_{ci}(x) \left[ V_j(x^i) + \sum_k \frac{\partial V_j}{\partial x_k}(x^i)(x_k - x^i_k) - V_j(x) \right] +
\]
\[
+ \sum_i \varphi_{\varepsilon i}(x) \left\{ \sum_k \left[ V_k(x^i) + \sum_l \frac{\partial V_k}{\partial x_l}(x^i)(x_l - x'_l) - V_k(x) \right] \frac{\partial \tilde{u}_{\varepsilon k}}{\partial x_j}(x) + \right\}
\]
\[
+ \sum_i \varphi_{\varepsilon i}(x) \left\{ \sum_l \left[ \frac{\partial V_k}{\partial x_l}(x^i) - \frac{\partial V_k}{\partial x_l}(x) \right] \tilde{u}_{\varepsilon l}(x) \right\} +
\]
\[
+ \sum_i \varphi_{\varepsilon i}(x) \left\{ \sum_{k,l} \left[ \frac{\partial V_k}{\partial x_l}(x^i) - \frac{\partial V_k}{\partial x_l}(x) \frac{\partial \tilde{u}_{\varepsilon k}}{\partial x_j}(x) \tilde{u}_{\varepsilon l}(x) \right] + \right\}
\]
\[
V_j(x) + \sum_k V_k(x) \frac{\partial \tilde{u}_{\varepsilon k}}{\partial x_j}(x) + \sum_l \frac{\partial V_j}{\partial x_l}(x) \tilde{u}_{\varepsilon l}(x) +
\]
\[
+ \sum_{k,l} \frac{\partial V_k}{\partial x_l}(x) \frac{\partial \tilde{u}_{\varepsilon k}}{\partial x_j}(x) \tilde{u}_{\varepsilon l}(x). \tag{4.19}
\]

The derivatives of order \( r \) of the expressions in the brackets in the first two terms in (4.19) have the estimate \( O(\varepsilon^{2-r}) \) and in the third and the fourth have the estimate \( O(\varepsilon^{1-r}) \). Therefore, using the estimates
\[
D^r \tilde{u}_{\varepsilon k} = O(\varepsilon^{1-r}), \quad D^r \varphi_{\varepsilon i} = O(\varepsilon^{-r}), \quad r = 0, 1,
\]
and the equality
\[
|\text{curl}(\tilde{u}_{\varepsilon l} \nabla \tilde{u}_{\varepsilon k})| = \left| \frac{\partial \tilde{u}_{\varepsilon l}}{\partial x_1} \frac{\partial \tilde{u}_{\varepsilon k}}{\partial x_2} - \frac{\partial \tilde{u}_{\varepsilon k}}{\partial x_1} \frac{\partial \tilde{u}_{\varepsilon l}}{\partial x_2} \right|
\]
and taking into account (4.6), (4.7) we get
\[
\int_{Q^C} (\text{div} V_\varepsilon)^2 < C_1, \tag{4.20}
\]
\[
\int_{Q^C} (|V_\varepsilon|^2 + |\text{curl} V_\varepsilon|^2) < C_2, \tag{4.21}
\]
where the constants \( C_1, C_2 \) does not depend on \( \varepsilon \).

It follows from the formula (4.19), that \( V_\varepsilon(x) \) converges weakly in \( (L^2(Q^C))^2 \) as \( \varepsilon \to 0 \) to \( V(x) \). Actually, using the above estimates for the equality
\[
\int_{n^\varepsilon} \frac{\partial \tilde{u}_{\varepsilon k}}{\partial x_j} = 0
\]
and (4.19) we conclude that for any smooth in \( Q^C \) vector- function \( W \) the equality
\[
\lim_{\varepsilon \to 0} \int_{Q^C} V_\varepsilon W = \int_{Q^C} V W
\]
holds. Together with (4.21) it provides the weak convergence of \( V_\varepsilon \) to \( V \) in \( (L^2(Q^C))^2 \).

Note that the estimate (4.21) also means that vector-function \( V_\varepsilon(x) \) is uniformly in \( \varepsilon \) bounded in the norm of the space \( H(Q^C, \text{curl}) \) [see[16]]. Due to the properties of function \( \Psi_\varepsilon(x) \)

\[
\text{div} P_\varepsilon(x) = 0 \text{ in } Q^C
\]

and hence, the vector-function \( P_\varepsilon(x) \) is uniformly in \( \varepsilon \) bounded in the norm of the space \( H(Q^C, \text{div}) \) [16]. Therefore, taking into account the weak convergence in \( (L^2(Q^C))^2 \) of \( V_\varepsilon(x) \) to \( V(x) \) and the convergence of \( P_\varepsilon(x) \) to \( P(x) \), with a help of the compensated compactness theorem ([7],[8],[16]), we conclude

\[
\lim_{\varepsilon \to 0} \int_{Q^C} P_\varepsilon V_\varepsilon = \int_{Q^C} PV = \int_{\Omega} \sum_l P_l V_l. \tag{4.22}
\]

Integrating by parts and taking into account (4.15) we can write

\[
\int_{Q^C} P_\varepsilon V_\varepsilon = - \int_{Q^C} \Psi_\varepsilon \text{div} V_\varepsilon = - \int_{Q^C} \tilde{\Psi}_\varepsilon D_\varepsilon, \tag{4.23}
\]

where the function \( D_\varepsilon(x) \) is determined by

\[
D_\varepsilon(x) = \begin{cases} 
\text{div} V_\varepsilon(x), & \text{when } x \in Q^C, \\
0, & \text{when } x \in \bigcup_i B_i^\varepsilon.
\end{cases}
\]

Taking into account (4.15), (4.20) and (3.15) in the same way as in the previous Section we conclude that \( D_\varepsilon(x) \) converges weakly in \( L^2(Q^C) \), as \( \varepsilon \to 0 \), to the function

\[
\sum_{k,l} a_{kl} \frac{\partial V_l}{\partial x_k}(x).
\]

We now pass to the limit as \( \varepsilon = \varepsilon_{\nu} \to 0 \) in (4.23). Using the strong convergence of \( \Psi_\varepsilon(x) \) to \( \Psi_\varepsilon(x) \) in \( L^2(Q^C) \), we then obtain

\[
\lim_{\varepsilon = \varepsilon_{\nu} \to 0} \int_{Q^C} P_\varepsilon V_\varepsilon = \lim_{\varepsilon = \varepsilon_{\nu} \to 0} \int_{Q^C} \tilde{\Psi}_\varepsilon D_\varepsilon = - \int_{Q^C} \Psi \sum_{k,l} a_{kl} \frac{\partial V_l}{\partial x_k}.
\]

As was proven above, \( \Psi \) is a smooth function in \( Q^C \). Taking into account, that \( V_l \) have compact support in \( Q^C \) and integrating by parts, we get

\[
\lim_{\varepsilon = \varepsilon_{\nu} \to 0} \int_{Q^C} (P_\varepsilon V_\varepsilon) = \int_{Q^C} \sum_l \left( \sum_k a_{kl} \frac{\partial \Psi}{\partial x_k} \right) V_l.
\]

Compare this equality with (4.11) and taking into account that the function \( V(x) \) was chosen arbitrarily and the symmetry of the matrix \( \{a_{kl}, k, l = 1,2\} \), we obtain the equality (4.14). Thus Lemma 3 is proven.
Now we can pass to the limit in the equalities (4.13). According to Corollary 1 (Section 3) and Lemma 3 we obtain

\[
\begin{align*}
&\sum a_{1l} \frac{\partial \Psi}{\partial x_1} = - \frac{\partial \Phi}{\partial x_2} \\
&\sum a_{2l} \frac{\partial \Psi}{\partial x_1} = - \frac{\partial \Phi}{\partial x_1}
\end{align*}
\] in $Q^C$.

(4.24)

Since $\Phi \in C^\infty(Q)$ and $\Phi$ is a constant $C$ on $\partial G$ and $\Phi$ equals zero on $\partial \Omega$, these equalities imply that $\Psi$ is a smooth function in $Q^C$ up to the boundaries $\partial \Omega$, $\partial G$ and $L^\pm$, and satisfies the conditions

\[
\frac{\partial \Psi}{\partial \nu_a} = 0 \text{ on } \partial \Omega \text{ and } \partial G, \quad \left[ \frac{\partial \Psi}{\partial \nu_a} \right] = 0 \text{ on } L.
\]

Hence we have proven that $\Psi$ is a solution of the problem (4.1). The uniqueness of this solution can be obtained in a standard way. Consequently, there exists a limit $\lim_{\epsilon \to 0} \Psi_\epsilon = \Psi$ (not only for a subsequence $\epsilon = \epsilon_\nu \to 0$). Finally by using standard techniques of calculus of variations we see that the problem (4.1) is equivalent to the minimization problem (4.2) in the class $E^d$.

6. The end of proof of Theorem 1. Let $u_\epsilon(x)$ be a solution of the problem (1.2). As was shown in Section 2 $u_\epsilon = e^{i\Psi\epsilon}$, where $\Psi\epsilon$ is the corresponding solution of the problem (2.3). Then we can choose a real $\theta_\epsilon$ so that the equality $e^{i\theta_\epsilon}u_\epsilon = e^{i\Psi_\epsilon}$ holds, where $\Psi_\epsilon$ is the solution of the problem (2.3), satisfying the condition (2.4). It follows now from Theorem 3 that $e^{i\theta_\epsilon}u_\epsilon$ converges, as $\epsilon \to 0$, in the norm of $L_2(Q_\epsilon)$ to $u = e^{i\Psi}$, where $\Psi$ is the solution of the problem (4.1). Taking this into account we conclude that $u$ is a smooth function in $Q^C$ and satisfies in $Q^C$ the equation (1.8) and the conditions $[u] = 0$, $\left[ \frac{\partial u}{\partial \nu_a} \right] = 0$ across the cut. Hence $u(x)$ is a smooth function everywhere in $Q$ and satisfies in $Q$ the equation (1.8). From the boundary conditions on $\partial \Omega$ and $\partial G$ for $\Psi$ we deduce the boundary condition (1.9) for $u$.

Taking into account (1.1), we obtain

\[
\deg(u, \partial G) = \frac{1}{2\pi} \int \left( u \times \frac{\partial u}{\partial \tau} \right) = \frac{1}{2\pi} \int \frac{\partial \Psi}{\partial \tau} = - \frac{1}{2\pi} [\Psi] = d,
\]

and in the same way $\deg(u, \partial \Omega) = d$. Thus $u$ belongs to the class $E_d$. Therefore it follows from (1.8), (1.9) that $u(x)$ is the solution of the minimization problem (1.7). The same conclusion can be obtained from the equality

\[
\int_Q \sum_{k,l} a_{kl} \frac{\partial u}{\partial x_k} \frac{\partial \bar{u}}{\partial x_l} = \int_{Q^C} \sum_{k,l} a_{kl} \frac{\partial \Psi}{\partial x_k} \frac{\partial \bar{\Psi}}{\partial x_l},
\]

taking into account that $\Psi$ minimizes the functional (4.2) in the class $E_d$ and that the formula $u = e^{i\Psi}$ gives a one-to-one correspondence:

\[
E_d \leftrightarrow E^d \cap \left\{ \Psi : \left[ \frac{\partial \Psi}{\partial \nu_a} \right] = 0 \text{ on } L \right\}
\]
We now show the convergence of the flux density (1.10). First we note that
\[ j_\varepsilon(x) = P_\varepsilon(x), \quad x \in Q \]
where \( P_\varepsilon(x) \) is a vector-function defined by equalities (4.12). Using Lemma 3 we conclude that \( j_\varepsilon(x) \) converges weakly in \( L_2(Q^C) \) to the vector-function
\[ j(x) = \sum_{k,l} a_{kl} \frac{\partial \Psi}{\partial x_l} e^k. \]
This together with the equality
\[ \frac{\partial \Psi}{\partial x_k} = \frac{1}{2i} \left( \bar{u} \frac{\partial u}{\partial x_k} - u \frac{\partial \bar{u}}{\partial x_k} \right) = \text{Im} \left[ \bar{u} \frac{\partial u}{\partial x_k} \right] \]
(which follows from the equality \( u = e^{i\Psi} \)), proves the formula (1.11). Thus Theorem 1 is proven.

Corollary 3. It follows from the formula for the homogenized flux density \( j(x) \) obtained above, that the total flux of the superfluid liquid through any cross-section \( L \) of the domain \( Q = \Omega \setminus \bar{G} \), joining \( \partial G \) and \( \partial \Omega \) is equal to \( C \), where \( C \) is the constant in the problem (3.2).

Actually, due to (1.11) and (4.24)
\[ I = \int_L j_\nu = \int_L \sum_{k,l} a_{kl} \frac{\partial \Psi}{\partial x_l} \cos(\nu, x_k) = \int_L \frac{\partial \Phi}{\partial \tau} \]
and taking into account that \( \Phi = C \) on \( \partial G \) and equals 0 on \( \partial \Omega \), we conclude that \( I = C \).

7. On the relation between conductivity and polarization tensors. The results obtained above allow us to write the formula, connecting the elements of the conductivity tensor \( a = \{a_{kl}\} \) and elements of the polarization tensor \( b = \{b_{kl}\} \) of the form
\[ b_{kl} = \frac{a_{kl}}{\det a} \quad (6.1) \]
To prove it we consider the system (4.24). Solving it with respect to \( \left( \frac{\partial \Psi}{\partial x_1}, \frac{\partial \Psi}{\partial x_2} \right) \) and taking into account the symmetry of the matrix \( a \), we obtain
\[ \begin{align*}
\frac{\partial \Psi}{\partial x_1} &= \sum_l \hat{a}_{1l} \frac{\partial \Phi}{\partial x_l} \\
\frac{\partial \Psi}{\partial x_2} &= -\sum_l \hat{a}_{1l} \frac{\partial \Phi}{\partial x_l} \quad \text{in } Q^C, \quad (6.2)
\end{align*} \]
where \( \hat{a} = \frac{\partial a_{kl}}{\partial \det a} \).

Hence, since \( \Psi \) is a solution of the problem (4.1), we conclude that \( \Phi \) satisfies the equations
\[ \sum_{k,l} \hat{a}_{kl} \frac{\partial^2 \Phi}{\partial x_k \partial x_l} = 0 \text{ in } Q, \quad (6.3) \]
\[ \int_{\partial G} \sum_{k,l} \hat{a}_{kl} \frac{\partial \Phi}{\partial x_l} \cos(\nu, x_k) = 2\pi d. \] (6.4)

On the other hand, according to (3.2)

\[ \sum_{k,l} b_{kl} \frac{\partial^2 \Phi}{\partial x_k \partial x_l} = 0 \text{ in } Q, \] (6.5)

\[ \int_{\partial G} \sum_{k,l} b_{kl} \frac{\partial \Phi}{\partial x_k} \cos(\nu, x_l) = 2\pi d. \] (6.6)

It follows from (6.3), (6.4) that

\[ b_{kl} = C \hat{a}_{kl}, \] (6.7)

where \( C \) is an arbitrary constant.

Indeed, since matrix \( a \) and \( b \) are symmetric and positive, they can be diagonalized by the same non-degenerate matrix \( A \): \( \hat{a}' = A a A^T \), \( b' = A b A^T \) where \( \hat{a}'_{kk} > 0 \), \( b'_{kk} > 0 \), \( \hat{a}'_{kl} = 0 \), \( b'_{kl} = 0 \) (\( k \neq l \)).

Changing for a new variable \( y = Ax \) and setting \( \tilde{\Phi}(y) = \Phi(A^{-1}y) \), we write the equations (4.17), (4.19) as follows

\[ \sum_{i,j} \left( \sum_{k,l} A_{ik} \hat{a}'_{kl} A_{lj}^T \right) \frac{\partial^2 \tilde{\Phi}}{\partial y_i \partial y_j} = 0, \]

\[ \sum_{i,j} \left( \sum_{k,l} A_{ik} b_{kl} A_{lj}^T \right) \frac{\partial^2 \tilde{\Phi}}{\partial y_i \partial y_j} = 0, \]

or

\[ \begin{cases} \hat{a}'_{11} \frac{\partial^2 \tilde{\Phi}}{\partial y_1^2} + \hat{a}'_{22} \frac{\partial^2 \tilde{\Phi}}{\partial y_2^2} = 0, \\ b'_{11} \frac{\partial^2 \tilde{\Phi}}{\partial y_1^2} + b'_{22} \frac{\partial^2 \tilde{\Phi}}{\partial y_2^2} = 0 \end{cases} \] (6.8)

The system (6.8) has a non-zero solution \( \left( \frac{\partial^2 \tilde{\Phi}}{\partial y_1^2}, \frac{\partial^2 \tilde{\Phi}}{\partial y_2^2} \right) \) only when its determinant is zero. Hence

\[ b'_{kk} = C \hat{a}'_{kk}, \text{ } k = 1, 2, \]

where \( C \) is an arbitrary constant.

Thus, the matrices \( b' = \text{diag}(b'_{kk}) \), and \( \hat{a}' = \text{diag}(\hat{a}'_{kk}) \) are proportional, and since \( \hat{a} = A^{-1} \hat{a}'(A^T)^{-1} \) and \( b = A^{-1}b'(A^T)^{-1} \) we obtain (6.7).

Substitute \( b_{kl} = C \hat{a}_{kl} \) in (6.6) and compare it with (6.4) we deduce that \( C = 1 \). Thus the formula (6.1) is established.

Notice, that this result can be obtained by using the duality principle for the variational formulation of the cell problems with finite conductivities \( \delta \) and \( \delta^{-1} \) of the
set $B$. Then one can take the limit, as $\delta \to 0$ (see for example [12], where an analogous limiting procedure has been used for a different purpose). However this way requires justification of changing the order of taking the limit in $\delta$ and the minimization.

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