Homogenization of Harmonic Maps with Large Number of Vortices and Applications in Superconductivity and Superfluidity.

Leonid Berlyand  
Department of Mathematics, Penn State University  
University Park, Pennsylvania 16802, United States

Evgenii Khruslov  
Institute of Low Temperature and Engineering  
Ukrainian Academy of Science,  
Lenin Ave 47, Kharkov 310164, Ukraine

We study a nonlinear homogenization problem of harmonic maps which describes an ideal superconducting or an ideal superfluid medium with large number of vortices and the degree conditions prescribed at the external insulating boundary. We derived the homogenized problem which describes the limiting behavior of the fluxes when the total number of vortices tends to infinity. The homogenized problem is described in terms of the effective vorticity and the effective anisotropy tensor. The calculation of this tensor amounts to solving a linear cell problem, which is well studied in the homogenization literature and can be solved by using existing numerical packages. The convergence of the fluxes is rigorously proved. We also discuss unusual features of the homogenized limit for the wave functions. The proofs are based on a variational approach which does not require periodicity and can be used in more general situations.

Keywords: homogenization, vortices, degree of a map, harmonic map, superconductivity, superfluidity.

INTRODUCTION

We consider a homogenization problem for harmonic maps with vortices which models three physical problems two of which are closely related.

Let $\Omega \subset \mathbb{R}^2$ be a smooth bounded domain. For simplicity of presentation we also assume that $\Omega$ is simply connected and therefore $\partial \Omega$ is a closed smooth curve. Let $\left\{ \bigcup_{i=1}^{N} D_{i}, \; i = 1, \ldots, N \right\}$ be a collection of small disks of diameter $\epsilon$ centered at the lattice points of a two-dimensional periodic lattice with periods $\epsilon h_{1}, \; \epsilon h_{2}$ ($h_{1} > 1, \; h_{2} > 1$) along axes $OX_{1}$ and $OX_{2}$ respectively. Then we denote by $\Omega_{\epsilon}$ the domain $\Omega$ with deleted disks $D_{i}$ i.e. $\Omega_{\epsilon} = \Omega \setminus \bigcup_{i=1}^{N} D_{i}$. Figure 2 illustrates this geometry in a simple case when $\Omega$ is a disk.
Consider the following boundary value problem

(1) \[ \Delta u_\varepsilon + u_\varepsilon |\nabla u_\varepsilon|^2 = 0 \quad \text{in } \Omega_\varepsilon, \]

(2) \[ |u_\varepsilon| = 1 \quad \text{in } \Omega_\varepsilon, \]

(3) \[ \frac{\partial u_\varepsilon}{\partial \nu} = 0 \quad \text{on } \partial \Omega_\varepsilon = \partial \Omega \cup \left( \bigcup_i \partial D_i^\varepsilon \right), \]

(4) \[ \deg(u_\varepsilon, \partial \Omega) = d_\varepsilon, \]

(5) \[ \deg(u_\varepsilon, \partial \Delta_i^\varepsilon) = d_{i\varepsilon}, \]

where \( u(x) = u(x_1, x_2) \) is a complex-valued function \( Q_\varepsilon \to S^1 \), \( d_\varepsilon, d_{i\varepsilon} \) are integer numbers, \( \deg(u, \Gamma) \) is the Brouwer degree or winding number of a function \( v \) considered as a map from a closed curve \( \Gamma \) into the unit sphere \( S^1 \), \( \nu \) is the unit outward normal. Equation (1) is called the harmonic map equation and its solution \( u_\varepsilon \) can be represented as

(6) \[ u_\varepsilon(x) = e^{i\psi_\varepsilon(x)}, \]

where \( \psi_\varepsilon(x) \) is a real-valued function (phase). Since the domain \( \Omega_\varepsilon \) is multiply connected the phase \( \psi_\varepsilon(x) \) is not single-valued and is multiplied by a factor of \( 2\pi \) when going around the holes \( D_i^\varepsilon \).

Our main goal is to describe asymptotic behavior of the fluxes

(7) \[ v_\varepsilon = \text{Im}(u_\varepsilon^* \nabla u_\varepsilon) \]

as \( \varepsilon \to 0 \), i.e., when the number of holes becomes large and their diameters become small. Here \( \text{Im} \) stands for the imaginary part of a complex number and \( ^* \) denotes the complex conjugate. This problem arises in several physical situations.

First consider the angular rotation of a superfluid (condensate) in a long vertical cylinder \( \Omega \) with a large number \( N_\varepsilon \) of very thin rods \( D_i^\varepsilon, i = 1, 2, \ldots, N_\varepsilon \). The superfluid is described by a complex-valued wave function \( u_\varepsilon \) ([17], [21]). We only consider an ideal superfluid which implies that \( |u_\varepsilon| = 1 \). Since the cylinder is long, the wave function \( u \) does not depend on the vertical variable \( x_3 \) and the problem becomes planar. We keep the same notations \( \Omega, D_i^\varepsilon, \partial \Omega, \partial D_i^\varepsilon \) for the corresponding two-dimensional cross-sections \( x_3 = 0 \) (see Fig. 2). Due to the quantization, the circulation of the superfluid velocity \( v_\varepsilon \) along the boundaries \( \partial \Omega \) and \( \partial D_i^\varepsilon \), is an integer

(8) \[ \frac{1}{2\pi} \int_{\partial \Omega} (v_\varepsilon \cdot \tau) d\tau = d_\varepsilon, \]
\( \frac{1}{2\pi} \int_{\partial D_{\kappa}} (v_{\kappa} \cdot \tau) d\tau = d_{\kappa}, \)

where \( \tau \) is the unit tangent vector. The velocity of the superfluid is given by [17]

\[ v_{\kappa} = \frac{\hbar}{m} \left( u_{\kappa}^* \nabla u_{\kappa} \right), \tag{10} \]

where \( \hbar \) is Planck's constant and \( m \) is the mass of a condensate particle. Due to (6) this implies that the wave function \( u_{\kappa} \) satisfies (4)–(5). It is also known ([7]) that \( u_{\kappa} \) satisfies the harmonic map equation (1) and

\[ \sum d_{\kappa} = d_c. \tag{11} \]

The integers \( d_{\kappa} \) can be positive, negative or zero. The latter means that the corresponding hole (rod) is not a real vortex. The boundary condition (3) means that there is no flux across the boundary of the cylinder and the boundaries of the rods.

It is also known ([7], [17]) that the wave function minimizes the Dirichlet integral \( \int_{Q_{\kappa}} |\nabla u_{\kappa}|^2 \) in the class of functions from \( H^1(Q_{\kappa}, S^1) \) satisfying the conditions (4)–(5), which implies ([7], [11]) that \( u_{\kappa} \) satisfies (1).

We next consider a composite which is confined inside a long vertical cylinder \( \Omega \). The composite consists of an ideal superconductor reinforced by a large number \( N_{\kappa} \) of very thin insulating rods \( B^i_{\kappa}, i = 1, 2, \ldots, N_{\kappa} \) (see Fig 1, Fig 2). Such rods can be used to improve the mechanical properties of the superconductor. Again the superconductor can be described by the complex-valued wave function (order parameter) \( u_{\kappa} \), and the supercurrent is determined by the formula ([17]: Ch. 5, Sect. 44),

\[ j_{\kappa} = \frac{\hbar e}{2m} \text{Im}(u_{\kappa}^* \nabla u_{\kappa}), \tag{12} \]

where \( e, m \) are the charge and the mass of superconducting charge carriers respectively. If this composite sample is exposed to an external vertically oriented magnetic field and the temperature is decreased below critical
then a supercurrent in the equatorial direction (perpendicular to the magnetic field) appears in the superconductor and the magnetic field is expelled from the superconductor almost completely (Meissner effect). After the applied magnetic field is terminated, a part of this field is trapped by some of the rods, \( D_{ik} \) (which are not in the superconducting state). More precisely due to quantization an integer number \( d_{ik} \) of quanta of the magnetic field \( H = \text{curl} \ A \) is trapped by the rod \( D_{ik} \), where \( d_{ik} \) can be positive, negative or zero. If for some \( i \) \( d_{ik} = 0 \) then the magnetic field did not penetrate into the rod \( D_{ik} \). The vector-potential \( A \) of the magnetic field satisfies relations analogous to (8)–(9), which in turn imply the boundary conditions (4), (5). Since the superconductor is ideal condition (2) holds. The boundary of the cylinder and the boundaries of the rods are insulating which in view of (12) implies (3). Finally the wave function (order parameter) satisfies (1)–(2) ([7]) and we again arrive at the problem (1)–(5).

The situation becomes a bit more complicated when the magnetic field \( H_\epsilon = \text{curl} \ A_\epsilon \) is present. Then an additional term proportional to \( |u_\epsilon|^2 A_\epsilon \) should be added to the RHS of (12). However it is also possible to show (analogously to [5], [4]) that the wave function \( u_\epsilon = e^{i\psi_\epsilon} \ (A_\epsilon = \nabla \psi_\epsilon) \) is the solution of the problem (1)–(5).

The third and perhaps most interesting problem which can be modeled by (1)–(5) is an array vortices in a superconducting thin film. The vortices are pinned down by materials defects, which form a periodic array. For
example, in the experimental work [22] a periodic array of such vortices has been obtained in Nb thin-film. The presence of such vortices drastically changes the properties of the superconducting medium and therefore such film is a superconducting composite and it is necessary to describe its effective properties. Of course the harmonic maps equation is an idealized model for such experiments but it still captures some realistic feature of the physical phenomenon and serves as a basis for a more sophisticated mathematical model based on the Ginzburg-Landau equation, which will be presented elsewhere.

The homogenization problem in the above context means the following. Given information about the limiting distribution as $\varepsilon \to 0$ of the vortices, describe the limiting behavior of the fluxes $\nu_\varepsilon(x)$ ((7)). Hereafter we refer to each disk $D_{\varepsilon i}$ with nonzero degree $d_{\varepsilon i}$ as a vortex.

Roughly speaking our main result can be described as follows. We introduce a parameter $\alpha (\frac{2}{3} \leq \alpha \leq 2)$ so that average distances between vortices are of order $\varepsilon^{\alpha/2}$. We characterize the limiting density of the vortices (normalized by $\varepsilon^\alpha$) by a scalar function $\rho(x)$. Then we study asymptotics of the vector-functions $\nu_{\varepsilon i}(x) = \varepsilon^{\alpha/2} \tilde{\nu}_i(x)$, where $\tilde{\nu}(x) = \text{Im}(u^* \nabla u)$ in $\Omega_{\varepsilon}$ and extended by zero into the holes $D_{\varepsilon i}$ ($i = 1, \ldots, N_{\varepsilon}$).

We show that if the vortices are distributed somewhat regularly (e.g. do not form clusters, which is possible after the rescaling by $\varepsilon^\alpha$ if $\alpha < 2$) then $\nu_{\varepsilon i}(x)$ converges weakly in $L^2(\Omega)$ to a vector function $\nu(x) = Au(x)$, where $u(x)$ is the solution of the following homogenized problem

\begin{align}
\text{curl } u &= \rho(x) \quad \text{in } \Omega, \\
\text{div } (Au) &= 0 \quad \text{in } \Omega, \\
(Au)_n &= 0 \quad \text{on } \partial \Omega,
\end{align}

Here $A$ is a matrix with constant entries $a_{ik}$ ($i, k = 1, 2$) which are determined by solutions of some standard linear problems in the rescaled periodicity cell of unit size also known as a virtual mass tensor, the subscript $\nu$ stands for the normal component of a vector. We show that the problem (13)-(15) has the unique solution $u(x) \in H^1(\Omega)$. This problem describes the vortical motion of an incompressible fluid in an anisotropic medium in $\Omega$ with the given vorticity density $\rho(x)$ when the fluid does not penetrate the boundary $\partial \Omega$.

Finally, we note that this work is a continuation of the work [5] where homogenization of harmonic maps in an annular domain with a large number of small holes has been studied. In [5] nonzero (finite, i.e. independent of $\varepsilon$) degrees were prescribed at the inner and outer boundaries of the annulus were prescribed and all holes had zero degrees so that there are no vortices inside the domain. By contrast the main focus of the present work is to study impact of a large number of vortices inside the domain. In [5]
the homogenized equations were derived for the wave functions (harmonic maps) and the convergence of the fluxes was obtained as a corollary. Here the main result describes the convergence of the fluxes. In fact, in the last section we discuss the limiting (homogenized) equations for the wave function \( u_{e} \). Because of the presence of a large number of vortices inside the domain \( O(e^{-\alpha}) \), the wave function \( u_{e} \) "spins around very fast" and the phase \( \psi_{e} \) is multi-valued with a high multiplicity. In order to make it single-valued a system of cuts inside the domain must be chosen which corresponds to a choice of one single-valued branch of the phase \( \psi_{e} \). As a result the limiting homogenized problem depends on the choice of this branch and therefore it is not invariant in usual physical sense. That is why in this work we study the convergence of fluxes, which are single-valued and represent measurable physical quantities such as the velocity of superfluid and supercurrent.

Remark 1. If \( e^{\alpha} \sim \frac{h}{m} \), then our result means that the velocity field \( \mathbf{v}(x) \) of the rotating superfluid with rods (or the supercurrent in the superconducting composite if \( e^{\alpha} \sim \frac{h}{2m} \)) can be approximated for small \( \epsilon \) by the homogenized field \( v(x) = A u(x) \). The homogenizing or averaging is performed over a small mesoscale \( \delta \) such that \( \epsilon \ll \delta \ll 1 \) (\( \epsilon \) is the microscale, 1 is the macroscale).

Note that there were many mathematical studies of vortices for both harmonic map and Ginzburg-Landau equations. It is impossible to provide a complete list of references and we address the reader to the work and references therein [7], [2], [10], [12], [20]. In particular we do not address here interesting aspects of vortex dynamics.

The paper is organized as follows:

In Section 1 we give a rigorous formulation of the problem. Then we introduce a generalized distribution function \( F_{\alpha e}(x) \) of vortices. We formulate conditions under which the fluxes \( u_{\alpha e} = e^{\alpha} \psi_{e}(x) \) converge to the homogenized flux in terms of the function \( F_{\alpha e}(x) \), introduce the virtual mass tensor for the unit cell of periodicity and formulate the main result.

In Section 2 we obtain a suitable representation for the solution of the problem (1)-(5). We use this representation to establish the existence of the solution of the problem (1)-(5). In the Section 3 and 4 we use this representation in the proof of our main result. In Section 5 we discuss convergence of the wave functions \( u_{e}(x) \).

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1. Formulation of the problem and the main result.

We shall consider domains in \( \mathbb{R}^2 \). Let \( \vec{e}_1, \vec{e}_2 \) be the unit vectors of the Cartesian coordinate system in \( \mathbb{R}^2 \) so that \( x = x_1 \vec{e}_1 + x_2 \vec{e}_2 \) for any point \( x \in \mathbb{R}^2 \). Let \( \Omega \subset \mathbb{R}^2 \) be a smooth bounded domain, which contains the origin. Again for the sake of simplicity we assume that \( \Omega \) is simply connected. It will be clear from our further consideration that if the domain \( \Omega \) is, for example, an annulus then our techniques still apply with minor technical modifications. We now describe the periodic microstructure. Let \( \Pi \) be a rectangle

\[
\Pi = \left\{ x \in \mathbb{R}^2, \ |x_1| < \frac{h_1}{2}, \ |x_2| < \frac{h_2}{2} \right\}
\]

and \( D \) is a smooth bounded simply connected domain containing the origin such that its closure \( \overline{D} \subset \Pi \).

We introduce the microscopic “holes” \( D_{ie} \) and the periodicity cells \( P_{ie} \) as follows (\( i = 1, 2, \ldots, N_e \)):

\[
D_{ie} = \epsilon D + x_e^i, \quad P_{ie} = \epsilon (\Pi \setminus D) + x_e^i,
\]

where \( x_e^i = \epsilon (m_i h_1 \vec{e}_1 + n_i h_2 \vec{e}_2) \) are the lattice points of a two dimensional periodic lattice with periods \( \epsilon h_1, \epsilon h_2 \) along the directions \( \vec{e}_1 \) and \( \vec{e}_2 \) respectively, \( m_i \) and \( n_i \) are integer numbers such that \( \overline{P_{ie}} \subset \Omega \), \( \epsilon \) is a small parameter.

We consider the perforated domain

\[
\Omega_{\epsilon} = \Omega \setminus \left( \bigcup_i \overline{D_{ie}} \right).
\]

Its boundary \( \partial \Omega_{\epsilon} \) consists of non intersecting closed curves \( \partial \Omega \) and \( \partial D_{ie} \) \( (i = 1, \ldots, N_e) \). Let \( \{d_1, d_2, \ldots, d_{N_e}\} \) be a collection of integer numbers which satisfies (11). Consider the set \( \mathcal{E}_\epsilon \) of smooth maps \( v(x) \) from the domain \( \Omega_{\epsilon} \) into the unit sphere \( S^1 \) which satisfy the following condition

\[
\deg(v, \partial \Omega) = d_\epsilon, \quad \deg(v, \partial D_{ie}) = d_{ie} \quad (i = 1, \ldots, N_e).
\]

Here, \( \deg(v, \Gamma) \) is the Brouwer degree or rotation number of the vector field \( v \) on a smooth closed curve \( \Gamma \subset \mathbb{R}^2 \).

If \( \tau \) is the unit tangent vector to the curve \( \Gamma \) which together with the unit outward normal \( \nu \) forms the properly oriented frame \( (\nu, \tau) \) then the following formula holds ([7]):

\[
\deg(v, \Gamma) = \frac{1}{2\pi} \int_{\Gamma} v \times \frac{\partial v}{\partial \tau}.
\]

We denote by \( \mathcal{E} \) the closure of the set \( \mathcal{E}_\epsilon \) in the metric \( H^1(\Omega_{\epsilon}) \). We shall identify vector fields from \( H^1(\Omega_{\epsilon}, S^1) \) with complex-valued functions on \( \Omega_{\epsilon} \) which take values on the unit circle in the complex plane \( \mathbb{C} \).
Consider the minimization problem

\begin{equation}
\min_{u_{\epsilon} \in \mathcal{E}_\epsilon} \int_{\Omega_\epsilon} |\nabla u_{\epsilon}|^2.
\end{equation}

We shall show that there exists a unique (up to a complex multiplicative constant of modulus 1) solution \( u_{\epsilon} \) of this problem and it satisfies in \( \Omega_\epsilon \) the following Euler-Lagrange equation

\begin{equation}
\Delta u_{\epsilon} + u_{\epsilon} |\nabla u_{\epsilon}|^2 = 0,
\end{equation}

and the Neumann boundary condition on \( \partial \Omega_\epsilon \)

\begin{equation}
\frac{\partial u_{\epsilon}}{\partial \nu} = 0,
\end{equation}

which establishes the existence of the solution \( u_{\epsilon}(x) \) of the problem (1)–(5). Our main goal is to study the asymptotic behavior of the fluxes \( u_{\epsilon} = \text{Im}(u_{\epsilon}^* \nabla u_{\epsilon}) \) as \( \epsilon \to 0 \). For this purpose we make the following three assumptions about the distribution of the “holes” \( D_{\epsilon,i} \).

(A1) We assume that the degrees \( d_{\epsilon,i} \) are uniformly bounded:

\begin{equation}
\max_i |d_{\epsilon,i}| < C,
\end{equation}

where \( C > 0 \) does not depend on \( \epsilon \).

(A2) It is possible to inscribe non intersecting rectangles \( \Pi_{\epsilon,i} \) into the domain \( \Omega_\epsilon \) such that each of these rectangles contains not more than one hole \( D_{\epsilon,i} \) with nonzero vorticity \( d_{\epsilon,i} \) and side lengths \( h_{1\epsilon,i}, h_{2\epsilon,i} \) of these rectangles satisfy the following inequalities

\begin{equation}
C_1 \epsilon^\frac{2}{3} \leq h_{1\epsilon,i}, h_{2\epsilon,i} \leq C_2 \epsilon^\frac{2}{3},
\end{equation}

where \( \alpha \geq \frac{2}{3} \) and the constants \( c_1 \) and \( c_2 \) are independent on \( i \) and \( \epsilon \).

From the above description of the geometry of the domain \( \Omega_\epsilon \), it follows that \( \alpha \leq 2 \) and if \( \alpha = 2 \) then the condition (21) always holds (i.e., (21) imposes certain restrictions on the distribution of the vortices only in the case when \( \alpha < 2 \)). We choose the smallest possible \( \alpha \) and by abuse of notation call it \( \alpha \). Roughly speaking this number characterizes the density of vortices (holes with nonzero degree). The rectangles \( \Pi_{\epsilon,i} \) partition the domain \( \Omega \) except at the boundary \( \partial \Omega \) where some of them may get cut. We only consider the geometries when \( \alpha \geq \frac{2}{3} \), i.e., the vortices are not “too rare.” The reason why the fraction \( \frac{2}{3} \) appears in our consideration will be explained later in the Remark 3, Section 4.

We next introduce a generalized function \( R_{\epsilon}(x) \) which characterizes the distribution of the vortices. Set

\begin{equation}
R_{\epsilon}(x) = 2\pi \epsilon^\alpha \sum_{i=1}^{N_\epsilon} d_{\epsilon,i} \delta(x - x_{\epsilon,i}^i),
\end{equation}
where $\delta(x)$ is the Dirac delta function and $x^i_\epsilon \in D_\epsilon$ are defined in (16).

(A3) There exists a weak limit (in the sense of distributions) $w = \lim_{\epsilon \to 0} R_\epsilon(x) = \rho(x) \in L_2(\Omega)$ i.e. for any $\varphi \in C^\infty(\Omega)$

$$
\lim_{\epsilon \to 0} \int_\Omega R_\epsilon(x) \varphi(x) = \int_\Omega \rho(x) \varphi(x)
$$

(23)

The function $\rho(x) \in P_2(\Omega)$ is called the \textit{limiting vorticity}.

\textbf{Remark 1.1} The relation (11) together with the condition (A3) imply that

$$
\lim_{\epsilon \to 0} \epsilon^3 d_\epsilon = \frac{1}{2\pi} \int_\Omega \rho(x)
$$

i.e. $d_\epsilon \sim O(\epsilon^{-3})$.

We introduce a tensor which provides a quantitative characteristic of the influence of the holes $D_\epsilon$. Consider the following boundary value problems ($k = 1, 2$) in the rescaled cell of periodicity with unit size.

$$
\Delta u_k = 0 \quad \text{on } \Pi \setminus \overline{D}
$$

$$
\frac{\partial u_k}{\partial n} = 0 \quad \text{on } \partial D
$$

(25)

where $u_k - x_k$ and $\nabla (u_k - x_k)$ are equal on the opposite side of the rectangle $\Pi$.

It is well known that there exists a unique (up to an additive constant) solution of this problem. We define the entries of a matrix $A$ by the following relations

$$
a_{k\ell} = \frac{1}{[\Pi]} \int_{\Pi \setminus D} \nabla u_k, \nabla u_\ell \quad k, \ell = 1, 2,
$$

(26)

where $[\Pi]$ is the area of the rectangle $\Pi$. The collection $\{a_{k\ell}, k, \ell = 1, 2\}$ forms a symmetric positive definite tensor in $\mathbb{R}^2$. This tensor is called the virtual mass tensor (sometimes it is also called the conductivity tensor). It characterizes the influence of the rods $D_\epsilon$ on the movement of the superfluid in $\Omega_\epsilon$ (or the influence of the insulating rods or defects on the superconductor $\Omega_\epsilon$).

In order to obtain a homogenization approximation for the velocity field of the superfluid (or the supercurrent in the superconducting composite) we introduce vector-functions $v_{\alpha\gamma}(x)$ which represent the rescaled fluxes

$$
v_{\alpha\gamma}(x) = \begin{cases} 
\epsilon^3 v_\epsilon(x), & x \in \Omega_\epsilon, \\
0, & x \in \cup D_\epsilon,
\end{cases}
$$

(27)

where $u_\epsilon(x)$ is the solution of the problem (1)–(5) and $v_\epsilon = \text{Im}(u_\epsilon^* \nabla u_\epsilon)$. 
**Theorem 1.1.** If the conditions (A1)–(A3) hold then \( v_{\epsilon_0}(x) \) converges weakly in \( L_2(\Omega) \) to the vector-function \( v(x) = Au(x) \) as \( \epsilon \to 0 \), where \( u(x) \) is the solution of the homogenized problem (13)–(15) and the entries of the matrix \( A \) are determined in (26).

This theorem will be proven in Sections 3 and 4.

2. **Representation of the solution through an auxiliary linear problem**

Consider the boundary value problem:

\[
\begin{align*}
\Delta \phi_\epsilon &= 0 & \text{in } \Omega_\epsilon, \\
\phi_\epsilon &= 0 & \text{on } \partial \Omega, \\
\phi_\epsilon &= C_{i\epsilon} & \text{on } \partial D_{i\epsilon}, \\
\int_{\partial D_{i\epsilon}} \frac{\partial \phi_\epsilon}{\partial \nu} &= 2\pi \epsilon^\alpha d_{i\epsilon}, & i = 1, \ldots, N_\epsilon,
\end{align*}
\]

where integer numbers \( d_{i\epsilon}(i = 1, \ldots, N_\epsilon) \) and the parameter \( \alpha \) satisfy the conditions of Theorem 1.1, \( v \) is the outward normal to the contour \( \partial D_{i\epsilon} \), \( \frac{\partial}{\partial \nu} \) is the normal derivative and the constants \( C_{i\epsilon} \) are unknown and determined in the course of solving the problem. Due to linearity the existence and uniqueness of the solution \( \{\phi_\epsilon, C_{i\epsilon}, i = 1, \ldots, N_\epsilon\} \) can be established by standard techniques. We extend the function \( \phi_\epsilon(x) \) by constant values \( C_{i\epsilon} \) on the holes \( D_{i\epsilon} \) and keep the same notation for \( \phi_\epsilon(x) \in H^1(\Omega, \mathbb{R}) \). This function minimizes the functional

\[
F[\phi_\epsilon] = \frac{1}{2} \int_\Omega |\nabla \phi_\epsilon|^2 + 2\pi \epsilon^\alpha \sum_{i=1}^{N_\epsilon} \frac{d_{i\epsilon}}{|D_{i\epsilon}|} \int_{D_{i\epsilon}} \phi_\epsilon,
\]

in the class

\[
V_\epsilon(\Omega) = \{ \phi_\epsilon \in H^1(\Omega, \mathbb{R}), \phi_\epsilon = 0 \text{ on } \partial \Omega, \phi_\epsilon = \text{const on } D_{i\epsilon} \},
\]

where \( |D_{i\epsilon}| \) is the area of \( D_{i\epsilon} \). We make cuts in the domain \( \Omega_\epsilon \) so that the obtained cut domain \( \Omega_\epsilon^C \) is simply connected. It is always possible using vertical and horizontal segments of straight lines passing through some of the lattice points \( x_\epsilon^i \). For instance, we first draw all vertical segments with the end points on \( \partial D_{i\epsilon} \) and then add necessary horizontal segments: one between each to neighboring vertical lines with end points on \( \partial D_{i\epsilon} \) and one more, which has its endpoint on \( \partial \Omega \) (see Fig. 2). Denote the obtained cuts by \( S_{r\epsilon} \). As a result we have obtained the cut domain,

\[
\Omega_\epsilon^C = \Omega \setminus F_\epsilon, \quad F_\epsilon = \left( \bigcup_{i\epsilon} D_{i\epsilon} \right) \cup \left( \bigcup_{r\epsilon} S_{r\epsilon} \right)
\]
Since $\overline{D}_\epsilon \cap \partial \Omega = \emptyset$ this domain is simply connected. Choose the orientation of a contour $L$, which encompasses the set $F_\epsilon$ according the arrows on Fig. 2, i.e. when moving along $L$ the set $F_\epsilon$ stays on the right. Each segment of the cut has two sides. Mark by the “−” sign the side which one passes first when going around the contour and by “+” sign the opposite side.

Consider the following Cauchy-Riemann system for the function $\psi_\epsilon$ in the domain $\Omega_\epsilon C$

$$\begin{align*}
\frac{\partial \psi_\epsilon}{\partial x_1} &= \frac{\partial \phi_\epsilon}{\partial x_2} \\
\frac{\partial \psi_\epsilon}{\partial x_2} &= -\frac{\partial \phi_\epsilon}{\partial x_1}
\end{align*}$$

(34)

where the function $\phi_\epsilon(x)$ is the solution of the problem (28)–(31). Since $\Delta \phi_\epsilon = 0$ in $\Omega_\epsilon^c \subset \Omega_\epsilon$ this system is compatible and therefore it is locally solvable (in any simply connected subdomain of $\Omega_\epsilon$). Thus given $\phi_\epsilon$ we can construct $\psi_\epsilon$ in the simply connected domain $\Omega_\epsilon^c$ by integration along paths lying in $\Omega_\epsilon^c$ which start from some fixed point $x_0^\phi \in \Omega_\epsilon^c$.

Due to the properties of the function $\phi_\epsilon(x)$ (see (28)–(31) and (34) we obtain a single-valued in $\Omega_\epsilon^c$ function $\psi_\epsilon(x)$, which satisfies the following conditions:

11
\( \Delta \psi_c = 0 \) \quad \text{in } \Omega_c^c, \\
\frac{\partial \psi_c}{\partial n} = 0 \quad \text{on } \partial \Omega \text{ and } \partial D_{ic}, \\
[\nabla \psi_c]^e = 0 \quad \text{on } S_{rc}, \\
[\psi_c]^e = 2\pi \alpha \sum_{i} \gamma_{ij} \quad \text{on } S_{rc},

where \([u]^e\) denotes the difference between limiting values of the function \(u\) on two opposite sides of a cut \(S_{rc}\), \text{i.e.} \([u]^e = u^+ - u^-\) and the sum \(\sum_{i} \gamma_{ij}\) is taken over all indices \(i\) that correspond to those holes \(D_{ic}\), which are encompassed by the piece of the contour \(L\) with the starting point \(x^- \in S_{rc}^-\) and the final point \(x^+ \in S_{rc}^+\).

First observe that (35) follows from (34) and the smoothness of \(\phi_c\), which implies that mixed derivatives are equal. Furthermore (36) follows from (29)–(30) since \(\frac{\partial \psi_c}{\partial n} = \frac{\partial \phi_c}{\partial n}\); (37) follows from the continuity of \(\nabla \phi_c\). To obtain (38) we note that (28) implies that \(\text{curl } \nabla \psi_c = 0\). Therefore the integral \(\int \nabla \psi_c \cdot d\tau\) along any nonselfintersecting contour in \(\Omega_c\), which joins two opposite sides of a cut \(S_{rc}\), is equal to the integral along the special contour which consists of the of boundaries of the cuts and boundaries of the disks \(D_{ic}\). Since the integrals along two opposite sides of each cut cancel each other we left with the boundaries of the holes and therefore (31) implies (38).

Clearly \(\psi_c(x)\) is uniquely determined up to an additive constant. We define this constant by imposing the following normalization condition

\[
\int_{\Omega_c} \psi_c = \hat{\psi}_c = 0.
\]

It follows from (35)–(38) and (39) that \(\psi_c(x)\) is the solution of the following minimization problem:

\[
\min_{E_c} \int_{\Omega_c} |\nabla \psi_c|^2,
\]

\[
E_c = \{ \psi_c \in H^1(\Omega_c, R), \quad [\psi_c] = 2\pi \alpha \sum_{i} \gamma_{ij} \quad \text{on } S_{rc},
\]

\[
\hat{\psi}_c = 0 \}.
\]

Consider a complex-valued function

\[
u_c(x) = e^{i \kappa \varphi_c(x)},
\]
Using properties of \( \psi_\varepsilon(x) \) we conclude that \( u_\varepsilon(x) \in \overline{\Omega}_\varepsilon \) and since

\[
(41) \quad \int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^2 = e^{-2\alpha} \int_{\Omega_\varepsilon} |\nabla \psi_\varepsilon|^2,
\]

we see that \( u_\varepsilon(x) \) is the minimizer of the problem (17). Moreover, (35)–(38) and (40) imply that \( u_\varepsilon \) is defined in \( \Omega_\varepsilon \) and is a solution of the problem (1)–(5). The latter problem, however, has many solutions which differ from each other by a constant factor of absolute value 1. We consider here the solution which is represented by the formulas (40), (35)–(38) and (39). Note that the choice of the system of cuts in the domain \( \Omega_\varepsilon \) is not uniquely determine. There are many other choices which also produce a simply connected domain. The change in the system of cuts corresponds to the multiplication of the solution \( u_\varepsilon \) by a complex constant factor \( \theta \) \( (|\theta| = 1) \). Thus we have established the solvability of the (1)–(5) and the representation of the solution by formulas (40), (34), (39), (28)–(31). Finally we provide a heuristic explanation of the scaling in (27). Since \( d_\varepsilon = O(e^{-\alpha}) \) the maximum jump of the phase \( \Psi_\varepsilon (u_\varepsilon(x) = e^{i\Psi_\varepsilon(x)}) \) inside the domain is also \( O(e^{-\alpha}) \). To ”suppress” this jump one may introduce a new phase \( \Psi_\varepsilon = c_\alpha \psi_\varepsilon \) so that \( u_\varepsilon(x) = e^{i\beta^\alpha \psi_\varepsilon(x)} \). Then the scaling factor \( e^\alpha \) appears in \( v_{\alpha} \equiv i m(e^{-i\Psi_\varepsilon} \nabla e^{i\Psi_\varepsilon}) \).

3. HOMOGENIZATION OF THE AUXILIARY PROBLEM (28)–(31)

Consider the following cell problem in the rescaled periodicity cell \( \Pi \setminus \overline{D} \):

\[
(42) \quad \Delta v_k = 0 \quad \text{in} \quad \Pi \setminus \overline{D},
\]

\[
(43) \quad v_k = \text{const} = c_k \quad \text{in} \quad \Pi \setminus \overline{D},
\]

\[
(44) \quad \int_{\partial D} \frac{\partial v_k}{\partial \nu} = 0
\]

\( v_k - x_\varepsilon \) and \( D(v_k - x_\varepsilon) \) are equal on the opposite sides of the rectangle \( \Pi \). The constants \( c_k (k = 1, 2) \) are unknown and determined in the course of solving the problem. In fact this problem is equivalent to a problem with \( c_k = 0 \) and no integral condition condition present. There exists a solution of this linear problem and it is unique up to an additive constant. In order to ensure the uniqueness we impose the normalization condition \( v_k = 0 \) on \( \partial D \). Set

\[
(44) \quad b_{k\ell} = \frac{1}{|\Pi|} \int_{\Pi \setminus D} \left( \nabla v_k, \nabla v_\ell \right).
\]

The collection of numbers \( \{ b_{k\ell}, k, \ell = 1, 2 \} \) defines a symmetric positive definite tensor in \( \mathbb{R}^2 \), which is called the polarization tensor. There is a
simple algebraic relation between the entries of the polarization tensor and the entries of the virtual mass tensor \( A \) (see Sect. 2)

\[
(45) \quad b_{k\ell} = \frac{a_{k\ell}}{\det A} \quad k, \ell = 1, 2.
\]

This relation can be obtained by using Keller-Dykne duality [15], [11] and it was also obtained directly in [5]. We shall use the polarization tensor \( B = \{ b_{k\ell}, k, \ell = 1, 2 \} \) to describe the homogenized problem which describes the limiting behavior of the solution \( \{ \phi_i, c_i, i = 1, \ldots, N_c \} \) of the problem (28)–(31) as \( \epsilon \to 0 \). We extend functions \( \phi_i(x) \) into the holes \( D_{i\epsilon} \) by the equalities \( \phi_i(x) = C_{i\epsilon}, x \in D_{i\epsilon} \) and keep the same notation for \( \phi_i(x) \in H^1(\Omega) \).

**Theorem 3.1.** If the conditions (A1)–(A3) (see Sect. 1) hold then functions \( \phi_i(x) \) as \( \epsilon \to 0 \) converge in \( L^2(\Pi) \) (and weakly in \( H^1(\Omega) \)) to the solution \( \phi(x) \) of the following homogenized problem:

\[
\sum_{k,\ell=1}^{2} b_{k\ell} \frac{\partial^2 \phi}{\partial x_k \partial x_\ell} = \rho(x) \text{ in } \Omega,
\]

\[
(46) \quad \phi = 0 \text{ on } \partial \Omega,
\]

where \( \rho(x) \) is defined in (23).

The following lemma which will be used in the proof of Theorem 3.1. The proof of this lemma is presented in the Appendix.

**Lemma 3.1.** Let \( \Pi_{i\epsilon} \) be a rectangle defined in the condition (A2), Sect. 1. Then for any \( \phi_i \in H^1(\Pi_{i\epsilon}) \) and \( \alpha(0 < \alpha \leq 2) \) the following inequality holds:

\[
\int_{D_{i\epsilon}} |\phi_i|^2 \, dx \leq C_1 \epsilon^{2-\alpha} \int_{\Pi_{i\epsilon}} |\phi_i|^2 \, dx + C_2 \epsilon^{1+\frac{\alpha}{2}} \int_{\Pi_{i\epsilon}} |\nabla \phi_i|^2 \, dx, \quad (L1)
\]

where \( C_1 \) and \( C_2 \) do not depend on \( \epsilon \) and \( i \).

**Proof of Theorem 3.1.**

It was observed in Section 2 that \( \phi_i(x) \) minimizes the functional (32) in the class (33). Since \( O \in V_\epsilon(\Omega) \) (see (33)) \( F(\phi_i) \leq F(0) = 0 \) and therefore

\[
\int_{\Omega} |\nabla \phi_i|^2 \leq 2\pi \epsilon^\alpha \left( \sum_{i=1}^{N_c} \frac{d_{i\epsilon}}{|D_{i\epsilon}|} \int_{D_{i\epsilon}} \phi_i \right).
\]

Using the condition (A1) from Section 1 and the Cauchy-Schwartz inequality \( |\int \phi| \leq (\int |\phi|^2)^{1/2} |\phi|^1/2 \) we obtain

\[
(47) \quad \int_{\Omega} |\nabla \phi_i|^2 \leq C \epsilon^{\alpha-1} \sum_i \left( \int_{D_{i\epsilon}} \phi_i^2 \right)^{1/2}.
\]
where \( c \) does not depend on \( \epsilon \) and \( \sum' \) stands for the summation over indices \( i \) for which \( d_{ie} \neq 0 \). According to the condition (A2) from Section 2 each hole \( D_{ic} \) with nonzero degree \( d_{ie} \) (i.e., each vortex is located inside a rectangle \( \Pi_{ie}^{\prime} \)). The sides of the rectangle \( \Pi_{ie}^{\prime} \) satisfy (21) and each \( \Pi_{ie}^{\prime} \) contains only one vortex.

Using Lemma 3.1, inequality (47), the Cauchy inequality and taking into account the fact that the rectangles \( \Pi_{ie}^{\prime} \) do not intersect and their total number is of order \( O(\epsilon^{-\alpha}) \), we get,

\[
\int_{\Omega} |\nabla \phi_e|^2 \leq C_1 \epsilon^{\alpha-1} \sum' \left( C_1 \epsilon^{2-\alpha} \int_{\Pi_{ie}^{\prime}} |\phi_e|^2 \right) + C_2 \epsilon^{1+\frac{\alpha}{2}} \int_{\Pi_{ie}^{\prime}} (\nabla \phi_e)^2 \right)^{1/2} \leq \\
\leq C_1 \epsilon^{\alpha-1} \sum' \left( \int_{\Pi_{ie}^{\prime}} |\phi_e|^2 \right)^{1/2} + C_2 \epsilon^{\alpha-1} \sum' \left( \int_{\Pi_{ie}^{\prime}} |\nabla \phi_e|^2 \right)^{1/2} \leq \\
\leq C_1 \epsilon^{\alpha-1} \left( \sum' \int_{\Pi_{ie}^{\prime}} |\phi_e|^2 \right)^{1/2} + C_2 \epsilon^{\alpha-1} \left( \sum' \int_{\Pi_{ie}^{\prime}} |\nabla \phi_e|^2 \right)^{1/2} \leq \\
\leq C_3 \left( \int_{\Omega} |\phi_e|^2 \right)^{1/2} + C_4 \epsilon^{\alpha-2} \left( \int_{\Omega} |\nabla \phi_e|^2 \right)^{1/2}
\]

(48)

where constants \( C_1, C_2, C_3 \) and \( C_4 \) do not depend on \( \epsilon \).

Since \( \phi_e(x) \in H_0^1(\Omega) \) the Friedrichs inequality holds

(49)

\[
\int_{\Omega} |\phi_e|^2 dx \leq C \int_{\Omega} |\nabla \phi_e|^2 dx,
\]

where \( C \) depends on the domain \( \Omega \) only.

It follows from (48) and (49) that for \( \alpha \geq \frac{2}{3} \)

\[
\|\phi_e\|_{H_0^1(\Omega)} < C,
\]

where \( C \) does not depend on \( \epsilon \).
Thus, the family \( \{ \phi_\epsilon(x), \; \epsilon \to 0 \} \) is weakly compact in \( H_0^1(\Omega) \) and therefore we can extract a subsequence \( \{ \phi_{\epsilon_\nu}, \epsilon_\nu \to 0, \; \nu = 1, 2, \ldots \} \) which converges weakly to some function \( \phi(x) \in H_0^1(\Omega) \). Due to the embedding theorem this subsequence converges to \( \phi(x) \) strongly in \( L_2(\Omega) \). Note that this is only true if \( \alpha \geq \frac{3}{4} \). If \( \alpha < \frac{3}{4} \), then the energy \( \int_\Omega |\nabla \phi_\epsilon|^2 \) becomes unbounded as \( \epsilon \to 0 \).

Physically it means that the current density \( \int_\epsilon^2 J_\epsilon^2 = \int |\nabla u_\epsilon|^2 = e^{-2\alpha} \int |\nabla \phi_\epsilon|^2 \) near holes with non zero degrees (vortices) becomes quite large. Since the density of such vortices and their intensity \( (e^\alpha d_\epsilon) \) are related due to the condition (A3), the total energy is no longer bounded (see also Remark 3, Sect.4).

We have to prove that \( \phi(x) \) is the solution of the problem ((46)). For simplicity of the presentation we divide the proof into steps.

**Step 1.** Introduce the following class of (trial) functions
\[
F_\epsilon(\Omega) = \left\{ \zeta_\epsilon(x) \in H^1(\Omega) \cap C^2(\Omega), \right. \\
\zeta_\epsilon(x) = \text{const} = \zeta_{\text{ic}} \text{ on } D_{\epsilon}, \right. \\
\left. \int_{\partial D_{\epsilon}} \frac{\partial \zeta_\epsilon}{\partial \nu} = 0, \quad i = 1, \ldots, N_\epsilon, \right. \\
\zeta_\epsilon(x) = 0 \text{ on } \partial \Omega \}
\]

Here \( \zeta_{\text{ic}} \) are arbitrary constants and derivatives functions \( \zeta_\epsilon(x) \) may have jumps across boundaries of disks \( D_{\epsilon} \).

Since \( \phi_\epsilon(x) \) minimizes the functional (32) in the class (33), for any \( \zeta_\epsilon(x) \in F_\epsilon(\Omega) \) the following equality holds
\[
\int_\Omega (\nabla \phi_\epsilon, \nabla \zeta_\epsilon) + 2\pi \; e^{\alpha} \sum_{i=1}^{N_\epsilon} \frac{d_{i\epsilon}}{|D_{\epsilon}|} \int_{D_{\epsilon}} \zeta_\epsilon = 0,
\]
(first variation is equal zero).

Integrating by parts and taking into account (29), (30) and \( \zeta_\epsilon(x) = \zeta_{\text{ic}} = \text{const} \), we get
\[
\int_{D_{\epsilon}} \phi_\epsilon \Delta \zeta_\epsilon = 2\pi \; e^{\alpha} \sum_{i=1}^{N_\epsilon} d_{i\epsilon} \zeta_{i\epsilon}.
\]

(50)

So we have shown that \( \phi_\epsilon \) satisfies the integral identity (50) for any \( \zeta_\epsilon \in F_\epsilon(\Omega) \). Our next goal is to pass to the limit \( \epsilon_\nu \to 0 \) in (50) and to obtain the integral identity
\[
\int_{\Omega} \phi \sum b_{k\ell} \frac{\partial^2 \zeta}{\partial x_k \partial x_\ell} = \int_{\Omega} \rho \zeta,
\]
(51)

for any \( \zeta \in C_0^\infty(\Omega) \). Since the limiting function \( \phi \in H_0^1(\Omega) \) and \( \rho \in L^2(\Omega) \) (51) implies that \( \phi \in W_2^2(\Omega) \) and satisfies the equation (46) (see [6]).
Step 2. We now start from an arbitrary function \( \zeta \in C_0^\infty(\Omega) \) and construct a special sequence of test functions \( \zeta_\epsilon(x) \) so that taking limit \( \epsilon \to 0 \) in (50) would give us (51). The functions \( \zeta_\epsilon(x) \) are constructed with the help of the solution \( v_k(x) \) of the problem (42) so that the function
\[
f_\epsilon = \begin{cases} 
\Delta \zeta_\epsilon & x \in \Omega_\epsilon \\
0 & x \in \cup_i D_{i\epsilon}
\end{cases}
\]
converges weakly in \( L^2(\Omega) \) to the function
\[
f(x) = \sum_{i=\epsilon} b_{i\epsilon} \frac{\partial \zeta}{\partial x_i},
\]
where \( b_{i\epsilon} \) are components of the polarization tensor. Since as \( \epsilon \phi_k \to \phi \) strongly in \( L^2(\Omega) \), this would enable us to get (51) by taking the limit \( \epsilon \to 0 \) in (50).

We construct functions \( \zeta_\epsilon(x) \) in the following way. Let us cover the space \( \mathbb{R}^2 \) by rectangles \( \Pi_{i\epsilon} \) centered at the points \( x^i = \epsilon(h_1m + i) \epsilon_1 + h_2n \epsilon_2, \) \( (m, n \in \mathbb{Z}) \) with sides \( ch_1 + \epsilon \), \( ch_2 + \epsilon \) where \( \gamma > 1 \) will be chosen later. We also introduce rectangles \( \Pi_{-i\epsilon} \) centered at the same points \( x^i \) with sides \( ch_1, ch_2 \) and \( ch_1 - \epsilon, ch_2 - \epsilon \) respectively. Thus \( \Pi_{\epsilon} \supset \Pi_{i\epsilon} \supset \Pi_{-i\epsilon} \). Let \( \{ \varphi_{\epsilon i}(x), x \in \mathbb{R}^2 \} \) be a partition of unity which corresponds to the covering \( \cup_{i\epsilon} \), i.e. \( \{ \varphi_{\epsilon i}(x) \} \) satisfies the following conditions \( \varphi_{\epsilon i}(x) = C^2, 0 \leq \varphi_{\epsilon i} \leq 1, \varphi_{\epsilon i} = 0 \) outside \( \Pi_{i\epsilon} \) and \( \varphi_{\epsilon i} = 1 \) inside \( \Pi_{-i\epsilon} \), \( \sum_{i\epsilon} \varphi_{\epsilon i}(x) = 1 \) and \( D^m \varphi_{\epsilon i}(x) = 0(\epsilon^{-m}) \).

Let \( v_k(x) \) be the solution of the cell problem (42). Consider the function \( \epsilon v_k(x) = x_k \) in the rectangle \( \Pi_{i\epsilon} \) which is centered at the origin. From (42) we see that this function can be extended on \( \mathbb{R}^2 \) by the following formula
\[
v_{\epsilon k}(x) = \epsilon v_k \left( \frac{x}{\epsilon} - x_{\epsilon k} \right) - (x_k - x_{\epsilon k}) , \quad x \in \Pi_{i\epsilon}.
\]
As the result we obtain function \( v_{\epsilon k}(x) \) which satisfies the equation \( \Delta v_{\epsilon k} = 0 \) everywhere outside the periodic in \( \mathbb{R}^2 \) set \( \cup_{i\epsilon} D_{i\epsilon}, D_{i\epsilon} = \epsilon D + x_k \epsilon \). Introduce
\[
v_{\epsilon k}(x) = v_{\epsilon k}(x) + (x_k - x_{\epsilon k}) \text{ for } x \in \Pi_{i\epsilon}.
\]
Then the following equalities hold
\[
\Delta v_{\epsilon k}(x) = 0 \quad \text{in } \Pi_{i\epsilon} \setminus D_{i\epsilon}
\]
\[
v_{\epsilon k} = 0 \quad \text{in } D_{i\epsilon}
\]
\[
\int_{\Omega_{\epsilon}} \frac{\partial v_{\epsilon k}}{\partial v} = 0
\]
Let $\zeta(x)$ be an arbitrary function from $C^\infty_0(\Omega)$. We define the function $\zeta_\epsilon(x) \in F_\epsilon(\Omega)$ in the following way

$$
\zeta(x) = \sum_i \zeta(x_i^i) + \sum_i \frac{\partial \zeta}{\partial x_i}(x_i^i) + \\
\quad \quad \quad \quad \quad \quad \quad \quad \quad + \frac{1}{2} \sum_{k \neq \ell} v_{\epsilon_k}^i(x) v_{\epsilon_\ell}^j(x) \frac{\partial^2 \zeta}{\partial x_k \partial x_\ell}(x_i^i) \varphi_\epsilon(x) \tag{55}
$$

Note that (55) is obtained from first three terms of the Taylor series for $\zeta_\epsilon(x)$ if $v_{\epsilon_k}^i(x)$ is replaced by $x_k - x_k^i$. It follows from the first and second equalities in (54) that $\zeta_\epsilon(x) \in F_\epsilon(\Omega)$ and $\zeta_\epsilon = \zeta(x_\epsilon^i)$.

Introduce the function $f_\epsilon(x)$ which is defined in $\Omega$ by the following equalities

$$
f_\epsilon(x) = \begin{cases} 
\Delta \zeta_\epsilon(x), & x \in \Omega_\epsilon \\
0, & x \in \bigcup_i \overline{D_\epsilon^i}
\end{cases}
$$

**Lemma 3.2.** As $\epsilon \to 0$ function $f_\epsilon(x)$ converges weakly in $L_2(\Omega)$ to the function

$$
f(x) = \sum_{k \neq \ell} b_{k\ell} \frac{\partial^2 \zeta}{\partial x_k \partial x_\ell},
$$

where $b_{k\ell}$ are the entries of the polarization tensor (44).

**Proof of Lemma 3.2.**

We first show that $L_2(\Omega)$ norms of the functions $f_\epsilon(x)$ are bounded uniformly in $\epsilon$. Taking into account (50) and the properties of the partition
of unity, we represent \( \zeta_e \) in the following form

\[
\zeta_e(x) = \sum_i \left\{ \zeta_i + \sum_k \zeta_{i,k}^l(x_k - x_{e,k}^l) + \frac{1}{2} \sum_{k,l} \zeta_{i,k}^l(x_k - x_{e,k}^l)(x_l - x_{e,l}^i) - \zeta \right\} \phi_{ei} + \sum_i \left\{ \sum_k \left[ \zeta_k + \sum_l \zeta_{k,l}^i(x_l - x_{e,l}^i) - \zeta_k \right] \nu_{ek} \right\} \phi_{ei} \right.

\[
+ \frac{1}{2} \sum_i \left\{ \sum_{k,l} \left( \zeta_{k,l}^i - \zeta_{k,l}^e \right) \nu_{ek} \nu_{el} \right\} \phi_{ei} + \zeta + \sum_k \zeta_k \nu_{ek} + \frac{1}{2} \sum_{k,l} \zeta_{k,l}^e \nu_{ek} \nu_{el}
\]

(56)

Here we have used the following notations:

\[
\zeta = \zeta(x), \quad \zeta_k = \frac{\partial \zeta}{\partial x_k}(x), \quad \zeta_{k,l}^i = \frac{\partial \zeta}{\partial x_k \partial x_l}(x)
\]

and the superscript \( i \) means that the functions \( \zeta, \zeta_k, \zeta_{k,l}^e \) are evaluated at the points \( x_{e,i}^l \).

Note that due to the Taylor formula the derivative of order \( m \) of the expression in the first curly bracket in (56) is of order \( O(\epsilon^{3-m}) \) \( (m = 0, 1, 2) \). Due to (52) we also have \( D^m \nu_{ek} = O(\epsilon^{3-m}) \) \( (m = 0, 1, 2) \) and therefore the second and third curly brackets in (56) are also \( O(\epsilon^{3-m}) \). Due to our choice of the partition of unity \( D^m \phi_{ei} = O(\epsilon^{m\gamma}) \), the total number of terms in the sums in (56) is \( O(\epsilon^{-2}) \), the area inside each square where \( D\phi \neq 0 \) is of order \( O(\epsilon^{\gamma}) \). Taking this into account we get \( \int_{\Omega_e} |\Delta \zeta| = O(\epsilon^{5-3\gamma}) \) and choosing \( \gamma \) so that

(57)

\[
1 < \gamma < \frac{5}{3}
\]

we get:

(58)

\[
\int_{\Omega_e} \frac{f^2}{\Delta \zeta} \leq \int_{\Omega_e} |\Delta \zeta|^2 < C,
\]

where \( C \) does not depend on \( \epsilon \) and only three last terms in (56) provide a contribution which does not vanish as \( \epsilon \to 0 \). For any function \( \varphi \in C^4(\Omega) \)
we write
\begin{equation}
\int f_{\epsilon} \varphi = \int_{\Omega} \Delta \zeta_{\epsilon} \varphi = \sum_{i} \int_{\Gamma_{i \epsilon} \setminus D_{i \epsilon}} \Delta \zeta_{\epsilon} \varphi + \sum_{i} \int_{\Pi_{i \epsilon} \setminus \Pi_{i \epsilon}^{*}} \Delta \zeta_{\epsilon} \varphi
\end{equation}

Due to (55) and harmonicity of $v_{\epsilon k}$
\[
\Delta \zeta_{\epsilon}(x) = \sum_{k, \ell} \frac{\partial^{2} \zeta}{\partial x_k \partial x_\ell}(x^i_k \nabla v_{\epsilon k}^i(x), \nabla v_{\epsilon \ell}^i(x)), \quad x \in \Pi_{i \epsilon} \setminus D_{i \epsilon}
\]
and therefore
\begin{equation}
\int_{\Pi_{i \epsilon} \setminus D_{i \epsilon}} \Delta \zeta_{\epsilon} \varphi = \sum_{k, \ell} \frac{\partial^{2} \zeta}{\partial x_k \partial x_\ell}(x^i_k) \int_{\Pi_{i \epsilon} \setminus D_{i \epsilon}} (\nabla v_{\epsilon k}^i \nabla v_{\epsilon \ell}^i)(1 + O(\epsilon))
\end{equation}

Due to (52) and (53) we have $v_{\epsilon k}^i(x) = \varphi(x_k^i - x_k^i)$. Therefore using the definition of $\Pi_{i \epsilon}^{*}$ we obtain
\[
\int_{\Pi_{i \epsilon} \setminus D_{i \epsilon}} (\nabla v_{\epsilon k}^i \nabla v_{\epsilon \ell}^i) = \epsilon^2 \int_{\Pi_{D}} (\nabla v_k \nabla v_\ell)(1 + O(\epsilon^{-1})) =
\]
\begin{equation}
= \epsilon^2 |\Pi| b_{\epsilon k}(1 + O(\epsilon^{-1}))
\end{equation}

From (60), (61) we conclude the first sum in the RHS of (59) converges to the integral
\[
\int_{\Omega} \sum_{k, \ell} b_{\epsilon k} \frac{\partial^{2} \zeta}{\partial x_k \partial x_\ell} \varphi
\]
as $\epsilon \to 0$. The second sum in the RHS of (59) tends to zero due to (58) and the following estimate
\[
\text{meas} \left[ \cup_{i}(\Pi_{i \epsilon} \setminus \Pi_{i \epsilon}^{*}) \right] = 0(\epsilon^{-2} c^i \epsilon) = O(\epsilon^{-1})
\]

Indeed $\int_{\Omega} f_{\epsilon} \chi_{G_{\epsilon}} \leq \left( \int_{\Omega} f_{\epsilon}^2 \right)^{1/2} \left( \int_{G_{\epsilon}} \chi_{G_{\epsilon}}^2 \right)^{1/2}$, where $\chi_{G_{\epsilon}}$ is the indicator function of the set $G_{\epsilon} = \cup(\Pi_{i \epsilon} \setminus \Pi_{i \epsilon}^{*})$.

Thus for any $\varphi \in C^1(\Omega)$
\begin{equation}
\lim_{\epsilon \to 0} \int_{\Omega} f_{\epsilon} \varphi = \int_{\Omega} \sum_{k, \ell} b_{\epsilon k} \frac{\partial^{2} \zeta}{\partial x_k \partial x_\ell} \varphi
\end{equation}

Using (58) and density of $C^1(\Omega)$ we obtain the claim of Lemma 3.2.

Now we are able to finish the proof of Theorem 3.1. Rewrite (50) in the form
\begin{equation}
\int_{\Omega} \phi_{\epsilon} f_{\epsilon} = R_{\epsilon}(\zeta)
\end{equation}
where $R_\epsilon(z)$ is a distribution (a generalized function) (22) acting on the function $z \in C^\infty$. Using Lemma 3.2, the condition (23) and strong $L_2$ convergence $\phi_\epsilon$ to $\phi$, we pass to the limit in (63) and obtain (60). Thus $\phi$ is the solution of (46) (in the sense of distributions).

Since this problem has a unique solution the sequence $\phi_\epsilon(x)$ also converges to $\phi(x)$ and Theorem 3.1 is proven.

**Corollary 3.1.** The derivatives $\frac{\partial \phi_\epsilon}{\partial x_k}$ converge weakly in $L_2(\Omega)$ to $\frac{\partial \phi}{\partial x_k}$ $(k = 1, 2)$.

4. **End of the Proof of Theorem 1.1**

The solution of the problem (1)-(5) can be represented as follows:

$$u_\epsilon(x) = e^{i\epsilon \alpha} \psi_\epsilon,$$

where the function $\psi_\epsilon$ is defined in the cut domain (see Fig. 2) by the system (34), $\phi_\epsilon(x)$ is the solution of the problem (28)-(31) in $\Omega_\epsilon$ (see Section 2). We extend $\phi_\epsilon(x)$ into the holes $D_\epsilon$ by constant values $C_\epsilon$. Then according to (27), (64) and (34)

$$v_\epsilon = e^{\alpha} v_\epsilon(x) \quad \text{in } \Omega,$$

where

$$P = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

is a constant matrix.

Due to Theorem 3.1 (Corollary 3.1) this implies that $v_\epsilon(x)$ converges as $\epsilon \to 0$ weakly in $L_2(\Omega)$ to the vector-function $v(x)$, which is determined by the following equality

$$v(x) = P \nabla \phi(x),$$

where $\phi(x)$ is the solution of the problem (46). Combining (65) and (46) we conclude that $v \in H^1(\Omega)$ and is a solution of the following boundary value problem

$$\text{div } v = 0, \quad \text{in } \Omega$$

$$- \sum_{k, \ell=1}^{c} \text{div } \{BPv\} - \rho(x), \quad \text{in } \Omega$$

$$v_\nu = 0 \quad \text{on } \partial \Omega,$$

where $B = \{b_{k, \ell}, \ k, \ell = 1, 2, \}$ is the polarization tensor (see Section 3). Using the duality relation (45) we can easily check that this problem is equivalent to the problem (13)-(15) for $v = Au$. Existence and uniqueness of the solution of these problems in $H^1(\Omega)$ follows from the solvability of the problem (46) in $H^2(\Omega)$ (since $\rho(x) \in L_2(\Omega)$). Thus Theorem 1 is proved.
Remark 2. Introduce the following metric in the domain $\Omega$

$$ds^2 = \sum_{k \ell=1}^{2} a^{k \ell} dx_k dx_\ell,$$

where $a^{k \ell}$ are elements of the matrix $A^{-1}$ and consider $\Omega$ as a Riemannian manifold with boundary. Then we can directly check that the problem (13)–(15) can be written as follows

$$du = \rho, \quad d^* u = 0, \quad u_{\partial \Omega} = 0,$$

where $d$ is the operator of external differentiation (generalized curl), $d^*$ is the dual (with respect to the metric (67)) operator.

Remark 3. The restriction $\alpha \geq \frac{2}{3}$ appeared due to the use of Sobolev spaces $H^1$ since the functionals (17) and (32) are quadratic. As we have explained in Section 3 if $\alpha < \frac{2}{3}$ the $H^1(\Omega)$ norm of the function $\phi_\epsilon(x)$ grows as $\epsilon \to 0$. Therefore to obtain similar results for $\alpha < \frac{2}{3}$ one can try to use the techniques of Sobolev spaces $W^1_r(\Omega)$ with $r = r(\alpha) < 2$.

Problem (example). We present here a probabilistic version of the same problem. Let the degrees $d_\epsilon$ be independent, identically distributed random variables which take values 1, −1 and 0 with probabilities $p(\epsilon) = p\epsilon^\beta$, $q(\epsilon) = q\epsilon^\beta$ and $1 - p(\epsilon) - q(\epsilon)$ respectively, where $\beta \geq 0$. Then the vector field $v_{\partial \Omega}(x)$ (see (27)) is a random vector field. The problem then is to prove that when $\alpha = 2 - \beta$ this field converges in distribution (weakly in $L_2(\Omega)$) to a nonrandom (homogenized) field $v(x) = Au(x)$, where $u(x)$ is the solution of the boundary value problem (13)–(15) with $\beta(x) = p - q = \text{const}$. Using Theorem 1.1 one can prove this fact for $\beta = 0$.

5. On Convergence of Harmonic Maps $u_\epsilon(x)$

We consider the solution $u_\epsilon(x)$ of the problem (1)–(5). We have shown above that up to a constant multiplier $\theta$ this solution can be represented by formulas (40), (35)–(38), (39) and this multiplier depends on the choice of the system of cuts which produce the simply connected cut domain $\Omega_\epsilon^c \subset \Omega_\epsilon$. Since $|u_\epsilon| = 1$ in $\Omega_\epsilon$ the family of functions $\{u_\epsilon\}$ extended by the value zero into the holes is simply connected in $L_2(\Omega)$. However the description of weak limits in $L_2(\Omega)$ for such function is quite difficult task. It follows from (40), (41) that for any prolongation of the functions $u_\epsilon(x)$ into the holes the obtained family $\{u_\epsilon(x)\}$ is neither weakly compact in $H^1(\Omega)$ ($e^{-2\alpha}$ grows as $\epsilon \to 0$, see (41)) nor it is strongly compact in
\( L_2(\Omega) \) (the phase in (40) oscillates rapidly), i.e. there are no convergent subsequences.

In this section we discuss the convergence of the wave functions \( u_\varepsilon(x) \). In fact it turns out that \( u_\varepsilon(x) \) do not converge but the branches of “root” of order \( e^{-\alpha} \) do converge and the limiting problem depends on the choice of system of cuts. This is an interesting mathematical phenomenon and at this point it is not clear to us whether this limit provides any significant physical insight. That is why we consider in the domain \( \Omega^c_\varepsilon \) the following functions (“complex roots”)

\[
(68) \quad w_{\varepsilon \alpha}(x) = (u_\varepsilon(x))^e \equiv e^{i\psi_\varepsilon(x)},
\]

where \( \psi_\varepsilon(x) \) is the solution of the problem (35)-(38), (39) and therefore depends on the choice of the system of cuts in the domain \( \Omega^c_\varepsilon \). Functions \( (u_\varepsilon(x))^e \) takes many values at each point \( x \in \Omega^c_\varepsilon \) (order \( e^{-\alpha} \), since this is “\( e^{-\alpha} \) root”). It is then natural to choose one continuous branch by fixing a system of cuts which makes the cut domain simply connected. One of such branches is determined by the second equality in (68).

It is possible to show that these functions can be extended into the holes \( D_\varepsilon \) so that the obtained family of extended functions \( \{\tilde{w}_{\varepsilon \alpha}(x) \in L_2(\Omega), \varepsilon > 0\} \) is compact in \( L_2(\Omega) \). It turns out that for a given choice of the system of cuts there exists the limit

\[
w = \lim_{\varepsilon \to 0} \tilde{w}_{\varepsilon \alpha},
\]

which describes the asymptotics of the chosen branch of the function \( (u_\varepsilon(x))^e \). Consider one of such limit. For the sake of simplicity we assume that \( \Omega \) is a disk of radius \( R \) centered at the origin. Then the system of cuts can be chosen as it shown on Fig. 3, i.e. so that all horizontal cuts lie on one horizontal line \( x_2 = 0 \).

Denote the upper and lower half of the disk by \( \Omega_+ \) and \( \Omega_- \) respectively by \( \Gamma_+ \) and \( \Gamma_- \) corresponding semicircles so that \( \Gamma_+(x_1) \) and \( \Gamma_-(x_1) \) are their graphs. Then we can describe the homogenized limit as follows.

Assume that the conditions (20), (21), (23) hold and \( \rho(x) \in C^1(\Omega) \) (see the definition of \( P(x) \) below). Then the functions \( w_{\varepsilon \alpha}(x) \) as \( \varepsilon \to 0 \) converge in the norm \( L_2(\Omega_\varepsilon) \) to the function \( w(x) \) which solves the
following conjugation problem.

\[
\sum_{k\ell} a_{k\ell} \frac{\partial^2 w}{\partial x_k \partial x_\ell} + w \sum_{k\ell} \frac{\partial w}{\partial x_k} \frac{\partial w}{\partial x_\ell} +
\]

\[+ iP(x)w = 0 \quad \text{in } \Omega_+ \text{ and } \Omega_- ,
\]

\[|w| = 1 \text{ in } \Omega_+ \text{ and } \Omega_- ,
\]

\[
\sum_{k\ell} a_{k\ell} \frac{\partial w}{\partial x_k} \cos(x_{k\ell}) = 0 \text{ on } \Gamma_+ \text{ and } \Gamma_-
\]

\[w^* \sum_{\ell} a_{2\ell} \frac{\partial w}{\partial x_\ell} = 0, \quad x_2 = 0,
\]

\[w_+ = w_- e^{iQ(x)}, \quad x_2 = 0 \text{ (conjugation)}
\]

\[\deg(w^*, \partial \Omega) = d ,
\]
where
\[
d = \lim_{\varepsilon \to 0} \varepsilon^2 a_\varepsilon = \int_\Omega \rho dx,
\]
\[
P(x) = P(x_1, x_2) = \left\{ \frac{\partial}{\partial x_1} \int_{\Gamma_-(x_1)}^{x_2} \rho(x, \eta) d\eta \right\} x_2 < 0,
\]
\[
P(x) = P(x_1, x_2) = \left\{ \frac{\partial}{\partial x_1} \int_{\Gamma_+(x_1)}^{x_2} \rho(x, \eta) d\eta \right\} x_2 > 0,
\]
\[
Q(x) = \int_{-R}^{R} \int_{\Gamma_-(\zeta)}^{\Gamma_+(\zeta)} \rho(\zeta, \eta) d\eta d\zeta,
\]
\{a_\varepsilon\} is the virtual mass tensor, \(\nu\) is the outward normal. Thus we see that the limit actually depends on the choice of the system of cuts (for our choice the conjugation occurs across the line \(x_2 = 0\) but it could be different straight or polygonal line for another system of cuts). The proof of this fact is quite cumbersome and will be presented elsewhere.

**APPENDIX A. PROOF OF LEMMA 3.1**

*Proof.* Without loss of generality we assume that \(\phi_\varepsilon\) is smooth in \(W_0^s\). Let \(x, y\) be two arbitrary points in \(W_0^s\) and \(x \in D_\varepsilon\). Then
\[
\phi_\varepsilon(y) = \phi_\varepsilon(x) + \int_0^{x-y} \frac{d\phi_\varepsilon(\zeta(r))}{dr} dr, \tag{I2}
\]
where
\[
\zeta = x + \frac{y-x}{|y-x|} r, \quad r \in \mathbb{R}. \tag{I3}
\]
Using the elementary inequality \(a^2 \leq 2b^2 + 2c^2\) with \(b = a + c\) we get
\[
|\phi_\varepsilon(x)|^2 \leq 2|\phi_\varepsilon(y)|^2 + 2 \left( \int_0^{x-y} \frac{d\phi_\varepsilon(\zeta)}{dr} dr \right)^2 \tag{IA}
\]
Estimate the second integral in the right hand side of this inequality. Using the Cauchy-Schwartz inequality (\(|\zeta - x| = r\)) we get
\[
\left( \int_0^{x-y} \frac{d\phi_\varepsilon(\zeta)}{dr} dr \right)^2 \leq \left( \int_0^{x-y} \frac{d\phi_\varepsilon(\zeta)}{dr} \frac{r^{3/2}}{\sqrt{r^{3/2}}} \frac{dr}{r^{3/2}} \right)^2 \leq \int_0^{x-y} \left( \frac{d\phi_\varepsilon}{dr} (\zeta) \right)^2 r^{3/2} dr \int_0^{x-y} \frac{dr}{|\zeta - x|^3} \tag{I5}
\]
Let \((\rho, \varphi)\) be polar coordinates of the point \(y\) with respect to the center \(x \in D_{ie}\). Substitute (L5) into (L4) and integrate in \(y \in \Pi'_{ie}:

\[
|\Pi'_{ie}| |\phi_e(x)|^2 \leq 2 \int_{\Pi'_{ie}} |\phi_e|^2 + 2 \int_0^{2\pi} d\varphi \int_0^{R_e(x, \varphi)} |\frac{\partial \phi_e}{\partial r}|^2 r^3 dr \cdot \int_0^{R_e} \left( \int_0^\rho \frac{dr}{|x - \zeta(r)|^\beta} \right) \rho d\rho
\]

\[
\leq 2 \int_{\Pi'_{ie}} |\phi_e|^2 + R_e^2 \int_0^{R_e} \frac{dr}{|x - \zeta(r)|^\beta} \max_{x} \int_0^{2\pi} d\varphi \int_0^{R_e(x, \varphi)} \left| \frac{\partial \phi_e}{\partial r} \right|^2 r^3 dr \quad (L6)
\]

where \(R_e(x, \varphi) = |y - x|\) for \(y \in \partial \Pi'_{ie}\), i.e., the distance from \(x\) to the boundary \(\partial \Pi'_{ie}\) along the ray \((L3)\) (see Fig 5), \(R_e = \max_{x,y} R_e(\varphi, x)\) is the diameter of \(\Pi'_{ie}\), \(\zeta(r)\) is any point of the form \((L3)\).

Our next step is to integrate (L6) in \(x\) over \(D_{ie}\) for fixed \(\zeta(r) \in \Pi'_{ie}\). In order to do this we first write

\[
\int_{D_{ie}} dx \int_0^{R_e} \frac{dr}{|x - \zeta(r)|^\beta} = \int_0^{R_e} dr \int_{D_{ie}} \frac{dx}{|x - \zeta(r)|^\beta} \quad (L7)
\]

so that 1D integral is replaced by a 2D integral. Furthermore using polar coordinates we see that

\[
\int_{D_{ie}} \frac{dx}{|x - \zeta|^\beta} \leq \frac{c}{2 - \beta} d_{ie}^{\frac{2\beta}{2 - \beta}}, \quad (L8)
\]
where $|D_c| \leq \epsilon^2$ is the area of the disk and $c$ is independent on $\epsilon$ and $\beta$. We now integrate (L6) in $x$. Taking into account (L7) and (L8) we get

$$\left| \Pi_{\epsilon} \right| \int_{D_{\epsilon}} \left| \phi_\epsilon \right|^2 \leq 2 |D_\epsilon| \int_{\Pi_{\epsilon}} \left| \phi_\epsilon \right|^2 + C \frac{R_{\epsilon}}{2 - \beta} |D_\epsilon|^{\frac{2-\beta}{\beta}}.$$  

$$\max_x \int_0^{2\pi} d\phi \int_0^{R_{\epsilon}(x,\phi)} \left| \frac{\partial \phi_\epsilon}{\partial r} \right|^2 r^2 dr$$  

(L9)

Using (21) we see that $R_\epsilon = O(\epsilon^{\frac{2}{\beta}})$ and $\Pi_\epsilon = O(\epsilon^\alpha)$. Since $|D_\epsilon| = O(\epsilon^2)$, (L9) implies (L1) when $\beta \to 1$.

REFERENCES


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