Commutator methods with applications to the spectral analysis of dynamical systems

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1 Commutator methods for self-adjoint operators

Commutator methods are a tool for the spectral theory and the scattering theory of self-adjoint operators in Hilbert spaces.

They have been introduced by Éric Mourre in the 80's for the study of Schrödinger operators in $\mathsf{L}^2(\mathbb{R}^d)$ (and further developed by Amrein, Boutet de Monvel, Georgescu, Gérard, Jensen, Perry, Sahbani, ...)

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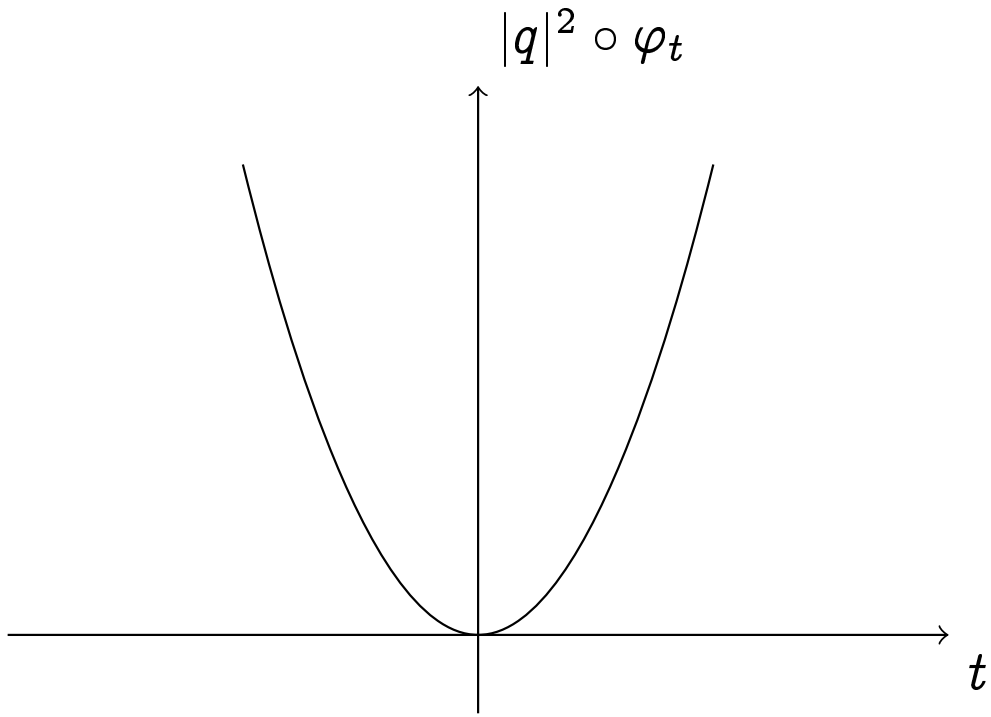
- $H \in C^\infty(M)$, Hamiltonian with complete flow $\{\varphi_t\}_{t\in\mathbb{R}}$
- Hamiltonian evolution equation for an observable $f \in C^\infty(M)$:

$$\frac{\mathrm{d}}{\mathrm{d}t} f \circ \varphi_t = \big\{f, H\big\} \circ \varphi_t, \qquad t \in \mathbb{R}.$$

For instance, if $H(q,p) := |p|^2 + V(q)$ on $M := T^*\mathbb{R}^d$ with $V \in C_c^\infty(\mathbb{R}^d)$, let's say that we don't want orbits bounded in $|q|^2$.

For instance, if $H(q,p) := |p|^2 + V(q)$ on $M := T^*\mathbb{R}^d$ with $V \in C_c^\infty(\mathbb{R}^d)$, let's say that we don't want orbits bounded in $|q|^2$.

We want something like:



Since, $\frac{\mathrm{d}^2}{\mathrm{d}t^2}|q|^2 \circ \varphi_t = \big\{\{|q|^2, H\}, H\big\} \circ \varphi_t$, it is sufficient to check that

$$\big\{\{|q|^2, H\}, H\big\} \geq \delta > 0.$$

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In the example $H(q,p) = |p|^2 + V(q)$, we get

$$\begin{aligned}\big\{\{|q|^2, H\}, H\big\} &= \big\{\big\{|q|^2, |p|^2 + V(q)\big\}, H\big\} \\ &= \big\{4(q \cdot p), |p|^2 + V(q)\big\} \\ &= 8\,|p|^2 - 4\,q \cdot (\nabla V)(q).\end{aligned}$$

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Thus, $|p|^2 > \frac{1}{2}\sup_{q\in\mathbb{R}^n}\big|q \cdot (\nabla V)(q)\big|$ implies $\lim\limits_{|t|\to\infty} |q|^2 \circ \varphi_t = +\infty$.

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Thus, $|p|^2 > \frac{1}{2}\sup_{q\in\mathbb{R}^n}\big|q \cdot (\nabla V)(q)\big|$ implies $\lim\limits_{|t|\to\infty} |q|^2 \circ \varphi_t = +\infty$.

(If the kinetic energy $|p|^2$ is large enough, all the trajectories go to infinity . . .)

To some extent, the idea behind commutators methods for self-adjoint operators is to translate the last example into the language of the (quantum) Hilbertian theory with the following heuristic dictionnary in mind:

Poisson manifold M	$\longleftrightarrow$	Hilbert space $\mathcal{H}$
Poisson bracket $\{\cdot,\cdot\}$	$\longleftrightarrow$	commutator $i[\cdot,\cdot]$
Hamiltonian $H \in C^\infty(M)$	$\longleftrightarrow$	self-adjoint operator H in $\mathcal{H}$
$\frac{\mathrm{d}}{\mathrm{d}t} f \circ \varphi_t = \{f, H\} \circ \varphi_t$	$\longleftrightarrow$	$\frac{\mathrm{d}}{\mathrm{d}t} \mathrm{e}^{-itH} F \mathrm{e}^{itH} = \mathrm{e}^{itH}[iF, H]\mathrm{e}^{-itH}$
bounded orbits of H	$\longleftrightarrow$	eigenvalues of H

1.2 Self-adjoint operators

References:

- W. O. Amrein, Hilbert Space Methods In Quantum Mechanics, EPFL Press, 2009
- M. Reed and B. Simon, Methods of modern mathematical physics. volumes I-IV, Academic Press, 1980
- J. Weidmann, Linear Operators in Hilbert Spaces, Springer Verlag, 1980

An operator H with dense domain $\mathcal{D}(H)$ in a Hilbert space $\mathcal{H}$ is symmetric if

$$\big\langle H\varphi, \psi \big\rangle = \big\langle \varphi, H\psi \big\rangle \quad \text{for all } \varphi, \psi \in \mathcal{D}(H).$$

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A vector $\eta \in \mathcal{H}$ belongs to $\mathcal{D}(H^*)$ if there exists $\eta^* \in \mathcal{H}$ such that

$$\big\langle \eta^*, \varphi \big\rangle = \big\langle \eta, A\varphi \big\rangle \quad \text{for all } \varphi \in \mathcal{D}(H).$$

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In this case, one sets $H^*\eta := \eta^*$ and one calls H^* the adjoint of H.

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A symmetric operator H is self-adjoint if

$$\big\{H, \mathcal{D}(H)\big\} = \big\{H^*, \mathcal{D}(H^*)\big\},$$

which is verified if and only if the ranges $\mathsf{Ran}(H \pm i) = \mathcal{H}$.

If H is self-adjoint, then the set $\mathcal{D}(H)$ equipped with the inner product

$$\langle \varphi, \psi \rangle_{\mathcal{D}(H)} := \langle \varphi, \psi \rangle + \langle H\varphi, H\psi \rangle, \quad \varphi, \psi \in \mathcal{D}(H),$$

and the induced norm

$$\|\varphi\|^2_{\mathcal{D}(H)} := \langle \varphi, \varphi \rangle_{\mathcal{D}(H)}, \quad \varphi \in \mathcal{D}(H),$$

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A subspace $\mathscr{D} \subset \mathcal{D}(H)$ is a core for H if the closure of $\mathscr{D}$ in $\mathcal{D}(H)$ is equal to $\mathcal{D}(H)$; that is,

$$\overline{\mathscr{D}}^{\|\cdot\|_{\mathcal{D}(H)}} = \mathcal{D}(H).$$

Example 1.1. *The multiplication operator Q in $\mathcal{H} := \mathsf{L}^2(\mathbb{R})$ given by*

$$(Q\varphi)(x) := x\,\varphi(x), \quad \varphi \in \mathcal{H}_1(\mathbb{R}) := \left\{ \varphi \in \mathcal{H} \mid \int_{\mathbb{R}} \left(1 + |x|^2\right) |\varphi(x)|^2 < \infty \right\},$$

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with $\mathscr{F}$ the 1-dimensional Fourier transform, is self-adjoint.

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The space $\mathscr{S}(\mathbb{R})$ of Schwartz functions on $\mathbb{R}$ is a core for Q and P, since $\mathscr{S}(\mathbb{R})$ is dense in the (Sobolev) spaces $\mathcal{H}_1(\mathbb{R})$ and $\mathcal{H}^1(\mathbb{R})$.

Let $\mathscr{B}(\mathcal{H})$ be the set of bounded operators on $\mathcal{H}$ and let H be a self-adjoint operator H in $\mathcal{H}$.

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The set

$$\rho(H) := \left\{ z \in \mathbb{C} \mid (H - z)^{-1} \text{ exists and belongs to } \mathscr{B}(\mathcal{H}) \right\}$$

is the resolvent set of H; it is an open subset of $\mathbb{C}$.

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The set $\sigma(H) := \mathbb{C} \setminus \rho(H)$ is the spectrum of H; it is a closed subset of $\mathbb{R}$.

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$$\big\langle \varphi, E(\mu)\varphi \big\rangle \leq \big\langle \varphi, E(\lambda)\varphi \big\rangle \quad \text{for all } \varphi \in \mathcal{H},\ \mu \leq \lambda \quad \text{(monotonicity)},$$

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For intervals, one defines the spectral measure

$$E\big((a,b]\big) := E(b) - E(a), \quad E\big((a,b)\big) := \text{s-}\lim_{\varepsilon \searrow 0} E(b-\varepsilon) - E(a), \quad \text{etc.}$$

and one extends these definitions to $E(\mathcal{V})$ for any Borel set $\mathcal{V} \subset \mathbb{R}$.

Theorem 1.3 (Spectral theorem). *A self-adjoint operator H in a Hilbert space $\mathcal{H}$ admits exactly one spectral family E^H such that*

$$H = \int_{\mathbb{R}} \lambda \, E^H(\mathrm{d}\lambda),$$

with the strong integral $\int_{\mathbb{R}} \lambda \, \mathrm{d}E^H(\mathrm{d}\lambda)$ satisfying

$$\left\langle \varphi, \int_{\mathbb{R}} \lambda \, E^H(\mathrm{d}\lambda) \, \psi \right\rangle := \int_{\mathbb{R}} \lambda \left\langle \varphi, E^H(\mathrm{d}\lambda) \, \psi \right\rangle, \quad \varphi \in \mathcal{H}, \ \psi \in \mathcal{D}(H).$$

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Furthermore, one has for $-\infty < a < b < \infty$ that

$$E^H\big((a,b]\big) = \frac{1}{\pi} \, \text{s-}\lim_{\delta \searrow 0} \, \text{s-}\lim_{\varepsilon \searrow 0} \int_{a+\delta}^{b+\delta} \mathrm{d}\lambda \, \mathrm{Im}(H - \lambda - i\varepsilon)^{-1}.$$

(Stone's Formula)

Two comments:

- The support of the spectral family E^H is the set of points of non-constancy and coincides with the spectrum of H

$$\operatorname{supp}\big(E^H\big) = \big\{\lambda \in \mathbb{R} \mid E^H(\lambda+\varepsilon) - E^H(\lambda-\varepsilon) \neq 0 \ \ \forall \varepsilon > 0\big\} = \sigma(H).$$

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- Formally, one has

$$\begin{aligned}
\|H\psi\|^2 = \langle H\psi, H\psi\rangle &= \int_{\mathbb{R}} \lambda \int_{\mathbb{R}} \mu \,\big\langle E^H(\mathrm{d}\mu)\,\psi, E^H(\mathrm{d}\lambda)\,\psi\big\rangle \\
&= \int_{\mathbb{R}} \lambda \int_{\mathbb{R}} \mu \,\big\langle \psi, E^H(\mathrm{d}\mu \cap \mathrm{d}\lambda)\,\psi\big\rangle \\
&= \int_{\mathbb{R}} \lambda^2 \,\big\langle \psi, E^H(\mathrm{d}\lambda)\,\psi\big\rangle,
\end{aligned}$$

so that $\psi \in \mathcal{D}(H)$ if and only if $\int_{\mathbb{R}} \lambda^2 \,\big\langle \psi, E^H(\mathrm{d}\lambda)\,\psi\big\rangle < \infty$.

Example 1.4. *The spectral projection* $E^Q(\lambda)$ *of the operator* Q *in* $\mathcal{H} := \mathsf{L}^2(\mathbb{R})$ *is the operator of multiplication by the characteristic function* $\chi_{(-\infty,\lambda]}$, i.e.,

$$E^Q(\lambda)\varphi := \chi_{(-\infty,\lambda]}\varphi, \quad \varphi \in \mathcal{H}.$$

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$$E^Q(\lambda)\varphi := \chi_{(-\infty,\lambda]}\varphi, \quad \varphi \in \mathcal{H}.$$

One verifies that

$$\sigma(Q) = \operatorname{supp}\big(E^Q\big) = \mathbb{R}.$$

Example 1.5. *The multiplication operator* $Q^2 := \sum_{j=1}^d Q_j^2$ *in* $\mathcal{H} := \mathsf{L}^2(\mathbb{R}^d)$ *given by*

$$\big(Q^2\varphi\big)(x) := x^2\,\varphi(x), \quad \varphi \in \mathcal{H}_2(\mathbb{R}^d),\ x^2 := \sum_{j=1}^d x_j^2,$$

is self-adjoint, and its spectral family is given by

$$E^{Q^2}(\lambda)\,\varphi := \begin{cases} \chi_{[-\lambda^{1/2},\lambda^{1/2}]}\,\varphi & \text{if } \lambda > 0 \\ 0 & \text{if } \lambda \le 0, \end{cases} \qquad \varphi \in \mathcal{H}.$$

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One verifies that

$$\sigma\big(Q^2\big) = \operatorname{supp}\big(E^{Q^2}\big) = [0,\infty).$$

The Laplacian $-\triangle$ in $\mathcal{H} := \mathsf{L}^2(\mathbb{R}^d)$ satisfies on $\mathscr{S}(\mathbb{R}^d)$ (and thus on $\mathcal{H}^2(\mathbb{R}^d)$)

$$-\triangle = \sum_{j=1}^{d} P_j^2 \equiv P^2 = \mathscr{F}^{-1} Q^2 \mathscr{F},$$

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$$-\triangle = \sum_{j=1}^{d} P_j^2 \equiv P^2 = \mathscr{F}^{-1} Q^2 \mathscr{F},$$

with $\mathscr{F}$ the d-dimensional Fourier transform. So, one has

$$E^{-\triangle} = E^{\mathscr{F}^{-1} Q^2 \mathscr{F}} \overset{\text{(Stone)}}{=} \mathscr{F}^{-1} E^{Q^2} \mathscr{F}.$$

Let $\mathcal{A}_{\mathrm{B}}$ be the Borel σ-algebra of $\mathbb{R}$ and $|\mathcal{V}|$ be the Lebesgue measure of $\mathcal{V} \in \mathcal{A}_{\mathrm{B}}$.

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If H is a self-adjoint operator in $\mathcal{H}$, one has the orthogonal decompositions

$$\mathcal{H} = \mathcal{H}_{\mathrm{p}}(H) \oplus \mathcal{H}_{\mathrm{sc}}(H) \oplus \mathcal{H}_{\mathrm{ac}}(H)$$
$$H = H|_{\mathcal{H}_{\mathrm{p}}(H)} \oplus H|_{\mathcal{H}_{\mathrm{sc}}(H)} \oplus H|_{\mathcal{H}_{\mathrm{ac}}(H)},$$

with

$$\mathcal{H}_{\mathrm{p}}(H) := \overline{\mathrm{Span}}\big\{\text{eigenvectors of } H\big\}$$
$$\mathcal{H}_{\mathrm{sc}}(H) := \big\{\varphi \in \mathcal{H} \mid \lambda \mapsto \|E^H(\lambda)\varphi\|^2 \text{ is continuous}$$
$$\text{and } \exists\, \mathcal{V} \in \mathcal{A}_{\mathrm{B}} \text{ with } |\mathcal{V}| = 0 \text{ and } E^H(\mathcal{V})\varphi = \varphi\big\}$$
$$\mathcal{H}_{\mathrm{ac}}(H) := \big\{\varphi \in \mathcal{H} \mid \lambda \mapsto \|E^H(\lambda)\varphi\|^2 \text{ is absolutely continuous}\big\}.$$

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The decomposition of $\mathcal{H}$ induces a decomposition of $\sigma(H)$

$$\sigma(H) = \sigma_{\mathrm{p}}(H) \cup \sigma_{\mathrm{sc}}(H) \cup \sigma_{\mathrm{ac}}(H),$$

with

$\sigma_{\mathrm{p}}(H) := \sigma\big(H|_{\mathcal{H}_{\mathrm{p}}(H)}\big)$ the pure point spectrum of H,

$\sigma_{\mathrm{sc}}(H) := \sigma\big(H|_{\mathcal{H}_{\mathrm{sc}}(H)}\big)$ the singular continuous spectrum of H,

$\sigma_{\mathrm{ac}}(H) := \sigma\big(H|_{\mathcal{H}_{\mathrm{ac}}(H)}\big)$ the absolutely continuous spectrum of H.

The subspaces $\mathcal{H}_{\mathrm{p}}(H)$, $\mathcal{H}_{\mathrm{sc}}(H)$, $\mathcal{H}_{\mathrm{ac}}(H)$ are the pure point subspace of H, the singular continuous subspace of H and the absolutely continuous subspace of H.

The decomposition of $\mathcal{H}$ induces a decomposition of $\sigma(H)$

$$\sigma(H) = \sigma_{\mathrm{p}}(H) \cup \sigma_{\mathrm{sc}}(H) \cup \sigma_{\mathrm{ac}}(H),$$

with

$\sigma_{\mathrm{p}}(H) := \sigma\big(H|_{\mathcal{H}_{\mathrm{p}}(H)}\big)$ the pure point spectrum of H,

$\sigma_{\mathrm{sc}}(H) := \sigma\big(H|_{\mathcal{H}_{\mathrm{sc}}(H)}\big)$ the singular continuous spectrum of H,

$\sigma_{\mathrm{ac}}(H) := \sigma\big(H|_{\mathcal{H}_{\mathrm{ac}}(H)}\big)$ the absolutely continuous spectrum of H.

The sets $\sigma_{\mathrm{p}}(H)$, $\sigma_{\mathrm{sc}}(H)$, $\sigma_{\mathrm{ac}}(H)$ are closed and (in general) not mutually disjoint.

Example 1.6. *For each* $\lambda \in \mathbb{R}$ *and* $\varphi \in \mathcal{H} := \mathsf{L}^2(\mathbb{R})$, *one has*

$$\begin{aligned}\left\|E^Q(\lambda)\,\varphi\right\|^2 &= \left\|\chi_{(-\infty,\lambda]}\,\varphi\right\|^2 \\ &= \int_{-\infty}^{\lambda} \mathrm{d}x\,|\varphi(x)|^2 \\ &= \text{integral of a } \mathsf{L}^1\text{-function} \\ &= \text{absolutely continuous function.}\end{aligned}$$

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So, $\mathcal{H} = \mathcal{H}_{\mathrm{ac}}(Q)$ *and* Q *has purely absolutely continuous spectrum* $\sigma(Q) = \sigma_{\mathrm{ac}}(Q) = \mathbb{R}$.

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So, $\mathcal{H} = \mathcal{H}_{\mathrm{ac}}(Q)$ and Q has purely absolutely continuous spectrum $\sigma(Q) = \sigma_{\mathrm{ac}}(Q) = \mathbb{R}$.

In fact, Q has Lebesgue spectrum since

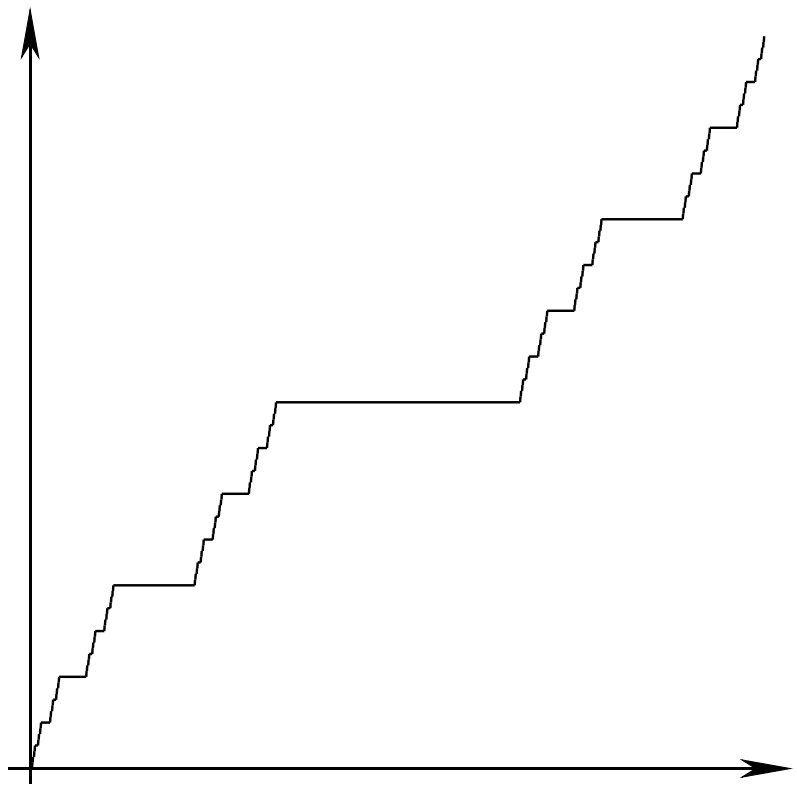
$$\mathrm{e}^{itP}\,\mathrm{e}^{isQ}\,\mathrm{e}^{-itP} = \mathrm{e}^{ist}\,\mathrm{e}^{isQ}, \quad s,t \in \mathbb{R} \quad \Longleftrightarrow \quad \mathrm{e}^{itP}\,Q\,\mathrm{e}^{-itP} = Q+t, \quad t \in \mathbb{R}.$$

(...Stone-von Neumann theorem...)

Example 1.7. *Let* $f : [0,1] \to [0,1]$ *be the Cantor function, and let*

$$M_f \varphi := f \varphi, \quad \varphi \in \mathcal{H} := \mathsf{L}^2([0,1]),$$

be the corresponding bounded multiplication operator.



The spectral family of M_f is

$$E^{M_f}(\lambda)\varphi := \begin{cases} \chi_{f^{-1}([0,\lambda])}\varphi & \textit{if} \quad \lambda \in [0,1] \\ 0 & \textit{if} \quad \lambda \in \mathbb{R} \setminus [0,1], \end{cases} \qquad \varphi \in \mathcal{H}.$$

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One verifies that

$$\sigma(M_f) = \mathrm{supp}\big(E^{Q^2}\big) = \text{Cantor ternary set}$$

and that the function

$$[0,1] \ni \lambda \mapsto \big\|E^{M_f}(\lambda)\varphi\big\|^2 = \big\|\chi_{f^{-1}([0,\lambda])}\varphi\big\|^2 = \int_0^1 \mathrm{d}x\, \chi_{f^{-1}([0,\lambda])}(x)|\varphi(x)|^2$$

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So, $\mathcal{H} = \mathcal{H}_{\mathrm{sc}}(M_f)$ and M_f has purely singular continuous spectrum $\sigma(M_f) = \sigma_{\mathrm{sc}}(M_f) =$ Cantor ternary set.

An interesting link between spectral theory and dynamics is provided by the following:

Theorem 1.8 (RAGE theorem). *Let H be a self-adjoint operator in a Hilbert space $\mathcal{H}$ and let $C \in \mathscr{B}(\mathcal{H})$ be such that $C(H+i)^{-1}$ is compact. Then,*

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \mathrm{d}t \left\| C\, \mathrm{e}^{-itH}\, \varphi \right\|^2 = 0 \qquad \textit{for all } \varphi \in \mathcal{H}_{\mathrm{sc}}(H) \oplus \mathcal{H}_{\mathrm{ac}}(H).$$

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(the typical example is when H is a Schrödinger operator in $\mathbb{R}^d$ and C the orthogonal projection onto a compact subset of $\mathbb{R}^d$)

1.3 Commutator methods for self-adjoint operators

References:

- W. O. Amrein, A. Boutet de Monvel and V. Georgescu, C_0-groups, commutator methods and spectral theory of N-body Hamiltonians, Birhäuser, 1996
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- J. Sahbani, The conjugate operator method for locally regular Hamiltonians, J. Operator Theory, 1997.

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- A, H, self-adjoint operators in $\mathcal{H}$ with domains $\mathcal{D}(A), \mathcal{D}(H)$, spectral families $E^A(\cdot)$, $E^H(\cdot)$ and spectra $\sigma(A), \sigma(H)$
- The adjoint space of a Banach space $\mathcal{B}$ is defined by

$$\mathcal{B}^* := \big\{\text{anti-linear continuous functions } \phi : \mathcal{B} \to \mathbb{C}\big\}$$

$$\|\phi\|_{\mathcal{B}^*} := \sup\big\{|\phi(\varphi)| \mid \varphi \in \mathcal{B},\ \|\varphi\|_{\mathcal{B}} \leq 1\big\}$$

Definition 1.9. *An operator $S \in \mathscr{B}(\mathcal{H})$ satisfies $S \in C^k(A)$ if the map*

$$\mathbb{R} \ni t \mapsto \mathrm{e}^{-itA}\, S\, \mathrm{e}^{itA} \in \mathscr{B}(\mathcal{H})$$

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is strongly of class C^k.

In other terms, $S \in C^k(A)$ if there exist

$$B_0(t) \equiv \mathrm{e}^{-itA}\, S\, \mathrm{e}^{itA}, B_1(t), B_2(t), \ldots, B_k(t) \in \mathscr{B}(\mathcal{H}), \quad t \in \mathbb{R},$$

such that

$$\lim_{h \to 0} \left\| \frac{B_j(t+h) - B_j(t)}{h} \varphi - B_{j+1}(t)\, \varphi \right\| = 0 \quad \text{for all } t \in \mathbb{R},\ \varphi \in \mathcal{H},$$

for $j = 0, 1, \ldots, k-1$.

$S \in C^1(A)$ if and only if the quadratic form

$$\mathcal{D}(A) \ni \varphi \mapsto \big\langle A\varphi, S\varphi \big\rangle - \big\langle \varphi, SA\varphi \big\rangle \in \mathbb{C}$$

is continuous for the topology induced by $\mathcal{H}$ on $\mathcal{D}(A)$; that is, if

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The bounded operator corresponding to the continuous extension of the quadratic form is denoted by $[A, S]$, and one has

$$-[iA, S] = \text{s-} \frac{\mathrm{d}}{\mathrm{d}t} \, \mathrm{e}^{-itA} \, S \, \mathrm{e}^{itA} \Big|_{t=0} \in \mathscr{B}(\mathcal{H}).$$

Example 1.10. *Let* $f \in \mathsf{L}^\infty(\mathbb{R})$ *be an absolutely continuous function with* $f' \in \mathsf{L}^\infty(\mathbb{R})$*, and let*

$$M_f \varphi := f \varphi, \quad \varphi \in \mathcal{H} := \mathsf{L}^2(\mathbb{R}),$$

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be the corresponding bounded multiplication operator.

Then, one has for each $\varphi \in \mathcal{H}$

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathrm{e}^{-itP} M_f \, \mathrm{e}^{itP} \varphi = \frac{\mathrm{d}}{\mathrm{d}t} M_{f(\,\cdot\,-t)} \varphi = -M_{f'(\,\cdot\,-t)} \varphi,$$

and thus $M_f \in C^1(P)$ with $[iP, M_f] = M_{f'}$.

In the case of (unbounded) self-adjoint operators, we have a similar definition:

Definition 1.11. *A self-adjoint operator H is of class $C^k(A)$ if $(H-z)^{-1} \in C^k(A)$ for some $z \in \rho(H)$.*

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If H is of class $C^1(A)$, then

$$\begin{aligned}\big[A,(H-z)^{-1}\big] &= (H-z)^{-1}[H-z,A](H-z)^{-1} \\ &= (H-z)^{-1}[H,A](H-z)^{-1},\end{aligned}$$

with $[H,A]$ the bounded operator from $\mathcal{D}(H)$ to $\mathcal{D}(H)^*$ associated with the continuous extension to $\mathcal{D}(H)$ of the quadratic form

$$\mathcal{D}(H)\cap\mathcal{D}(A) \ni \varphi \mapsto \big\langle H\varphi, A\varphi\big\rangle - \big\langle A\varphi, H\varphi\big\rangle \in \mathbb{C}.$$

Theorem 1.12 (Virial Theorem). *Let A, H be self-adjoint operators with H of class $C^1(A)$. Then,*

$$E^H(\{\lambda\})\,[A,H]\,E^H(\{\lambda\}) = 0$$

for each $\lambda \in \mathbb{R}$. In particular, one has $\langle \varphi, [A,H]\varphi \rangle = 0$ if φ is an eigenvector of H.

Proof. We must show that if $\varphi_1, \varphi_2 \in \mathcal{D}(H)$ satisfy $H\varphi_j = \lambda\varphi_j$ for some $\lambda \in \mathbb{R}$, then $\langle \varphi_1, [A, H]\varphi_2 \rangle = 0$.

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$$\begin{aligned}
&\langle \varphi_1, [A, H]\varphi_2 \rangle \\
&= \langle (\lambda - i)(H - i)^{-1}\varphi_1, [A, H](\lambda + i)(H + i)^{-1}\varphi_2 \rangle
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$\square$

Corollary 1.13 (Point spectrum of H). *Let A, H be self-adjoint operators with H of class $C^1(A)$. Assume there exist a Borel set $I \subset \mathbb{R}$, a number $a > 0$ and $K \in \mathscr{K}(\mathcal{H})$ such that*

$$E^H(I)\,[iH, A]\,E^H(I) \geq a\,E^H(I) + K. \tag{1.1}$$

Then, H has at most finitely many eigenvalues in I (multiplicities counted).

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Some comments:

- If I is bounded, one has

$$\underbrace{E^H(I)}_{\in\mathscr{B}(\mathcal{D}(H)^*,\mathcal{H})} \quad \underbrace{[iH, A]}_{\in\mathscr{B}(\mathcal{D}(H),\mathcal{D}(H)^*)} \quad \underbrace{E^H(I)}_{\in\mathscr{B}(\mathcal{H},\mathcal{D}(H))} \quad \in \mathscr{B}(\mathcal{H}).$$

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- If I is not bounded, the inequality (1.1) holds in the sense of quadratic forms on $\mathcal{D}(H)$.

- The inequality (1.1) is called a Mourre estimate.

Proof. If $\varphi \in \mathcal{H}$ is an eigenvector of H with $\|\varphi\| = 1$ and with eigenvalue in I, the Mourre inequality (1.1) implies that

$$0 \geq a \left\langle \varphi, E^H(I)\varphi \right\rangle + \left\langle \varphi, K\varphi \right\rangle \implies \left\langle \varphi, K\varphi \right\rangle \leq -a.$$

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Now, if the claim were false, there would exist an infinite orthonormal sequence $\{\varphi_j\}$ of eigenvectors of H in $E^H(I)\mathcal{H}$. In particular, one would have $\text{w-}\lim_{j\to\infty} \varphi_j = 0$.

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Note that the proof shows that if $K = 0$, then H is purely continuous in $I \cap \sigma(H)$.

Example 1.14 (Finite dimension). *If* $\dim(\mathcal{H}) < \infty$, *then* A, H *are hermitian matrices and* $H \in C^\infty(A)$.

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Furthermore, one has

$$[iH, A] = 1 + \big([iH, A] - 1\big) = 1 + \text{compact operator},$$

and the corollary implies (without surprise) that H *has at most finitely many eigenvalues in* $\sigma(H)$.

Definition 1.15. *$S \in C^{1+0}(A)$ if $S \in C^1(A)$ and*

$$\int_0^1 \frac{\mathrm{d}t}{t} \left\| \mathrm{e}^{-itA}[A,S]\,\mathrm{e}^{itA} - [A,S] \right\|_{\mathscr{B}(\mathcal{H})} < \infty.$$

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Similarly, a self-adjoint operator H is of class $C^{1+0}(A)$ if $(H-z)^{-1} \in C^{1+0}(A)$ for some $z \in \rho(H)$.

If we regard $C^1(A)$, $C^{1+0}(A)$ and $C^2(A)$ as subspaces of $\mathscr{B}(\mathcal{H})$, we have the inclusions

$$C^2(A) \subset C^{1+0}(A) \subset C^1(A) \subset \mathscr{B}(\mathcal{H}).$$

Example 1.16. *Let* $f \in \mathsf{L}^\infty(\mathbb{R})$ *be an absolutely continuous function with* $f' \in \mathsf{L}^\infty(\mathbb{R})$ *Dini-continuous, and let* M_f *be the corresponding multiplication operator in* $\mathcal{H} := \mathsf{L}^2(\mathbb{R})$.

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Then, we know that $M_f \in C^1(P)$ with $[iP, M_f] = M_{f'}$, and

$$\begin{aligned}\int_0^1 \frac{\mathrm{d}t}{t} \left\| \mathrm{e}^{-itP}[P, M_f]\,\mathrm{e}^{itP} - [P, M_f] \right\|_{\mathscr{B}(\mathcal{H})} &= \int_0^1 \frac{\mathrm{d}t}{t} \left\| M_{f'(\,\cdot\,-t)-f'} \right\|_{\mathscr{B}(\mathcal{H})} \\ &= \int_0^1 \frac{\mathrm{d}t}{t} \left\| f'(\,\cdot\,-t) - f' \right\|_{\mathsf{L}^\infty(\mathbb{R})} \\ &< \infty\end{aligned}$$

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due to the Dini-continuity of f'. So, one has $M_f \in C^{1+0}(P)$.

Spectral result of Mourre (and Amrein, Boutet de Monvel, Georgescu, Sahbani, . . .)

Theorem 1.17 (Spectral properties of H). *Let H be of class $C^{1+0}(A)$. Assume there exist an open set $I \subset \mathbb{R}$, a number $a > 0$ and $K \in \mathscr{K}(\mathcal{H})$ such that*

$$E^H(I)\,[iH, A]\,E^H(I) \geq a\,E^H(I) + K.$$

Then, H has at most finitely many eigenvalues in I (multiplicities counted), and H has no singular continuous spectrum in I.

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- if $K = 0$, then H is purely absolutely continuous in $I \cap \sigma(H)$
- if H has a spectral gap or satisfies an additional invariance assumption, then one can replace the condition $C^{1+0}(A)$ by a weaker condition $C^{1,1}(A)$

Sketch of the proof of Mourre (i)

One has for $\mu \in \sigma(H)$ and $\varepsilon \in \mathbb{R}$ that

$$\left\|(H-\mu-i\varepsilon)^{-1}\right\| = \left\|x \mapsto (x-\mu-i\varepsilon)^{-1}\right\|_{\mathsf{L}^\infty(\mathbb{R})} = |\varepsilon|^{-1}.$$

Thus, $(H-\mu-i\varepsilon)^{-1}$ cannot have a limit in $\mathscr{B}(\mathcal{H})$ as $\varepsilon \to \pm 0$.

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Thus, $(H-\mu-i\varepsilon)^{-1}$ cannot have a limit in $\mathscr{B}(\mathcal{H})$ as $\varepsilon \to \pm 0$.

However, for some $\varphi \in \mathcal{H} \setminus \{0\}$, the holomorphic function

$$F : \rho(H) \to \mathbb{C}, \quad z \mapsto \left\langle \varphi, (H-z)^{-1}\varphi \right\rangle,$$

may have a limit

$$F(\mu) := \lim_{\varepsilon \searrow 0} F(\mu + i\varepsilon)$$

uniformly on each interval $[a,b] \subset I$.

In such a case, Stone's Formula and Lebesgue's dominated convergence theorem imply for $\lambda \in (a, b]$ that

$$\begin{aligned}\left\| E^H\big((a,\lambda]\big)\varphi \right\|^2 = \left\langle \varphi, E^H\big((a,\lambda]\big)\varphi \right\rangle &= \frac{1}{\pi} \lim_{\delta \searrow 0} \int_{a+\delta}^{\lambda+\delta} \mathrm{d}\mu \ \mathrm{Im}\, F(\mu) \\ &= \frac{1}{\pi} \int_a^{\lambda} \mathrm{d}\mu \ \mathrm{Im}\, F(\mu).\end{aligned}$$

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But, F is continuous on $[a, b]$ due to the uniform convergence of the sequence $F_\varepsilon(\,\cdot\,) := \big\langle \varphi, \big(H - (\,\cdot\,) - i\varepsilon\big)^{-1}\varphi\big\rangle$. Thus,

$$\mathrm{Im}\, F(\mu) \in \mathsf{L}^1([a,b]) \quad \text{and} \quad E^H(I)\varphi \in \mathcal{H}_{\mathrm{ac}}(H).$$

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Therefore, if there is a dense set of vectors $\varphi \in \mathcal{H}$ satisfying what precedes, then $E^H(I)\mathcal{H} \subset \mathcal{H}_{\mathrm{ac}}(H)$ and H is purely absolutely continuous in $I \cap \sigma(H)$.

Sketch of the proof of Mourre (ii)

Let's show the existence of the limit $\lim_{\varepsilon \searrow 0} F(\mu + i\varepsilon)$ in the homogeneous case $[iH, A] = H$.

(in such case, one has $\mathrm{e}^{-itA}\, H\, \mathrm{e}^{itH} = \mathrm{e}^{t}\, H$, and thus we already know that H has Haar spectrum on $(0, \infty)$)

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$$z \frac{\mathrm{d}}{\mathrm{d}z} (H - z)^{-1} = z\,(H - z)^{-2} = (H - z)^{-1} H (H - z)^{-1} - (H - z)^{-1}$$
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$$= \left[iA, (H - z)^{-1}\right] - (H - z)^{-1}$$

which gives for $\varphi \in \mathcal{D}(A)$

$$z \frac{\mathrm{d}}{\mathrm{d}z} F(z) = -F(z) - \left\langle iA\varphi, (H - z)^{-1} \right\rangle - \left\langle (H - \bar{z})^{-1} \varphi, iA\varphi \right\rangle.$$

But, if $z = \mu + i\varepsilon$ with $\varepsilon > 0$, then

$$\begin{aligned}\left\|(H-\mu-i\varepsilon)^{-1}\varphi\right\|^2 &= \left\|(H-\mu+i\varepsilon)^{-1}\varphi\right\|^2 \\ &= \left\langle \varphi, |H-\mu-i\varepsilon|^{-2}\varphi \right\rangle\end{aligned}$$

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Thus, we get for $z = \mu + i\varepsilon$ with $\mu \neq 0$ fixed and $\varepsilon > 0$ that

$$\left| z \frac{\mathrm{d}}{\mathrm{d}z} F(z) \right| = \left| -F(z) - \left\langle iA\varphi, (H-z)^{-1} \right\rangle - \left\langle (H-\bar{z})^{-1}\varphi, iA\varphi \right\rangle \right|$$

$$\Longrightarrow |\mu + i\varepsilon| \left| \frac{\mathrm{d}}{\mathrm{d}\varepsilon} F(\mu + i\varepsilon) \right| \leq \left| F(\mu + i\varepsilon) \right| + 2 \|A\varphi\| \left\| (H - \mu - i\varepsilon)^{-1}\varphi \right\|$$

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Now,

$$\left|F(\mu+i\varepsilon)\right| \geq \left|\operatorname{Im} F(\mu+i\varepsilon)\right| = \varepsilon \left\|(H-\mu-i\varepsilon)^{-1}\varphi\right\|^2 > 0$$

if $\varepsilon > 0$ and $\varphi \neq 0$.

Now,

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So, one can divide the last inequality by $\big|F(\mu+i\varepsilon)\big|^{1/2}$ to get

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$$\left|\frac{\mathrm{d}}{\mathrm{d}\varepsilon}F(\mu+i\varepsilon)\right| \leq \frac{1}{|\mu|}\big(\|\varphi\|+2\,\|A\varphi\|\big)\,\varepsilon^{-1/2}\big|F(\mu+i\varepsilon)\big|^{1/2},$$

one gets for each $|\mu| \geq \delta > 0$ and $\varepsilon \in (0,1)$

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&\leq \mathrm{c}(\delta,\varphi)\,\varepsilon^{-1/2}\big(\|\varphi\|^2+\|A\varphi\|^2\big).
\end{aligned}$$

It follows that $\big\{F(\mu + i/m)\big\}_{m\in\mathbb{N}^*}$ is a Cauchy sequence since

$$\big|F(\mu + i/m) - F(\mu + i/n)\big| = \left|\int_{1/n}^{1/m} \mathrm{d}\varepsilon\, \frac{\mathrm{d}}{\mathrm{d}\varepsilon} F(\mu + i\varepsilon)\right|$$

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Thus, the limit $\lim_{\varepsilon\searrow 0} F(\mu + i\varepsilon)$ exists uniformly on $|\mu| \geq \delta$.

1.4 Schrödinger operators

Let M_V be the self-adjoint multiplication operator in $\mathcal{H} := \mathsf{L}^2(\mathbb{R})$ given by $V \in \mathsf{L}^\infty(\mathbb{R};\mathbb{R})$. Then, the 1-dimensional Schrödinger operator

$$H\varphi := -\triangle\varphi + M_V\varphi, \quad \varphi \in \mathcal{D}(H) := \mathcal{H}^2(\mathbb{R}),$$

is self-adjoint due to the Kato-Rellich theorem.

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In quantum mechanics, the operator H describes a non-relativistic particle in $\mathbb{R}$ in presence of a scalar (electric) potential V.

Can we (under some assumptions on V) determine the spectral nature of H ?

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Can we do it using commutator methods ?

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$$(U_t\,\varphi)(x) := \mathrm{e}^{t/2}\,\varphi(\mathrm{e}^t\,x), \quad \varphi \in \mathscr{S}(\mathbb{R}),\ x, t \in \mathbb{R},$$

defines a strongly continuous unitary group (the dilation group).

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defines a strongly continuous unitary group (the dilation group).

The self-adjoint generator $A := i\big(\mathrm{s}\text{-}\frac{\mathrm{d}}{\mathrm{d}t}\,U_t\big|_{t=0}\big)$ of $\{U_t\}_{t\in\mathbb{R}}$ acts as

$$A\varphi := \frac{1}{2}\big(QP + PQ\big)\varphi, \quad \varphi \in \mathscr{S}(\mathbb{R}).$$

The operator A is the quantum analogue of the classical observable $q \cdot p$ on $M := T^*\mathbb{R}$ which appeared at the beginning:

$$\begin{aligned}\big\{\big\{q^2, p^2 + V(q)\big\}, p^2 + V(q)\big\} &= \big\{4(q \cdot p), p^2 + V(q)\big\} \\ &= 8p^2 - 4q \cdot (\nabla V)(q).\end{aligned}$$

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... just replace the observables q and p on $M := T^*\mathbb{R}$ by the self-adjoint operators Q and P in $\mathcal{H}$, and be cautious with the domains of the unbounded operators ...

One has

$$\mathrm{e}^{-itA}\,(-\triangle + i)^{-1}\,\mathrm{e}^{itA}$$

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Thus,

$$
\mathrm{s}\text{-}\frac{\mathrm{d}}{\mathrm{d}t}\,\mathrm{e}^{-itA}\,(-\triangle+i)^{-1}\,\mathrm{e}^{itA}\,\Big|_{t=0}=(-\triangle+i)^{-1}\,2(-\triangle)(-\triangle+i)^{-1},
$$

and $-\triangle$ is of class $C^\infty(A)$ with $[iA,-\triangle]=-2(-\triangle)$.

Similarly, one has

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Thus, if V is absolutely continuous with $\mathrm{id}_{\mathbb{R}} \cdot V' \in \mathsf{L}^\infty(\mathbb{R})$, one has

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and $M_V \in C^1(A)$ with $[iA, M_V] = M_{\mathrm{id}_{\mathbb{R}} \cdot V'}$.

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and $M_V \in C^1(A)$ with $[iA, M_V] = M_{\mathrm{id}_{\mathbb{R}} \cdot V'}$.

Furthermore, if V' is Dini-continuous, one has $M_V \in C^{1+0}(A)$ since

$$\begin{aligned}
&\int_0^1 \frac{\mathrm{d}t}{t} \left\| \mathrm{e}^{-itA}[A, M_V]\, \mathrm{e}^{itA} - [A, M_V] \right\|_{\mathscr{B}(\mathcal{H})} \\
&= \int_0^1 \frac{\mathrm{d}t}{t} \left\| (\mathrm{id}_{\mathbb{R}} \cdot V')(\mathrm{e}^t\ \cdot\ ) - \mathrm{id}_{\mathbb{R}} \cdot V' \right\|_{\mathsf{L}^\infty(\mathbb{R})} \\
&< \infty.
\end{aligned}$$

We infer that H is of class $C^{1+0}(A)$, with

$$[iH, A] = 2(-\triangle) - M_{\mathrm{id}_{\mathbb{R}} \cdot V'} = 2H - M_{(2V - \mathrm{id}_{\mathbb{R}} \cdot V')}.$$

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Now, assume that

$$\lim_{|x| \to \infty} \big(2V - \mathrm{id}_{\mathbb{R}} \cdot V'\big)(x) = 0.$$

Then, a standard result tells us that

$$M_{(2V - \mathrm{id}_{\mathbb{R}} \cdot V')} (-\triangle + i)^{-1} \in \mathscr{K}(\mathcal{H}).$$

(the products $f(Q)\,g(P)$ with $f, g \in C(\mathbb{R})$ vanishing at infinity are compact operators)

Given an open bounded set $I \subset \mathbb{R}$, it follows that

$$E^H(I)[iH, A]E^H(I)$$
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Thus, Theorem 1.17 implies that H has at most finitely many eigenvalues in each open bounded set $I \subset (0, \infty)$ (multiplicities counted), and that H has no singular continuous spectrum in $(0, \infty)$.

(in fact, since $M_V(-\triangle + i)^{-1}$ is compact, one has $\sigma_{\mathrm{ess}}(H) = [0, \infty)$, so that $\sigma_{\mathrm{sc}}(H) = \varnothing$ and $\sigma_{\mathrm{ac}}(H) = [0, \infty)$)

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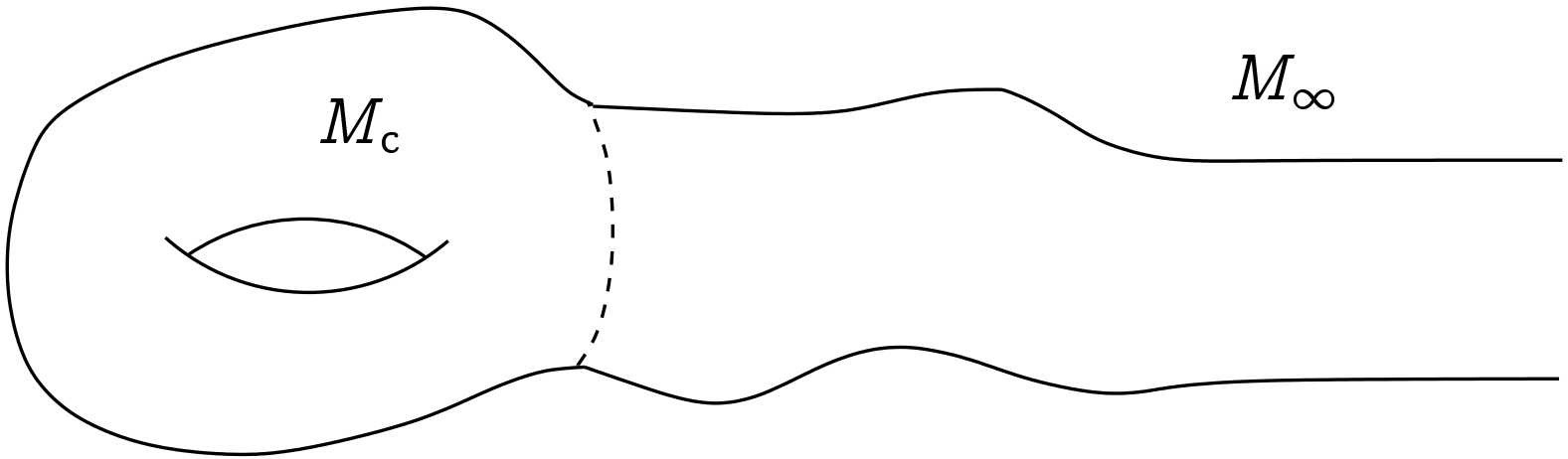
- the operator $-\triangle$ can be replaced by the Laplace-Beltrami operator (on functions or differential forms) on various types of non-compact manifolds



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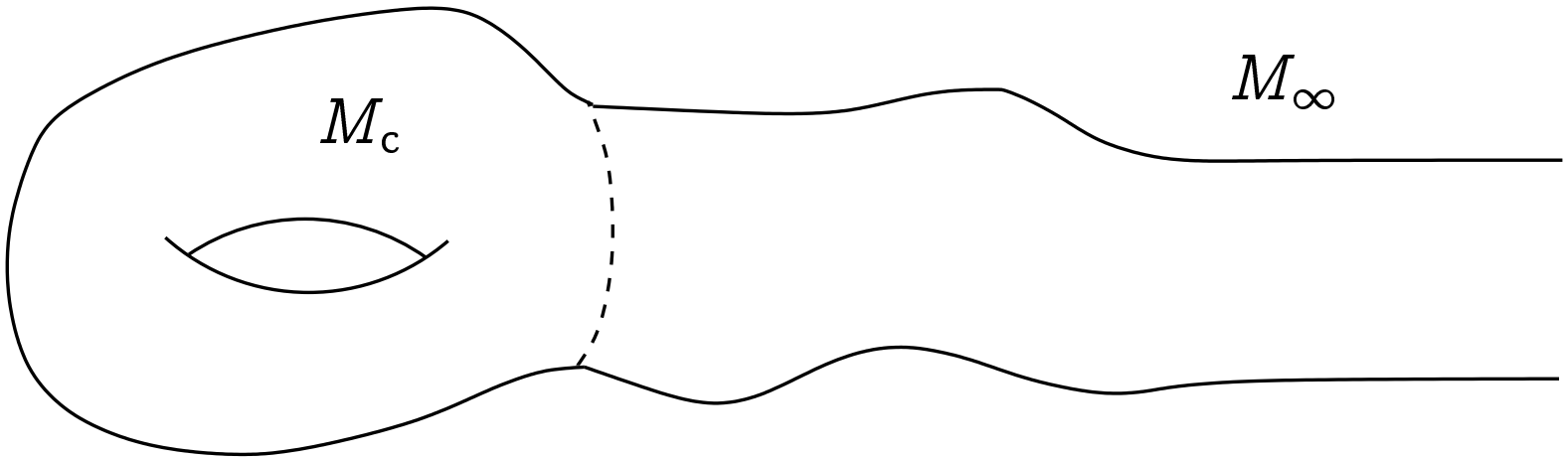

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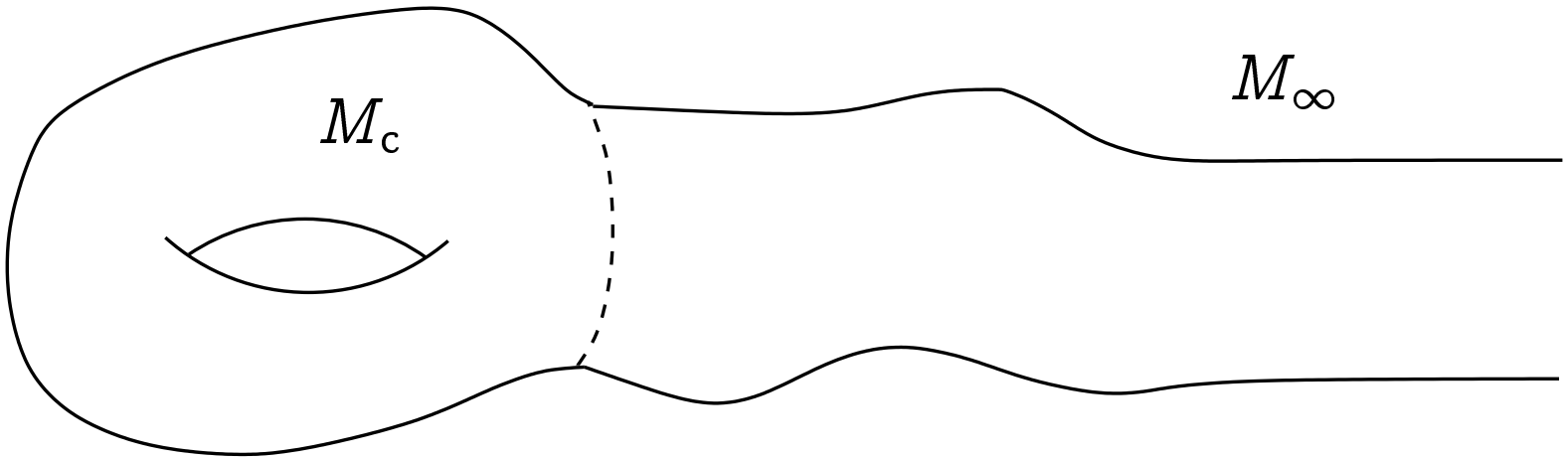

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- etc . . .

1.5 Time changes of horocycles flows

References:

- G. Forni and C. Ulcigrai, Time-changes of horocycle flows, J. Mod. Dyn., 2012
- R. Tiedra, Spectral analysis of time changes of horocycle flows, J. Mod. Dyn., 2012.
- R. Tiedra, Commutator methods for the spectral analysis of uniquely ergodic dynamical systems, preprint on arXiv

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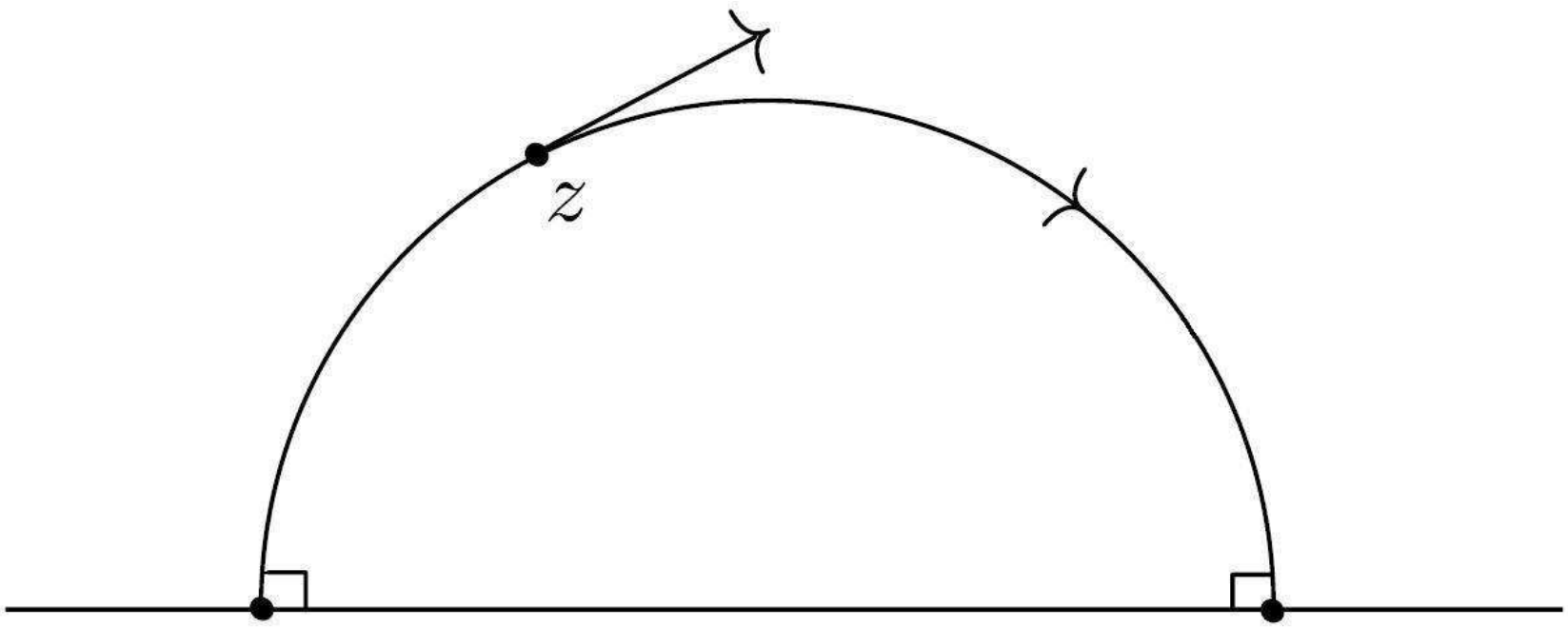
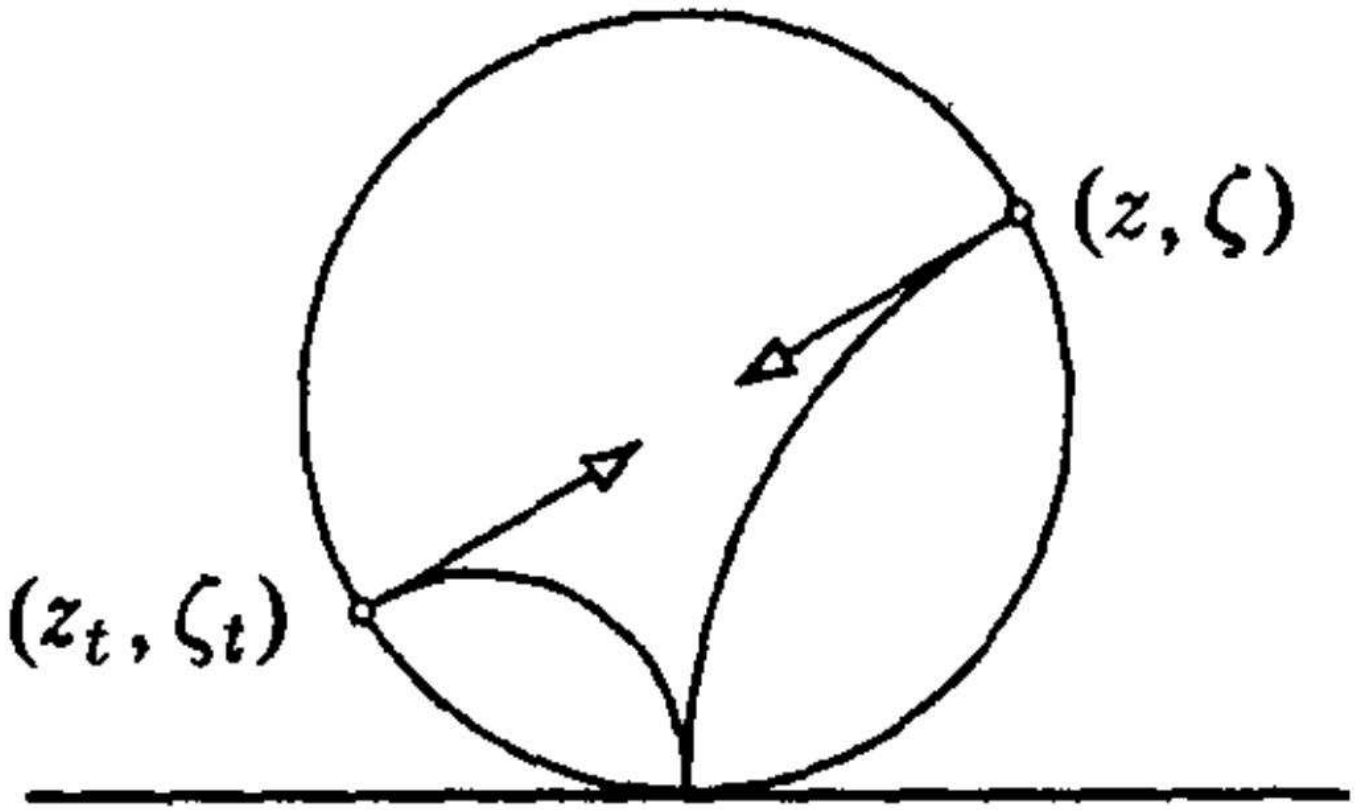
Both flows correspond to right translations on M when $M \simeq \Gamma \setminus \mathsf{PSL}(2;\mathbb{R})$, for some cocompact lattice Γ in $\mathsf{PSL}(2;\mathbb{R})$.



Geodesic in the Poincaré half plane



Horocycle flow in the Poincaré half plane

The operators

$$U_j(t)\varphi := \varphi \circ F_{j,t}, \quad t \in \mathbb{R},\ \varphi \in C(M),$$

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$$H_j\varphi := -i\mathscr{L}_{X_j}\varphi, \quad \varphi \in C^\infty(M),$$

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The horocycle flow $\{F_{1,t}\}_{t\in\mathbb{R}}$ is uniquely ergodic [Furstenberg 73], mixing of all orders [Marcus 78], and $U_1(t)$ has countable Lebesgue spectrum for each $t \neq 0$ [Parasyuk 53].

The horocycle flow and the geodesic flow satisfy the commutation relation (see for instance [Bachir/Mayer 00])

$$U_2(s)\,U_1(t)\,U_2(-s) = U_1(\mathrm{e}^s\,t), \quad s,t \in \mathbb{R}, \tag{1.2}$$

which is a consequence of the matrix identity in $\mathsf{SL}(2,\mathbb{R})$:

$$\begin{pmatrix} \mathrm{e}^{s/2} & 0 \\ 0 & \mathrm{e}^{-s/2} \end{pmatrix} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathrm{e}^{-s/2} & 0 \\ 0 & \mathrm{e}^{s/2} \end{pmatrix} = \begin{pmatrix} 1 & \mathrm{e}^s\,t \\ 0 & 1 \end{pmatrix}.$$

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Therefore, by applying the strong derivatives $\frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0}$ and $\frac{\mathrm{d}}{\mathrm{d}s}\big|_{s=0}$ to (1.2), one obtains that H_1 is of class $C^\infty(H_2)$ with

$$\big[iH_1, H_2\big] = H_1.$$

Time changes of horocycle flows

Consider a C^1 vector field with the same orientation and proportional to X_1; that is, fX_1 with $f \in C^1\big(M;(0,\infty)\big)$.

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The reparametrised time coordinate $h(p,t)$ given by

$$t = \int_0^{h(p,t)} \frac{\mathrm{d}s}{f\big(F_{1,s}(p)\big)}, \quad t \in \mathbb{R},\ p \in M,$$

is such that $h(p,0) = 0$, $\lim_{t\to\pm\infty} h(p,t) = \pm\infty$ and $\frac{\mathrm{d}}{\mathrm{d}t} h(p,t) = f\big(F_{1,h(p,t)}(p)\big)$.

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The function $\mathbb{R} \ni t \mapsto \widetilde{F}_{1,t}(p) \in M$ given by $\widetilde{F}_{1,t}(p) := F_{1,h(p,t)}(p)$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\widetilde{F}_1(p,t) = (fX_1)_{\widetilde{F}_1(p,t)}, \quad \widetilde{F}_1(p,0) = p,$$

and thus $\{\widetilde{F}_{1,t}\}_{t\in\mathbb{R}}$ is the flow of fX_1.

The operators

$$\widetilde{U}_1(t)\,\varphi := \varphi \circ \widetilde{F}_{1,t}\,, \quad t \in \mathbb{R},\ \varphi \in C(M),$$

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The generator $\widetilde{H} := -i\mathscr{L}_{fX_1}$ of $\{\widetilde{U}_1(t)\}_{t\in\mathbb{R}}$ is essentially self-adjoint on $C^1(M)$ and unitarily equivalent to the operator in $\mathcal{H}$ given by

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(...the unitary operator $\mathscr{U} : \mathcal{H} \to \widetilde{\mathcal{H}},\ \varphi \mapsto f^{1/2}\varphi$ realises the unitary equivalence...)

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- In 2006, Katok and Thouvenot conjecture that $\widetilde{H}$ has absolute continuous spectrum (and even countable Lebesgue spectrum) if f is sufficiently smooth.

Mourre estimate

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Mourre estimate

Let $z \in \mathbb{C} \setminus \mathbb{R}$ and assume for a moment that $f \equiv 1$, so that $H \equiv H_1$. Then, one has $(H+z)^{-1} \in C^1(H_2)$ with

$$\begin{aligned}\big[i(H+z)^{-1}, H_2\big] &= -(H+z)^{-1}[iH, H_2](H+z)^{-1} \\ &= -(H+z)^{-1}H(H+z)^{-1}.\end{aligned}$$

It follows that

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Thus H^2 is of class $C^\infty(H_2)$ with $[iH^2, H_2] = 2H^2$, and

$$E^{H^2}(I)\big[iH^2, H_2\big]E^{H^2}(I) = E^{H^2}(I)\,2H^2\,E^{H^2}(I) \geq 2\inf(I)\,E^{H^2}(I)$$

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Therefore, in the case $f \equiv 1$, Mourre's theorem applies to the operator H^2 on the interval $(0, \infty)$.

So, let's try the same approach in the case $f \not\equiv 1 \ldots$

If $f \not\equiv 1$, one has $(H+z)^{-1} \in C^1(H_2)$ with

$$\left[i(H+z)^{-1}, H_2\right] = -(H+z)^{-1}(Hg+gH)(H+z)^{-1}$$

and

$$g := \frac{1}{2} - \frac{1}{2}\mathscr{L}_{X_2}\big(\ln(f)\big).$$

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A calculation as in the case $f \equiv 1$ shows that

$$\big[i\big(H^2+1\big)^{-1}, H_2\big] = -\big(H^2+1\big)^{-1}\big(H^2g+2HgH+gH^2\big)\big(H^2+1\big)^{-1},$$

which means that $(H^2+1)^{-1} \in C^1(H_2)$ with

$$\big[iH^2, H_2\big] = H^2g + 2HgH + gH^2.$$

If $g > 0$ and f is of class C^2, one has

$$H^2 g + g H^2$$
$$= \left[H^2, g^{1/2}\right] g^{1/2} + 2\, g^{1/2} H^2 g^{1/2} + g^{1/2} \left[g^{1/2}, H^2\right]$$

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&= H \left[H, g^{1/2}\right] g^{1/2} + \left[H, g^{1/2}\right] g^{1/2} H + \left[H, g^{1/2}\right] \left[H, g^{1/2}\right] \\
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&= H \left[H, g^{1/2}\right] g^{1/2} + \left[H, g^{1/2}\right] g^{1/2} H + 2 \left[H, g^{1/2}\right]^2 + g^{1/2} \left[g^{1/2}, H\right] H \\
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Thus, making everything rigorous, one obtains that

$$\begin{aligned}
&E^{H^2}(I)\big[iH^2, H_2\big]E^{H^2}(I)\\
&= E^{H^2}(I)\big(H^2 g + 2HgH + gH^2\big)E^{H^2}(I)\\
&\geq aE^{H^2}(I) \quad \text{with} \quad a := 2\inf(I)\cdot \inf_{p\in M} g(p) > 0
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Standard arguments then imply that H has the same spectral properties as H^2.

Summing up:

Theorem 1.18. *Under Kushnirenko's condition and for time changes f of class C^2, the self-adjoint operator $\widetilde{H}$ associated with the vector field fX_1 has purely absolutely continuous spectrum, except at 0, where it has a simple eigenvalue.*

Summing up:

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In fact, this also holds for noncompact surfaces Σ of finite volume.

Fine, but... Forni and Ulcigrai have obtained the same result (and also Lebesgue maximal spectral type) without assuming Kushnirenko's condition (for compact surfaces and for time changes in a Sobolev space of order $> 11/2$).

So, can we get rid off Kushnirenko's condition ?

Mourre estimate (one more time)

Lemma 1.19 (Conjugate operator). *Let* $f \in C^3\big(M;(0,\infty)\big)$ *and* $L > 0$. *Then, the operator*

$$A_L\,\varphi := \frac{1}{L}\int_0^L \mathrm{d}t\ \mathrm{e}^{itH}\,H_2\,\mathrm{e}^{-itH}\,\varphi, \quad \varphi \in C^1(M),$$

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Idea of the proof. A calculation on $C^1(M)$ shows that

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(...if someone knows how to do it for f of class C^2...)

Replacing H_2 by A_L in the previous calculations and noting that

$$\begin{aligned}\frac{1}{L}\int_0^L \mathrm{d}t\ \mathrm{e}^{itH}\, g\, \mathrm{e}^{-itH} &= \frac{1}{L}\int_0^L \mathrm{d}t\ \mathrm{e}^{it\,\mathscr{U}^*\widetilde{H}\mathscr{U}}\ g\, \mathrm{e}^{-it\,\mathscr{U}^*\widetilde{H}\mathscr{U}}\\ &= \frac{1}{L}\int_0^L \mathrm{d}t\ \mathscr{U}^*\,\mathrm{e}^{it\widetilde{H}}\, g\, \mathrm{e}^{-it\widetilde{H}}\,\mathscr{U}\\ &= \frac{1}{L}\int_0^L \mathrm{d}t\,\big(g\circ\widetilde{F}_{1,-t}\big),\end{aligned}$$

we obtain that $(H^2+1)^{-1}\in C^2(A_L)$ with

$$\big[i(H^2{+}1)^{-1},A_L\big] = -(H^2{+}1)^{-1}\big(H^2 g_L{+}2Hg_LH{+}g_LH^2\big)(H^2{+}1)^{-1},$$

where

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The flow $\{\widetilde{F}_{1,t}\}_{t\in\mathbb{R}}$ is uniquely ergodic, since it is a reparametrised version of the uniquely ergodic flow $\{F_{1,t}\}_{t\in\mathbb{R}}$ [Humphries 74].

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$$\begin{aligned}
\lim_{L\to\infty} g_L = \int_M \mathrm{d}\widetilde{\mu}_\Omega\, g_L &= \frac{1}{2} - \frac{1}{2}\int_M \mathrm{d}\widetilde{\mu}_\Omega\, \mathscr{L}_{X_2}\big(\ln(f)\big) \\
&= \frac{1}{2} + \frac{1}{2\int_M f^{-1}\,\mathrm{d}\mu_\Omega}\int_M \mathrm{d}\mu_\Omega\, \mathscr{L}_{X_2}\big(f^{-1}\big) \\
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\end{aligned}$$

$$\Longrightarrow\ g_L > 0 \text{ if } L > 0 \text{ is big enough.}$$

So, we got rid off Kushnirenko's condition, and thus have proved the following:

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(...if someone knows how to prove Lebesgue spectrum...)

2 Commutator methods for unitary operators

Commutator methods for unitary operators is the unitary analogue of commutator methods for self-adjoint operators.

The theory applies to general unitary operators U (not necessarily of the type e^{iH}), up to the regularity class $C^{1+0}(A)$.

2.1 Unitary operators

A unitary operator U in a Hilbert space $\mathcal{H}$ is a surjective isometry; that is,

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Since $U^*U = UU^*$, the spectral theorem for normal operators implies that U admits exactly one complex spectral family E^U with support

$$\operatorname{supp}(E^U) = \sigma(U) \subset \mathbb{S}^1 := \big\{z \in \mathbb{C} \mid |z| = 1\big\}$$

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where $E^U(\lambda + i\mu) := E^{\mathsf{Re}(U)}(\lambda)\, E^{\mathsf{Im}(U)}(\mu)$ for each $\lambda, \mu \in \mathbb{R}$, and

$$\mathsf{Re}(U) := \frac{1}{2}(U + U^*) \qquad \text{and} \qquad \mathsf{Im}(U) := \frac{1}{2i}(U - U^*).$$

One has $U = \int_{\mathbb{R}} \mathrm{e}^{is}\ \widetilde{E}^U(\mathrm{d}s)$ with

$$\widetilde{E}^U(s) := \begin{cases} 0 & \text{if } s < 0 \\ E^U(\mathrm{e}^{is}) & \text{if } s \in [0, 2\pi) \\ 1 & \text{if } s \geq 2\pi. \end{cases}$$

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So, one can use the real spectral family $\widetilde{E}^U$ to obtain orthogonal decompositions

$$\mathcal{H} = \mathcal{H}_{\mathrm{p}}(U) \oplus \mathcal{H}_{\mathrm{sc}}(U) \oplus \mathcal{H}_{\mathrm{ac}}(U)$$
$$U = U|_{\mathcal{H}_{\mathrm{p}}(U)} \oplus U|_{\mathcal{H}_{\mathrm{sc}}(U)} \oplus U|_{\mathcal{H}_{\mathrm{ac}}(U)}$$

as in the self-adjoint case.

Example 2.1 (1-parameter groups of unitary operators). *If H is a self-adjoint operator in a Hilbert space $\mathcal{H}$, then*

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is a unitary operator for each $t \in \mathbb{R}$, and the family $\{U_t\}_{t\in\mathbb{R}}$ defines a strongly continuous 1-parameter group of unitary operators.

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$$U_t := \mathrm{e}^{-itH}$$

is a unitary operator for each $t \in \mathbb{R}$, and the family $\{U_t\}_{t\in\mathbb{R}}$ defines a strongly continuous 1-parameter group of unitary operators.

Example 2.2 (Koopman operator). *Let $T : X \to X$ be an automorphism of a probability space X with probability measure μ. Then, the Koopman operator U_T in $\mathcal{H} := \mathsf{L}^2(X, \mu)$ given by*

$$U_T : \mathcal{H} \to \mathcal{H}, \quad \varphi \mapsto \varphi \circ T,$$

is a unitary operator.

Ergodicity, weak mixing and strong mixing of an automorphism $T : X \to X$ are expressible in terms of spectral properties of the Koopman operator U_T :

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$$\boxed{\text{strong mixing} \implies \text{weak mixing} \implies \text{ergodicity}}$$

2.2 Commutator methods for unitary operators

References:

- M. A. Astaburuaga, O. Bourget, V. H. Cortés, and C. Fernández, Floquet operators without singular continuous spectrum. J. Funct. Anal., 2006.
- C. Fernández, S. Richard and R. Tiedra, Commutator methods for unitary operators, to appear in J. Spectr. Theory.
- C. R. Putnam, Commutation properties of Hilbert space operators and related topics, Springer-Verlag, 1967.

In [Astaburuaga/Bourget/Cortés/Fernández 06], the authors show an analogue of Mourre's theorem for a unitary operator U in a Hilbert space $\mathcal{H}$.

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However ...

- the regularity assumption is $U \in C^2(A)$,
- the proofs rely once more on differential inequalities for "resolvents" of U.

We want to obtain this result with the weaker assumption $U \in C^{1+0}(A)$ and with a simpler proof !

At the end of the day, we obtain:

Theorem 2.3 (Spectral properties of U). *Let $U \in C^{1+0}(A)$. Assume there exist an open set $\Theta \subset \mathbb{S}^1$, a number $a > 0$ and $K \in \mathscr{K}(\mathcal{H})$ such that*

$$E^U(\Theta)\, U^*[A, U]\, E^U(\Theta) \geq a\, E^U(\Theta) + K.$$

Then, U has at most finitely many eigenvalues in Θ (multiplicities counted), and U has no singular continuous spectrum in Θ.

Sketch of the proof (i)

Why the "commutator" $U^*[A, U]$ is the right expression to consider ?

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Imagine that $U = \mathrm{e}^{-iH}$ with $H \in C^1(A)$, then one has

$$U^*[A,U]$$

$$= i \left(\text{s-}\frac{\mathrm{d}}{\mathrm{d}t}\, \mathrm{e}^{iH}\, \mathrm{e}^{-itA}\, \mathrm{e}^{-iH}\, \mathrm{e}^{itA} \right)_{t=0}$$

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$$= i\left(\text{s-}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \mathrm{d}\mu\;\text{s-}\frac{\mathrm{d}}{\mathrm{d}\mu}\,\mathrm{e}^{i\mu H}\,\mathrm{e}^{-itA}\,\mathrm{e}^{-i\mu H}\,\mathrm{e}^{itA}\right)_{t=0}$$

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\begin{aligned}
&U^*[A,U] \\
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&= -\int_0^1 \mathrm{d}\mu\;\text{s-}\frac{\mathrm{d}}{\mathrm{d}t}\big(\,\mathrm{e}^{i\mu H}\,H\,\mathrm{e}^{-itA}\,\mathrm{e}^{-i\mu H}\,\mathrm{e}^{itA} - \mathrm{e}^{i\mu H}\,\mathrm{e}^{-itA}\,H\,\mathrm{e}^{-i\mu H}\,\mathrm{e}^{itA}\,\big)_{t=0}
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&= -\int_0^1 \mathrm{d}\mu\;\text{s-}\frac{\mathrm{d}}{\mathrm{d}t}\big(\,\mathrm{e}^{i\mu H}\,H\,\mathrm{e}^{-itA}\,\mathrm{e}^{-i\mu H}\,\mathrm{e}^{itA} - \mathrm{e}^{i\mu H}\,\mathrm{e}^{-itA}\,H\,\mathrm{e}^{-i\mu H}\,\mathrm{e}^{itA}\,\big)_{t=0} \\
&= -\int_0^1 \mathrm{d}\mu\,\big(\,\mathrm{e}^{i\mu H}\,H\big[i\,\mathrm{e}^{-i\mu H},A\big] - \mathrm{e}^{i\mu H}\,\big[iH\,\mathrm{e}^{-i\mu H},A\big]\big)
\end{aligned}
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&= -\int_0^1 \mathrm{d}\mu\,\big(\,\mathrm{e}^{i\mu H}\,H\big[i\,\mathrm{e}^{-i\mu H},A\big] - \mathrm{e}^{i\mu H}\,\big[iH\,\mathrm{e}^{-i\mu H},A\big]\big) \\
&= \int_0^1 \mathrm{d}\mu\;\mathrm{e}^{i\mu H}\,\big[iH,A\big]\,\mathrm{e}^{-i\mu H}\,.
\end{aligned}
$$

Thus,

$$U^*[A,U] = \int_0^1 \mathrm{d}\mu \; \mathrm{e}^{i\mu H} \left[iH, A\right] \mathrm{e}^{-i\mu H},$$

and positivity of $[iH, A]$ leads to positivity of $U^*[A,U]$ and vice versa.

Thus,

$$U^*[A,U] = \int_0^1 \mathrm{d}\mu \; \mathrm{e}^{i\mu H} \left[iH, A\right] \mathrm{e}^{-i\mu H},$$

and positivity of $[iH, A]$ leads to positivity of $U^*[A,U]$ and vice versa.

(the idea of using $U^*[A,U]$ dates back to Putnam in the 60's)

Sketch of the proof (ii)

As in the self-adjoint case, one can show a Virial theorem which implies the following:

Corollary 2.4 (Point spectrum of U). *Let U and A be respectively a unitary and a self-adjoint operator in $\mathcal{H}$, with $U \in C^1(A)$. Assume there exist a Borel set $\Theta \subset \mathbb{S}^1$, a number $a > 0$ and $K \in \mathscr{K}(\mathcal{H})$ such that*

$$E^U(\Theta)\, U^*[A, U]\, E^U(\Theta) \geq a E^U(\Theta) + K.$$

Then, U has at most finitely many eigenvalues in Θ (multiplicities counted).

If $U \in C^1(A)$ and

$$E^U(\Theta)\, U^*[A, U]\, E^U(\Theta) \geq a\, E^U(\Theta) + K,$$

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So, there exists $\theta \in \Theta$ which is not an eigenvalue of U, and the range $\mathrm{Ran}(1 - \bar{\theta} U)$ of $1 - \bar{\theta} U$ is dense in $\mathcal{H}$.

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So, there exists $\theta \in \Theta$ which is not an eigenvalue of U, and the range $\mathrm{Ran}(1-\bar{\theta}U)$ of $1-\bar{\theta}U$ is dense in $\mathcal{H}$.

Indeed, if $\psi \in \mathcal{H}$ is such that $\psi \perp \mathrm{Ran}(1-\bar{\theta}U)$, then

$$\begin{aligned} \big\langle \psi, (1-\bar{\theta}U)\varphi \big\rangle = 0 \quad \text{for all } \varphi \in \mathcal{H} \;&\Longrightarrow\; (1-\theta U^*)\,\psi = 0 \\ &\Longrightarrow\; U\psi = \theta\,\psi \\ &\Longrightarrow\; \psi = 0. \end{aligned}$$

Furthermore, the Cayley transform of U at the point θ; that is, the operator

$$H_\theta := -i\big(1+\bar{\theta}U\big)\big(1-\bar{\theta}U\big)^{-1}, \quad \mathcal{D}(H_\theta) := \mathsf{Ran}(1-\bar{\theta}U),$$

is self-adjoint.

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Indeed, H_θ is self-adjoint if and only if

$$\mathsf{Ran}(H_\theta + i) = \mathsf{Ran}(H_\theta - i) = \mathcal{H}$$

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$$\begin{aligned}
&\mathsf{Ran}(H_\theta + i) = \mathsf{Ran}(H_\theta - i) = \mathcal{H} \\
&\iff \mathsf{Ran}\Big(-2i\bar\theta U(1-\bar\theta U)^{-1}|_{\mathsf{Ran}(1-\bar\theta U)}\Big) \\
&\qquad\qquad = \mathsf{Ran}\Big(-2i(1-\bar\theta U)^{-1}|_{\mathsf{Ran}(1-\bar\theta U)}\Big) = \mathcal{H}
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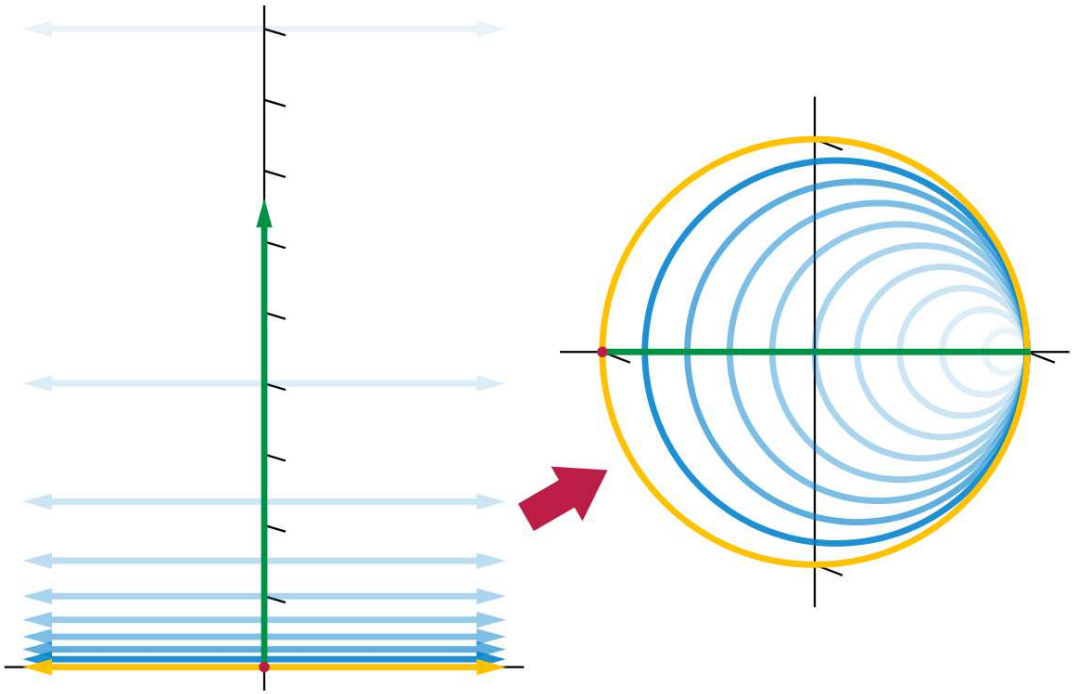
&\iff -2i\bar\theta U\mathcal{H} = -2i\mathcal{H} = \mathcal{H} \\
&\iff \mathcal{H} = \mathcal{H} = \mathcal{H}.
\end{aligned}$$

For any Borel set $\Theta \subset \mathbb{S}^1$, the spectral family E^{H_θ} of H_θ satisfies

$$E^{H_\theta}(I) = E^U(\Theta) \qquad \text{with} \qquad I := \left\{ -i\,\frac{1+\bar{\theta}z}{1-\bar{\theta}z} \mid z \in \Theta \right\}.$$



Cayley transform of $\mathbb{R}$ (for $\theta = -i$)

Sketch of the proof (iii)

One has

$$
\begin{aligned}
(H_\theta - i)^{-1} &= \Big\{\big(- i(1+\bar\theta U) - i(1-\bar\theta U)\big)(1-\bar\theta U)^{-1}\Big\}^{-1} \\
&= \Big\{ - 2i\big(1-\bar\theta U\big)^{-1}\Big\}^{-1} \\
&= -\frac{1}{2i}\big(1-\bar\theta U\big).
\end{aligned}
$$

Sketch of the proof (iii)

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Thus,

$$\big[A, (H_\theta - i)^{-1}\big] = \left[A - \frac{1}{2i}\big(1-\bar\theta U\big)\right] = \frac{\bar\theta}{2i}\big[A, U\big],$$

and the regularity condition $U \in C^{1+0}(A)$ implies the regularity condition $(H_\theta - i)^{-1} \in C^{1+0}(A)$.

Sketch of the proof (iv)

A calculation in $\mathscr{B}\big(\mathcal{D}(H_\theta), \mathcal{D}(H_\theta)^*\big)$ shows that

$$
\begin{aligned}
[iH_\theta, A] &= \big[\big(1+\bar\theta U\big)\big(1-\bar\theta U\big)^{-1}, A\big] \\
&= \big(1+\bar\theta U\big)\big[\big(1-\bar\theta U\big)^{-1}, A\big] + \big[\big(1+\bar\theta U\big), A\big]\big(1-\bar\theta U\big)^{-1} \\
&\;\;\vdots \\
&= 2\big\{(1-\bar\theta U)^{-1}\big\}^* U^* [A, U] (1-\bar\theta U)^{-1}
\end{aligned}
$$

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&= 2\big\{(1-\bar\theta U)^{-1}\big\}^* U^*[A,U](1-\bar\theta U)^{-1}
\end{aligned}
$$

So, the positivity of $U^*[A,U]$ on a Borel set $\Theta \subset \mathbb{S}^1$ implies the positivity of $[iH_\theta, A]$ on the corresponding set $I \subset \mathbb{R}$.

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\end{aligned}$$

So, the positivity of $U^*[A,U]$ on a Borel set $\Theta \subset \mathbb{S}^1$ implies the positivity of $[iH_\theta, A]$ on the corresponding set $I \subset \mathbb{R}$.

Since H_θ is of class $C^{1+0}(A)$, the usual (self-adjoint) Mourre's theorem implies that H_θ has no singular continuous spectrum in I.

Now, suppose by absurd that U has some singular continuous spectrum in $\Theta \setminus \{\theta\}$. Then, there exist $\varphi \in \mathcal{H} \setminus \{0\}$ and $\mathcal{V} \subset [0, 2\pi)$ such that

$$\text{closure}\big(\mathrm{e}^{i\mathcal{V}}\big) \subset \Theta \setminus \{\theta\}, \quad |\mathcal{V}| = 0 \quad \text{and} \quad \widetilde{E}^U(\mathcal{V})\varphi = \varphi.$$

Now, suppose by absurd that U has some singular continuous spectrum in $\Theta \setminus \{\theta\}$. Then, there exist $\varphi \in \mathcal{H} \setminus \{0\}$ and $\mathcal{V} \subset [0, 2\pi)$ such that

$$\text{closure}\big(\mathrm{e}^{i\mathcal{V}}\big) \subset \Theta \setminus \{\theta\}, \quad |\mathcal{V}| = 0 \quad \text{and} \quad \widetilde{E}^U(\mathcal{V})\varphi = \varphi.$$

This implies that

$$\widetilde{E}^U(\mathcal{V})\varphi = \varphi \iff E^U(\mathrm{e}^{i\mathcal{V}})\varphi = \varphi \iff E^{H_\theta}(J)\varphi = \varphi,$$

with

$$J := \left\{ -i\,\frac{1 + \bar{\theta}\,\mathrm{e}^{iv}}{1 - \bar{\theta}\,\mathrm{e}^{iv}} \mid v \in \mathcal{V} \right\} \subset I.$$

But, the function

$$\mathcal{V} \ni v \mapsto -i\,\frac{1+\bar{\theta}\,\mathrm{e}^{iv}}{1-\bar{\theta}\,\mathrm{e}^{iv}} \in J$$

has the Luzin N property. So $|J| = 0$, and thus $\varphi = 0$ since H_θ has no singular continuous spectrum in $J \subset I$.

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Since $\varphi \in \mathcal{H} \setminus \{0\}$, this is a contradiction. So, U has no singular continuous spectrum in $\Theta \setminus \{\theta\}$, and thus no singular continuous spectrum in Θ.

No need to re-do any proof with differential inequalities. We just used the Cayley transform and the pre-existing self-adjoint theory.

We also have the following perturbation result:

Corollary 2.5 (Perturbations of U). *Let U, V be unitary, with $U, V \in C^{1+0}(A)$. Assume there exist an open set $\Theta \subset \mathbb{S}^1$, a number $a > 0$ and $K \in \mathscr{K}(\mathcal{H})$ such that*

$$E^U(\Theta)\, U^*[A, U]\, E^U(\Theta) \geq a\, E^U(\Theta) + K. \tag{2.1}$$

Suppose also that $(V - 1) \in \mathscr{K}(\mathcal{H})$ is compact. Then, VU has at most finitely many eigenvalues in each closed subset of Θ (multiplicities counted), and VU has no singular continuous spectrum in Θ.

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- the Mourre estimate (2.1) depends on U only (V is the perturbation)
- UV and VU are unitarily equivalent since $UV = U(VU)U^*$

2.3 Perturbations of bilateral shifts

Let U be a bilateral shift on a Hilbert space $\mathcal{H}$ with wandering subspace $\mathcal{M} \subset \mathcal{H}$, *i.e.*,

$$\mathcal{M} \perp U^n(\mathcal{M}) \text{ for each } n \in \mathbb{Z} \setminus \{0\} \quad \text{and} \quad \mathcal{H} = \bigoplus_{n \in \mathbb{Z}} U^n(\mathcal{M}).$$

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Using the notation $\varphi \equiv \{\varphi_n\} \in \mathcal{H}$, define the (number) operator

$$A\varphi := \{n\varphi_n\}, \quad \varphi \in \mathcal{D}(A) := \left\{\psi \in \mathcal{H} \mid \sum_{n \in \mathbb{Z}} n^2 \|\psi_n\|^2 < \infty\right\},$$

which is self-adjoint since it is a maximal multiplication operator in a ℓ^2-space.

One has for each $\varphi \in \mathcal{D}(A)$

$$\begin{aligned}\big\langle A\varphi, U\varphi\big\rangle - \big\langle \varphi, UA\varphi\big\rangle &= \big\langle \{n\,\varphi_n\}, \{\varphi_{n+1}\}\big\rangle - \big\langle \{\varphi_n\}, U\{n\,\varphi_n\}\big\rangle \\ &= \big\langle \{\varphi_n\}, \{(n+1)\,\varphi_{n+1}\}\big\rangle - \big\langle \{\varphi_n\}, \{n\,\varphi_{n+1}\}\big\rangle \\ &= \big\langle \varphi, U\varphi\big\rangle,\end{aligned}$$

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meaning that $U \in C^\infty(A) \subset C^{1+0}(A)$ and $U^*[A,U] = U^*U = 1$.

Thus, Theorem 2.3 implies that U has purely absolutely continuous spectrum, as it is well known.

In fact, the conditions $U \in C^1(A)$ and $[A,U] = U$ imply that

$$\text{s-}\frac{\mathrm{d}}{\mathrm{d}t}\,\mathrm{e}^{-itA}\,U\,\mathrm{e}^{itA} = -i\,\mathrm{e}^{-itA}\,U\,\mathrm{e}^{itA} \quad\Longleftrightarrow\quad \mathrm{e}^{-itA}\,U\,\mathrm{e}^{itA} = \mathrm{e}^{-it}\,U.$$

So, U is unitarily equivalent to $\mathrm{e}^{-it}\,U$ for each $t \in \mathbb{R}$, and thus has purely Lebesgue spectrum covering the whole circle $\mathbb{S}^1$.

Let V be another unitary operator with $V \in C^{1+0}(A)$ and $(V-1) \in \mathscr{K}(\mathcal{H})$.

We deduce from Corollary 2.5 that VU has purely absolutely continuous spectrum except, possibly, at a finite number of points of $\mathbb{S}^1$, where VU may have eigenvalues of finite multiplicity.

2.4 Perturbations of the Schrödinger free evolution

The Schrödinger free evolution $\{U_t\}_{t\in\mathbb{R}}$ in $\mathcal{H} := \mathsf{L}^2(\mathbb{R}^d)$ given by

$$U_t := \mathsf{e}^{-itP^2}, \quad t \in \mathbb{R},$$

satisfies

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$$\sigma(U_t) = \sigma_{\mathrm{ac}}(U_t) = \mathbb{S}^1 \quad \text{for each } t \neq 0.$$

Indeed, one has for each $s \in [0, 2\pi)$ and $t \neq 0$ that

$$\begin{aligned} E^{\mathrm{e}^{-itP^2}}(\mathrm{e}^{is}) &= E^{\cos(-tP^2)}\big(\cos(s)\big)\, E^{\sin(-tP^2)}\big(\sin(s)\big) \\ &= E^{-tP^2}\big([0,s] + 2\pi\mathbb{Z}\big)\, E^{-tP^2}\big([0,s] + 2\pi\mathbb{Z}\big) \\ &= E^{P^2}\left([0, -s/t] + \frac{2\pi}{t}\,\mathbb{Z}\right). \end{aligned}$$

What can we say about perturbations of the type VU_t ?

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The operator

$$A := \frac{1}{2}\left\{(P^2+1)^{-1}P\cdot Q + Q\cdot P(P^2+1)^{-1}\right\}$$

is essentially self-adjoint on $C_{\mathrm{c}}^{\infty}(\mathbb{R}^d)$ (because the vector field $X_x := x(x^2+1)^{-1} \in \mathbb{R}^d$ is complete),

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is essentially self-adjoint on $C_{\mathrm{c}}^{\infty}(\mathbb{R}^d)$ (because the vector field $X_x := x(x^2+1)^{-1} \in \mathbb{R}^d$ is complete), and calculations on $C_{\mathrm{c}}^{\infty}(\mathbb{R}^d)$ show that $U \in C^1(A)$ with

$$\begin{aligned}
&(U_t)^*[A, U_t] \\
&= \frac{1}{2}\,\mathrm{e}^{itP^2}\sum_j \left\{(P^2+1)^{-1}P_j\left[Q_j, \mathrm{e}^{-itP^2}\right] + \left[Q_j, \mathrm{e}^{-itP^2}\right]P_j(P^2+1)^{-1}\right\} \\
&= t\,\mathrm{e}^{itP^2}\sum_j \left\{(P^2+1)^{-1}P_j^2\,\mathrm{e}^{-itP^2} + \mathrm{e}^{-itP^2}\,P_j^2(P^2+1)^{-1}\right\} \\
&= 2tP^2(P^2+1)^{-1}.
\end{aligned}$$

Further commutations on $C_c^\infty(\mathbb{R}^d)$ show that $U_t \in C^2(A)$.

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Moreover, if $t > 0$ and $\mathrm{closure}(\Theta) \cap \{1\} = \varnothing$, there exists $\delta > 0$ such that

$$E^{U_t}(\Theta)\,(U_t)^*[A, U_t]\,E^{U_t}(\Theta) \geq 2t\delta(\delta+1)^{-1}E^{U_t}(\Theta).$$

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So, all the assumptions for U_t are satisfied, and we have:

Lemma 2.6. *If $V \in C^{1+0}(A)$ and $(V-1) \in \mathscr{K}(\mathcal{H})$, then the eigenvalues of VU_t outside $\{1\}$ are of finite multiplicity and can accumulate only at $\{1\}$. Furthermore, VU has no singular continuous spectrum.*

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This extends previous results on the Schrödinger free evolution perturbed by "periodic kicks" ($V = \mathrm{e}^{iB}$ with $B = B^*$ of finite rank).

2.5 Skew products over translations

Let $\{y_t\}_{t\in\mathbb{R}}$ be a C^1 one-parameter subgroup of a compact metric abelian Banach Lie group X with normalised Haar measure μ (such group X is isomorphic to a subgroup of $\mathbb{T}^{\aleph_0} \equiv (\mathbb{R}/\mathbb{Z})^{\aleph_0}$).

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Let $\{F_t\}_{t\in\mathbb{R}}$ be the corresponding translation flow,

$$F_t(x) := y_t\, x, \quad t \in \mathbb{R},\ x \in X,$$

and let $\{V_t\}_{t\in\mathbb{R}}$ the corresponding strongly continuous unitary group in $\mathcal{H} := \mathsf{L}^2(X, \mu)$,

$$V_t\varphi := \varphi \circ F_t, \quad t \in \mathbb{R},\ \varphi \in C(X).$$

The generator H of $\{V_t\}_{t\in\mathbb{R}}$ given by

$$H\varphi := -i\mathscr{L}_Y\varphi, \quad \varphi \in C^\infty(X),$$

with Y the vector field associated with $\{F_t\}_{t\in\mathbb{R}}$ and $\mathscr{L}_Y$ the corresponding Lie derivative, is essentially self-adjoint on $C^\infty(X)$.

Let G be another compact metric abelian group with Haar measure ν and character group $\widehat{G}$, and let $\phi : X \to G$ be a measurable function (cocycle).

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We want to apply commutator methods to the Koopman operator

$$W\psi := \psi \circ T, \quad \psi \in \mathsf{L}^2(X \times G, \mu \times \nu),$$

with T the (measure-preserving invertible) skew product

$$T : X \times G \to X \times G, \quad (x, z) \mapsto \big(y_1 x, \phi(x) z\big).$$

The operator W is reduced by the orthogonal decomposition (given by Peter-Weyl theorem)

$$\mathsf{L}^2(X \times G, \mu \times \nu) = \bigoplus_{\chi \in \widehat{G}} L_\chi, \quad L_\chi := \{\varphi \otimes \chi \mid \varphi \in \mathcal{H}\},$$

and $W|_{L_\chi}$ is unitarily equivalent to the unitary operator

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Furthermore, the operator U_χ satisfies the following purity law:

If F_1 is ergodic, the spectrum of U_χ has uniform multiplicity and is either purely punctual, purely singular continuous or purely Lebesgue (see [Helson 86] in the case $X = G = \mathbb{T}$).

We assume the following:

Assumption 2.7. *The translation F_1 is ergodic and $\phi : X \to G$ satisfies $\phi = \xi\eta$, where*

(i) $\xi : X \to G$ is a continuous group homomorphism,

(ii) $\eta \in C(X;G)$ has a Lie derivative $\mathscr{L}_Y(\chi \circ \eta)$ which satisfies

$$\int_0^1 \frac{\mathrm{d}t}{t} \left\| \mathscr{L}_Y(\chi \circ \eta) \circ F_t - \mathscr{L}_Y(\chi \circ \eta) \right\|_{\mathsf{L}^\infty(X)} < \infty.$$

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Two comments:

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Two comments:

- $\chi \circ \xi$ encodes the "topological degree" of the cocycle $\chi \circ \phi$.
- (ii) means that $\mathscr{L}_Y(\chi \circ \eta)$ is of Dini-type along the flow $\{F_t\}_{t \in \mathbb{R}}$.

Define

$$\xi_0 := \frac{\mathrm{d}}{\mathrm{d}t}(\chi\circ\xi)(y_t)\Big|_{t=0}, \quad g := |\xi_0|^2 - \xi_0 \frac{\mathscr{L}_Y(\chi\circ\eta)}{\chi\circ\eta} \quad \text{and} \quad A := -i\xi_0 H,$$

and observe that $g : X \to \mathbb{R}$ is of Dini-type along $\{F_t\}_{t\in\mathbb{R}}$ and that A is self-adjoint with $\mathcal{D}(A) \supset \mathcal{D}(H)$.

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Since A and V_1 commute, we have for each $\varphi \in C^\infty(X)$ that

$$\begin{aligned}\big\langle A\varphi, U_\chi\varphi\big\rangle - \big\langle \varphi, U_\chi A\varphi\big\rangle &= \big\langle \varphi, \big[A, \chi\circ\phi\big] V_1\varphi\big\rangle \\ &= \big\langle \varphi, -\xi_0\,\mathscr{L}_Y(\chi\circ\phi) V_1\varphi\big\rangle,\end{aligned}$$

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with

$$\begin{aligned}\mathscr{L}_Y(\chi\circ\phi) &= \mathscr{L}_Y(\chi\circ\xi)\,(\chi\circ\eta) + (\chi\circ\xi)\,\mathscr{L}_Y(\chi\circ\eta) \\ &= \left(\xi_0 + \frac{\mathscr{L}_Y(\chi\circ\eta)}{\chi\circ\eta}\right)(\chi\circ\phi).\end{aligned}$$

It follows that

$$\langle A\varphi, U_\chi \varphi\rangle - \langle \varphi, U_\chi A\varphi\rangle = \langle \varphi, g U_\chi \varphi\rangle,$$

with $g \in \mathsf{L}^\infty(X)$. So, one has $U_\chi \in C^1(A)$ with $\left[A, U_\chi\right] = g U_\chi$ due to the density of $C^\infty(X)$ in $\mathcal{D}(A)$.

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$$\big\langle A\varphi, U_\chi\varphi\big\rangle - \big\langle \varphi, U_\chi A\varphi\big\rangle = \big\langle \varphi, g\,U_\chi\varphi\big\rangle,$$

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Since g is of Dini-type along $\{F_t\}_{t\in\mathbb{R}}$, the equalities

$$\begin{aligned}
&\int_0^1 \frac{\mathrm{d}t}{t}\,\big\|\,\mathrm{e}^{-itA}\,\big[A, U_\chi\big]\,\mathrm{e}^{itA} - \big[A, U_\chi\big]\big\|_{\mathscr{B}(\mathcal{H})}\\
&= \int_0^1 \frac{\mathrm{d}t}{t}\,\big\|\,\mathrm{e}^{-itA}\,g\,U_\chi\,\mathrm{e}^{itA} - g\,U_\chi\big\|_{\mathscr{B}(\mathcal{H})}\\
&= \int_0^1 \frac{\mathrm{d}t}{t}\,\big\|\big(\,\mathrm{e}^{-itA}\,g\,\mathrm{e}^{itA} - g\big)\,\mathrm{e}^{-itA}\,U_\chi\,\mathrm{e}^{itA} + g\big(\,\mathrm{e}^{-itA}\,U_\chi\,\mathrm{e}^{itA} - U_\chi\big)\big\|_{\mathscr{B}(\mathcal{H})}
\end{aligned}$$

imply that $U_\chi \in C^{1+0}(A)$.

If the function g were strictly positive, we would be able to apply Theorem 2.3 since

$$(U_\chi)^* \big[A, U_\chi\big] = (U_\chi)^* g U_\chi \geq \inf_{x \in X} g(x) > 0.$$

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But, this is a priori not the case since

$$g = |\xi_0|^2 - \xi_0 \frac{\mathscr{L}_Y(\chi \circ \eta)}{\chi \circ \eta} \equiv \text{positive constant} + \text{total derivative}.$$

Nonetheless, the same averaging of the conjugate operator A as the one used for horocycle flows may work and lead to a strictly positive function g.

Since $U_\chi \in C^1(A)$, we have $U_\chi^\ell \in C^1(A)$ and $U_\chi^\ell \mathcal{D}(A) = \mathcal{D}(A)$ for each $\ell \in \mathbb{Z}$, and thus the operator

$$A_n \varphi := \frac{1}{n} \sum_{\ell=0}^{n-1} U_\chi^{-\ell} A U_\chi^\ell \varphi = \frac{1}{n} \sum_{\ell=0}^{n-1} U_\chi^{-\ell} \big[A, U_\chi^\ell\big] \varphi + A\varphi, \quad \varphi \in \mathcal{D}(A_n) := \mathcal{D}(A),$$

is self-adjoint since $\frac{1}{n} \sum_{\ell=0}^{n-1} U_\chi^{-\ell} \big[A, U_\chi^\ell\big]$ is bounded.

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Doing the same calculations as before with A_n instead of A, one obtains that $U_\chi \in C^{1+0}(A_n)$ with

$$\big[A_n, U_\chi\big] = \frac{1}{n} \sum_{\ell=0}^{n-1} U_\chi^{-\ell} \big[A, U_\chi\big] U_\chi^\ell = \frac{1}{n} \sum_{\ell=0}^{n-1} U_\chi^{-\ell} \big(g U_\chi\big) U_\chi^\ell = g_n U_\chi$$

and

$$g_n := \left(\frac{1}{n} \sum_{\ell=0}^{n-1} U_\chi^{-\ell} g U_\chi^\ell \right) = \frac{1}{n} \sum_{\ell=0}^{n-1} g \circ F_{-\ell}.$$

Since F_1 is ergodic, we know (see [Cornfeld/Fomin/Sinaĭ82]) that the flow $\{F_\ell\}_{\ell\in\mathbb{Z}}$ is uniquely ergodic and that

$$\xi_0 = \frac{\mathrm{d}}{\mathrm{d}t}(\chi\circ\xi)(y_t)\Big|_{t=0} \neq 0 \quad \text{if} \quad \chi\circ\xi \not\equiv 1.$$

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$$\xi_0 = \frac{\mathrm{d}}{\mathrm{d}t}(\chi\circ\xi)(y_t)\Big|_{t=0} \neq 0 \quad \text{if} \quad \chi\circ\xi \not\equiv 1.$$

Using the notation $\chi\circ\eta = \mathrm{e}^{if_{\chi,\eta}}$, we infer that

$$\begin{aligned}\lim_{n\to\infty} g_n = \int_X \mathrm{d}\mu\, g &= |\xi_0|^2 - \xi_0 \int_X \mathrm{d}\mu\, \frac{\mathscr{L}_Y(\chi\circ\eta)}{\chi\circ\eta} \\ &= |\xi_0|^2 + \xi_0 \left\langle 1, Hf_{\chi,\eta}\right\rangle \\ &= |\xi_0|^2\end{aligned}$$

uniformly on X.

Thus, $g_n > 0$ if n is big enough, and

$$(U_\chi)^* \big[A_n, U_\chi\big] = (U_\chi)^* g_n U_\chi \geq \inf_{x \in X} g_n(x) > 0$$

as desired.

Thus, $g_n > 0$ if n is big enough, and

$$(U_\chi)^* \big[A_n, U_\chi\big] = (U_\chi)^* g_n U_\chi \geq \inf_{x \in X} g_n(x) > 0$$

as desired.

Putting everything together, we obtain the following:

Theorem 2.8 (Spectral properties of W)**.** *Let F_1 be ergodic and let ϕ satisfy Assumption 2.7 with $\chi \circ \xi \not\equiv 1$. Then, U_χ has purely Lebesgue spectrum. In particular, the restriction of W to the subspace $\bigoplus_{\chi \in \widehat{G},\, \chi \circ \xi \not\equiv 1} L_\chi \subset \mathsf{L}^2(X \times G, \mu \times \nu)$ has countable Lebesgue spectrum.*

Two remarks:

- In the case $X = \mathbb{T}^d$, $G = \mathbb{T}^{d'}$ with $d, d' \geq 1$, this complements previous results of [Iwanik/Lemańczyk/Rudolph 93-99], where $\mathscr{L}_Y(\chi \circ \eta)$ is of bounded variation instead of Dini-type.

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  (bounded variation and Dini-continuity are mutually independent)

- If we do not assume that $\mathscr{L}_Y(\chi \circ \eta)$ is of Dini-type, we can already infer that W has purely continuous spectrum in $\bigoplus_{\chi \in \widehat{G},\, \chi \circ \xi \not\equiv 1} L_\chi$ due to the corollary on the point spectrum (Corollary 2.4) and the purity law.