Test models for filtering and prediction of moisture-coupled tropical waves

John Harlima and Andrew J. Majda

Department of Mathematics, North Carolina State University, USA

Department of Mathematics and Center for Atmospheric and Ocean Science, Courant Institute of Mathematical Sciences, New York University, USA

Correspondence to: Department of Mathematics, North Carolina State University, BOX 8205, Raleigh, NC 27695, USA. E-mail: jharlim@ncsu.edu

The filtering/data assimilation and prediction of moisture coupled tropical waves is a contemporary topic with significant implications for extended range forecasting. The development of efficient algorithms to capture such waves is limited by the unstable multiscale features of tropical convection which can organize large scale circulations and the sparse observations of the moisture coupled wave in both the horizontal and vertical. The approach proposed here is to address these difficult issues of data assimilation and prediction through a suite of analog models which despite their simplicity capture key features of the observational record and physical processes in moisture coupled tropical waves. The analog models emphasized here involve the multicloud convective parameterization based on three cloud types, congestus, deep, and stratiform, above the boundary layer. Two test examples involving an MJO-like turbulent traveling wave and the initiation of a convectively coupled wave train are introduced to illustrate the approach. A suite of reduced filters with judicious model errors for data assimilation of sparse observations of tropical waves, based on linear stochastic models in a moisture coupled eigenmode basis is developed here and applied to the two test problems. Both the reduced filter and 3D-VAR with a full moist background covariance matrix can recover the unobserved troposphere humidity and precipitation rate; on the other hand, 3D-VAR with a dry background covariance matrix fails to recover these unobserved variables. The skill of the reduced filtering methods in recovering the unobserved precipitation, congestus, and stratiform heating rates as well as the front to rear tilt of the convectively coupled waves exhibits a subtle dependence on the sparse observation network and the observation time.

Key Words: tropical data assimilation, reduced stochastic filters, multicloud models, Madden-Julian Oscillation

Received …

Citation: …
1. Introduction

Observational data indicate that through the complex interaction of heating and moist convection, tropical atmosphere flows are organized on a hierarchy of scales (Nakazawa 1988) ranging from cumulus clouds of a few kilometers to mesoscale convective systems (Houze 2004) to equatorial synoptic-scale convectively coupled Kelvin waves and two-day waves (Kiladis et al. 2009) to planetary-scale intraseasonal organized circulations such as the Madden-Julian Oscillation (MJO, Zhang 2005). These moisture coupled tropical waves like the MJO exert a substantial influence on intraseasonal prediction in the tropics, sub-tropics, and midlatitudes (Moncrieff et al. 2007). Despite the continued research efforts by the climate community, the present coarse resolution GCM’s, used for prediction of weather and climate, poorly represent variability associated with tropical convection (Lau and Waliser 2005; Zhang 2005; Lin et al. 2006). Given the importance of moisture coupled tropical waves for short term climate and medium to long range weather prediction, new strategies for the filtering or data assimilation and prediction of moisture coupled tropical waves are needed and this is the topic of the present paper.

The approach proposed here is to address the issues of data assimilation and prediction through a suite of analog models which despite their simplicity capture key features of the observational record and physical processes in moisture coupled tropical waves. This approach is analogous to the use of various versions of the Lorenz-96 model (Lorenz 1996; Wilks 2005; Majda et al. 2005; Abramov and Majda 2007; Crommelin and Vanden-Eijnden 2008; Harlim and Majda 2008a, 2010a; Majda and Harlim 2012, and references therein) to gain insight into basic issues for midlatitude filtering, prediction, and parameterization. The viability of this approach for moisture coupled tropical waves rests on recent advances in simplified modelling of convectively coupled tropical waves and the MJO which predict key physical features of these waves such as their phase speed, dispersion relation, front to rear tilt (Kiladis et al. 2005, 2009), and circulation in qualitative agreement with observations (Khouider and Majda 2006a,b, 2007, 2008a,b; Majda et al. 2007; Majda and Stechmann 2009a,b, 2011) through simplified moisture-coupled models. The analog models emphasized here involve the multicloud convective parameterization based on three cloud types congestus, deep, and stratiform, above the boundary layer (Khouider and Majda 2006a,b, 2007, 2008a,b). The convective closure of the multicloud model takes into account the energy available for congestus and deep convection and uses a nonlinear moisture switch that allows for natural transitions between congestus and deep convection as well as for stratiform downdrafts which cool and dry the boundary layer. As a simplified two vertical baroclinic mode model, the multicloud model is very successful in capturing most of the spectrum of convectively coupled waves (Kiladis et al. 2009; Khouider and Majda 2008b; Han and Khouider 2010) as well as the nonlinear organization of large scale envelopes mimicking across scale interactions of the MJO and convectively coupled waves (Khouider and Majda 2007, 2008b). Furthermore, the multicloud parameterization has been used in the next generation NCAR-GCM (HOMME) and is very successful in simulating the MJO and convectively coupled equatorial waves, at a coarse resolution of 170km in the idealized case of a uniform SST (aquaplanet) setting (Khouider et al. 2011). A stochastic version of the multicloud model has been utilized recently as a novel convective parameterization to improve the physical variability of deficient deterministic convective parameterizations (Khouider et al. 2010; Frenkel et al. 2011b).
The filtering skill for the recovery of troposphere moisture, heating profiles, precipitation, and vertical tilts in circulation and temperature from sparse noisy partial observations is studied here for a turbulent MJO-like travelling wave (Majda et al. 2007) and for the temporal development of a convectively coupled wave train. A suite of filters with judicious model errors, based on linear stochastic models (Harlim and Majda 2008a, 2010a; Majda and Harlim 2012) in a moisture coupled eigenmode basis is developed here and applied to the two test problems as well as related 3D-V AR algorithms with a full moist background covariance matrix or a dry background covariance (Zagar et al. 2004b,a). These results are the first demonstration of the utility of the analog multicloud models for gaining insight for data assimilation and prediction of moisture coupled tropical waves.

The plan for the remainder of the paper is the following. In Section 2, the suite of simplified tropical models for filtering and prediction is reviewed; section 3 illustrates two simplified cases, an MJO analog wave (Majda et al. 2007) and the temporal development of a convectively coupled tropical wave train which illustrate phenomena in the models and also serve as examples for filtering in subsequent sections of the paper. The suite of filters with judicious model errors for moisture coupled tropical waves are introduced in Section 4. Filtering skill for these algorithms applied to the MJO analog wave and the development of a convectively coupled wave train is reported in Section 5. Section 6 is a concluding discussion and summary.

2. Test models with moisture coupled tropical waves

The test models proposed here begin with two coupled shallow water systems: a direct heating mode forced by a bulk precipitation rate from deep penetrative clouds (Neelin and Zeng 2000) and a second vertical baroclinic mode forced by the upper level heating (cooling) and lower level cooling (heating) of stratiform and congestus clouds, respectively (Khouider and Majda 2006a). Below, for simplicity in exposition, we present these equations without explicit nonlinear advection effects and coupling to barotropic winds. This allows us to emphasize moisture coupled tropical waves here but we comment later in this section about how nonlinear advection and barotropic winds enrich the dynamics of the test models. Thus, the test models begin with two equatorial shallow water equations

\[
\frac{\partial \mathbf{v}_j}{\partial t} + \mathbf{U} \cdot \nabla \mathbf{v}_j + \beta y \mathbf{v}_j^T - \theta_j = -C_d u_0 \mathbf{v}_j - \frac{1}{\tau_w} \mathbf{v}_j,
\]

\[
\frac{\partial \theta_1}{\partial t} + \mathbf{U} \cdot \nabla \theta_1 - \text{div} \mathbf{v}_1 = P + S_1, \tag{1}
\]

\[
\frac{\partial \theta_2}{\partial t} + \mathbf{U} \cdot \nabla \theta_2 - \frac{1}{4} \text{div} \mathbf{v}_2 = -H_s + H_c + S_2,
\]

for \( j = 1, 2 \). The equations in (1) are obtained by a Galerkin projection of the hydrostatic primitive equations with constant buoyancy frequency onto the first two baroclinic modes. More details of their derivation are found in (Neelin and Zeng 2000; Frierson et al. 2004; Stechmann and Majda 2009). In (1), \( \mathbf{v}_j = (u_j, v_j)_{j=1,2} \) represent the first and second baroclinic velocities assuming \( G(z) = \sqrt{2} \cos(\pi z/H_T) \) and \( G(2z) = \sqrt{2} \sin(2\pi z/H_T) \) vertical profiles, respectively, while \( \theta_j, j = 1, 2 \) are the corresponding potential temperature components with the vertical profiles \( G'(z) = \sqrt{2} \sin(\pi z/H_T) \) and \( 2G'(2z) = 2\sqrt{2} \sin(2\pi z/H_T) \), respectively. Therefore, the total velocity field is approximated by

\[
\mathbf{V} \approx \mathbf{U} + G(z) \mathbf{v}_1 + G(2z) \mathbf{v}_2,
\]

\[
w \approx -\frac{H_T}{\pi} \left[ G'(z) \text{div} \mathbf{v}_1 + \frac{1}{2} G'(2z) \text{div} \mathbf{v}_2 \right],
\]

where \( \mathbf{V} \) is the horizontal velocity and \( w \) is the vertical velocity. The total potential temperature is given
approximately by

$$\Theta \approx z + G'(z)\theta_1 + 2G''(2z)\theta_2.$$ 

Here $H_T \approx 16$ km is the height of the tropical troposphere with $0 \leq z \leq H_T$ and $v_j = (-v_j, u_j)$ while $\bar{U}$ is the incompressible barotropic wind which is set to zero hereafter, for the sake of simplicity. In (1), $P \geq 0$ models the heating from deep convection while $H_s$, $H_c$ are the stratiform and congestus heating rates. Conceptually, the direct heating mode has a positive component and serves to heat the whole troposphere and is associated with a vertical shear flow. The second baroclinic mode is heated by the congestus clouds, $H_c$, from below and by the stratiform clouds, $H_s$, from above and therefore cooled by $H_c$ from above and by $H_s$ from below. It is associated with a jet shear flow in the middle troposphere (Khouider and Majda 2006a,b). The terms $S_1$ and $S_2$ are the radiative cooling rates associated with the first and second baroclinic modes respectively.

The system of equations in (1) is augmented by an equation for the boundary layer equivalent potential temperature, $\theta_{eb}$, and another for the vertically integrated moisture content, $q$.

$$\frac{\partial \theta_{eb}}{\partial t} = \frac{1}{h_b}(E - D),$$

$$\frac{\partial q}{\partial t} + \bar{U} \cdot \nabla q + \text{div} (v_1q + \tilde{\alpha}v_2q) + \tilde{Q}\text{div} (v_1 + \tilde{\lambda}v_2) = -\frac{2\sqrt{2}}{\pi}P + \frac{1}{H_T}D.$$  \hspace{1cm} (2)

In (2), $h_b \approx 500$ m is the height of the moist boundary layer while $\tilde{Q}$, $\tilde{\lambda}$, and $\tilde{\alpha}$ are parameters associated with a prescribed moisture background and perturbation vertical profiles. According to the first equation in (2), $\theta_{eb}$ changes in response to the downdrafts, $D$, and the sea surface evaporation $E$. A detailed pedagogical derivation of the moisture equation starting from the equations of bulk cloud microphysics is presented in Khouider and Majda (2006b). The approximate numerical values of $\tilde{\lambda} = 0.8$ and $\tilde{\alpha} = 0.1$, follow directly from the derivation, while the coefficient $\tilde{Q}$ arises from the background moisture gradient. We use the standard value $\tilde{Q} \approx 0.9$ (Neelin and Zeng 2000; Frierson et al. 2004).

In full generality, the parametrizations in (1) and (2) automatically have conservation of an approximation to vertically integrated moist static energy. Notice that, the precipitation rate in (2), balances the vertical average of the total convective heating rate in (1), therefore leading to the conservation of the vertical average of the equivalent potential temperature $\langle \theta_{eb} \rangle = \langle Q(z) \rangle + q + \langle \Theta \rangle + \frac{\partial}{\partial z}\bar{q}_{eb}$ when the external forces, namely, the radiative cooling rates, $S_1$, $S_2$, and the evaporative heating, $E$, are set to zero. Also note that the sensible heating flux has been ignored in (1) for simplicity since this is a relatively small contribution in the tropics. Here and elsewhere in the text $\langle f \rangle = (1/H_T)\int_0^{H_T} f(z) dz$.

The equations in (1) and (2) for the prognostic variables $q, \theta_{eb}, \theta_j, v_j, j = 1,2$, are written in non-dimensional units where the equatorial Rossby deformation radius, $L_e \approx 1,500$ km is the length scale, the first baroclinic dry gravity wave speed, $c \approx 50$ m s$^{-1}$, is the velocity scale, $T = L_e/c \approx 8$ h is the associated time scale, and the dry-static stratification $\bar{\alpha} = H_T N^2 \bar{\theta}_s \approx 15$ K is the temperature unit scale. The basic bulk parameters of the model are listed in Table I for the readers convenience.

2.1. The convective parameterization

The surface evaporative heating, $E$, in (2) obeys an adjustment equation toward the boundary layer saturation equivalent potential temperature, $\theta_{eb}^*$.

$$\frac{1}{h_b}E = \frac{1}{\tau_e}(\theta_{eb}^* - \theta_{eb}),$$  \hspace{1cm} (3)
Table I. Bulk constants in two-layer mode model.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value/Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$H_T$</td>
<td>16 km height or the tropical troposphere.</td>
</tr>
<tr>
<td>$Q$</td>
<td>0.9</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>0.8</td>
</tr>
<tr>
<td>$\hat{\alpha}$</td>
<td>0.1</td>
</tr>
<tr>
<td>$\tau_w$</td>
<td>75 days Rayleigh-wind friction relaxation time</td>
</tr>
<tr>
<td>$\tau_R$</td>
<td>50 days Newtonian cooling relaxation time</td>
</tr>
<tr>
<td>$L_e$</td>
<td>1500 km equatorial deformation radius, length scale</td>
</tr>
<tr>
<td>$T = L_e/c$</td>
<td>$\approx$ 8 h time scale</td>
</tr>
<tr>
<td>$\bar{\alpha}$</td>
<td>$\approx$ 15 K dry static stratification, temperature scale</td>
</tr>
<tr>
<td>$C_d$</td>
<td>0.001</td>
</tr>
<tr>
<td>$\theta_0$</td>
<td>300 K Brunt-Väisala buoyancy frequency</td>
</tr>
<tr>
<td>$h_b$</td>
<td>500 m</td>
</tr>
<tr>
<td>$\alpha_2$</td>
<td>0.1 relative contribution of $\theta_2$ to the middle troposphere $\theta_e$</td>
</tr>
</tbody>
</table>

The value of $\bar{\alpha}$ represents a threshold below which the free troposphere is locally moist and “accepts” only deep convection while the value of $\theta^+$ defines complete dryness.

Therefore, the precipitation, $P$, and the downdrafts, $D$, obey

$$ P = \frac{1 - \Lambda}{1 - \Lambda^*} P_0 \quad \text{and} \quad D = \Lambda D_0, $$

while the stratiform and congestus heating rate, $H_s$ and $H_c$, solve the relaxation-type equations

$$ \frac{\partial H_s}{\partial t} = \frac{1}{\tau_s} (\alpha_s P - H_s) $$

and

$$ \frac{\partial H_c}{\partial t} = \frac{1}{\tau_c} \left( \alpha_c \frac{\Lambda - \Lambda^*}{1 - \Lambda^*} \frac{D}{H_T} - H_c \right), $$

with $\tau_e$ is the evaporative time scale. The middle tropospheric equivalent potential temperature anomaly is defined approximately by

$$ \theta_{em} \approx q + \frac{2\sqrt{2}}{\pi} (\bar{\theta}_1 + \alpha_2 \theta_2). $$

Notice that the coefficient $2\sqrt{2}/\pi$ in (4) results from the vertical average of the first baroclinic potential temperature, $\sqrt{2} \bar{\theta}_1 \sin(\pi z/H_T)$, while the small value for $\alpha_2$ adds a non-zero contribution from $\theta_2$ to $\theta_{em}$ to include its contribution from the lower middle troposphere although its vertical average is zero. The multicloud model closure is based on a moisture switch parameter $\Lambda$, Khouider and Majda (2006a, 2008a,b), which serves as a measure for the moistness and dryness of the middle troposphere. When the discrepancy between the boundary layer and the middle troposphere equivalent potential temperature is above some fixed threshold, $\theta^+$, the atmosphere is defined as dry. Moist parcels rising from the boundary layer will have their moisture quickly diluted by entrainment of dry air, hence losing buoyancy and stop to convect. In this case, we set $\Lambda = 1$ which automatically inhibits deep convection in the model (see below). When this discrepancy is below some lower value, $\theta^-$, we have a relatively moist atmosphere and we set $\Lambda = \Lambda^* < 1$. The function $\Lambda$ is then interpolated (linearly) between these two values. More precisely, we set

$$ \Lambda = \begin{cases} 
1 & \text{if } \theta_{eb} - \theta_{em} > \theta^+ \\
A(\theta_{eb} - \theta_{em}) + B & \text{if } \theta^- \leq \theta_{eb} - \theta_{em} \leq \theta^+ \\
\theta^* & \text{if } \theta_{eb} - \theta_{em} < \theta^-.
\end{cases} $$

The value of $\theta^-$ represents a threshold below which the free troposphere is locally moist and “accepts” only deep convection while the value of $\theta^+$ defines complete dryness.

Copyright © 0000 Royal Meteorological Society

Q. J. R. Meteorol. Soc. 00: 2–27 (0000)
respectively. The dynamical equations in (1), (2), (7), and (8) define the multicloud model. Notice that, as anticipated above, when the middle troposphere is dry, \( \Lambda = 1 \), deep convection is completely inhibited, even if \( P_0 \), i.e, CAPE is positive, whereas congestus heating is favored. Other variants of the equation in (8) for \( H_c \) can be utilized where changes in \( H_c \) respond to low-level CAPE (Khouider and Majda 2008a,b).

The quantities \( P_0 \) and \( D_0 \) represent respectively the maximum allowable deep convective heating/precipitation and downdrafts, independent of the value of the switch function \( \Lambda \). Notice that conceptually the model is not bound to any type of convective parametrization. A Betts-Miller relaxation type parametrization as well as a CAPE parametrization can be used to setup a closure for \( P_0 \). Here we let

\[
P_0 = \frac{1}{\tau_{\text{conv}}} \left[ a_1 \theta_{eb} + a_2 (q - \hat{q}) - a_0 (\theta_1 + \gamma_2 \theta_2) \right]^+, \tag{9}
\]

where \( f^+ = \max(f, 0) \) and \( \hat{q} \) is a threshold constant value measuring a significant fraction of the tropospheric saturation and \( \tau_{\text{conv}} \). \( a_1, a_2, a_0 \) are parameters specified below. In particular the coefficient \( a_0 \) is related to the inverse buoyancy relaxation time of Fuchs and Raymond (2002).

The downdrafts are closed by

\[
D_0 = \frac{m_0}{P} \left[ \hat{P} + \mu_2 (H_s - H_c) \right]^+ (\theta_{eb} - \theta_{em}), \tag{10}
\]

where \( m_0 \) is a scaling of the downdraft mass flux and \( \hat{P} \) is a prescribed precipitation/deep convective heating at radiative convective equilibrium. Here \( \mu_2 \) is a parameter allowing for stratiform and congestus mass flux anomalies (Majda and Shefter 2001; Majda et al. 2004). Finally the radiative cooling rates, \( S_1, S_2 \) in (1) are given by a simple Newtonian cooling model

\[
S_j = -Q_{R,j}^0 - \frac{1}{\tau_R} \theta_j, \quad j = 1, 2, \tag{11}
\]

where \( Q_{R,j}^0, j = 1, 2 \) are the radiative cooling rates at radiative convective equilibrium (RCE). This is a spatially homogeneous steady state solution where the convective heating is balanced by the radiative cooling. The basic constants in the model convective parametrization and the typical values utilized here are given in Table II. The physical features incorporated in the multi-cloud model are discussed in detail in (Khouider and Majda 2006a, 2007, 2008a,b).

2.2. Moisture coupled phenomena in the test models

As already noted in the introduction, the dynamic multicloud models in (1), (2), (7), (8) capture a number of observational features of equatorial convectively coupled waves and the MJO. These phenomena occur in multi-wave dynamical models with strong moisture coupling through (2), nonlinear on-off switches like (5), (9), (10) and nonlinear saturation of moisture coupled instabilities (Khouider and Majda 2006a, 2007, 2008a,b; Khouider et al. 2011). All of these features present major challenges for contemporary data assimilation and prediction strategies. Two detailed analog examples are presented in Section 3.

As described in detail in Khouider and Majda (2006b) the multicloud models in a limiting regime also include the quasi-equilibrium models (Neelin and Zeng 2000; Frierson et al. 2004; Pauluis et al. 2008) which mimic the Betts-Miller and Arakawa-Schubert parameterizations of GCM’s. Such models arise formally by keeping the first baroclinic mode in (1), retaining the moisture equation in (2) with \( D = 0 \), setting \( \Lambda = 1 \) in (6), and using \( P_0 \) in (9) with \( a_1 = 0 \) while ignoring all remaining
### Table II. Parameters in the convective parametrization. The parameters in the middle panel will be chosen differently for the MJO-analogue case in Section 3.1 and the temporal development of a convectively coupled wave train in Section 3.2. The parameters in the lower panel are determined at the RCE state.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\theta^*<em>{eb} - \theta</em>{eb}$</td>
<td>10 K</td>
<td>Discrepancy between boundary layer $\theta_c$ at its saturated value and at the RCE state</td>
</tr>
<tr>
<td>$\theta^\pm$</td>
<td>10, 20 K</td>
<td>Temperature threshold used to define the switch function $\Lambda$</td>
</tr>
<tr>
<td>$\alpha_s$, $\alpha_c$</td>
<td>0.25</td>
<td>Linear fitting constant interpolating the switch function $\Lambda$</td>
</tr>
<tr>
<td>$\gamma_2$</td>
<td>0.1</td>
<td>Stratiform heating adjustment coefficient</td>
</tr>
<tr>
<td>$Q_{R,1}^0$</td>
<td>1 K day$^{-1}$</td>
<td>Relative contribution of $\theta_2$ to convective parametrization</td>
</tr>
<tr>
<td>$\Lambda^*$</td>
<td></td>
<td>Second baroclinic radiative cooling rate</td>
</tr>
<tr>
<td>$\mu_2$</td>
<td></td>
<td>Lower threshold of the switch function $\Lambda$</td>
</tr>
<tr>
<td>$\alpha_e$</td>
<td></td>
<td>Relative contribution of stratiform and congestus mass flux anomalies to the downdrafts</td>
</tr>
<tr>
<td>$\tau_s$, $\tau_c$</td>
<td></td>
<td>Stratiform and congestus heating adjustment time</td>
</tr>
<tr>
<td>$a_0$</td>
<td></td>
<td>Inverse buoyancy time scale of convective parametrization</td>
</tr>
<tr>
<td>$a_1$, $a_2$</td>
<td></td>
<td>Relative contribution of $\theta_{eb}$ to convective parametrization</td>
</tr>
<tr>
<td>$\tau_{conv}$</td>
<td></td>
<td>Relative contribution of $q$ to convective parametrization</td>
</tr>
<tr>
<td>$\bar{\theta}<em>{eb} - \bar{\theta}</em>{em}$</td>
<td></td>
<td>Discrepancy between boundary and middle troposphere potential temperature at RCE value</td>
</tr>
<tr>
<td>$\tau_e$</td>
<td>$\approx$ 8 h</td>
<td>Evaporation time scale in the boundary layer</td>
</tr>
<tr>
<td>$Q_{R,2}^0$, $\hat{q}$, $m_0$</td>
<td></td>
<td>Second baroclinic radiative cooling rate, threshold beyond which condensation takes place in Betts-Miller scheme, Scaling of downdraft mass flux</td>
</tr>
</tbody>
</table>

boundary layer and cloud equations. There are many interesting exact solutions of the nonlinear dynamics with moisture switches in this quasi-equilibrium regime, large scale precipitation fronts, which serve as interesting test problems for filtering with nonlinear switches and moisture coupled waves (Frierson et al. 2004; Pauluis et al. 2008; Stechmann and Majda 2006); the behavior of finite ensemble Kalman filters (Evensen 1994; Anderson 2001; Bishop et al. 2001; Hunt et al. 2007) and particle filters (van Leeuwen 2010; Anderson 2010) are particularly interesting in this context with moisture coupled switches and exact solutions. See Zagar (2012) for other interesting use of similar models as tests for tropical data assimilation.

However, rigorous mathematical theory establishes that these quasi-equilibrium models have no instabilities or positive Lyapunov exponents (Majda and Souganidis 2010), unlike realistic tropical convection and the full multicloud models. More realism in the quasi-equilibrium tropical models can be achieved by allowing active barotropic dynamics and coupled nonlinear advection, which allows for tropical-extratropical wave interactions (Lin et al. 2000; Majda and Biello 2003; Biello and Majda 2004). Examples with these features are developed by Khouider and Majda (2005a,b).

### 3. Examples of moisture coupled tropical waves in the test model

In this section, we describe two concrete examples with solutions which will be used as the truth for generating synthetic observations (as we will describe in Section 4). The two specific examples include an MJO-like traveling wave (Majda et al. 2007) and the initiation of a convectively coupled wave train that mimics the solutions of explicit simulations with a Cloud Resolving Model (Grabowski and Moncrieff 2001). Following the basic setup in Khouider and Majda (2006a, 2007), we consider the multicloud model in (1), (2), (7), (8) on a periodic equatorial ring without rotation, $\beta = 0$, without barotropic wind, $\bar{U} = 0$, and with a uniform background sea surface temperature...
given by constant $\theta_{eb}$. With this setup, the wind velocity in (1), (2) has only the zonal wind component, $v_j = u_j$, resolved at every 40 km on an equatorial belt of 40,000 km.

3.1. An MJO-like turbulent traveling wave

In our first example, we consider the parameter regime for an intraseasonal MJO-like turbulent traveling wave. Following Majda et al. (2007), we set the bulk parameters in Table I, $\tilde{Q} = 1$, $\tilde{\lambda} = 0.6$, $C_d = 10^{-5}$, $\tau_w = 150$ days, $\tau_R = 50$ days and the convective parameters in Table II, $\tilde{\theta}_{em} = 12$ K, $a_0 = 12$, $a_1 = 0.1$, $a_2 = 0.9$, $\mu_2 = 0.5$, $\alpha_c = 0.5$, $\Lambda^* = 0.2$. The intraseasonal timescale is generated through $\tau_{conv} = 12$ hours which is consistent with the current observational estimates for large-scale consumption of CAPE and $\tau_s = \tau_c = 7$ days which is also consistent with the current observational record for low-level moistening and congestus cloud development in the MJO.

The linear stability analysis for this parameter regime has been studied in detail in Majda et al. (2007). Here, we summarize some of the important features for eastward propagating waves for the readers convenience: the unstable wavenumbers 2 and 3 have growth rates of roughly $(30 \text{ days})^{-1}$ and phase speed of 6.9 and 5.8 ms$^{-1}$, respectively. These unstable modes have westward, tilted vertical structure for heating, velocity, and temperature, with clear first and second baroclinic mode contributions and low-level warmer potential temperature leading and within the deep convection (see Figure 9 below). In Figure 1, we show the contour plot of the precipitation $P$ (which is exactly the deep heating rate for this model) at the statistical steady state from a numerical simulation between 5000-5200 days. The main feature here is an eastward moving wavenumber-2 waves MJO-like wave with phase speed 6.1 ms$^{-1}$. Within the envelope of this wave are intense westward moving small scale fluctuations. These fluctuations occur irregularly and there are often long breaks between intense deep convective events. All of these features are observed in the MJO (Zhang 2005).

3.2. Initiation of a convectively coupled wave train

In this second example, we consider the three cloud model with enhanced congestus heating (Khouider and Majda 2008a) with slightly different parameterization than the above. In particular, the total precipitation, $P$, is different from the deep convection heating rate, $H_d$, and is defined as follows,

$$P = \frac{2\sqrt{2}}{\pi}(H_d + \xi_s H_s + \xi_c H_c),$$

(12)

allowing for stratiform and congestus rain. The key feature in this new parameterization is attributed to the asymmetric heating rate contribution in the upper and lower level atmosphere with nonzero $\xi_s$ and $\xi_c$, respectively. This new feature replaces the first baroclinic heating equation in (1) with

$$\frac{\partial \theta_1}{\partial t} + \frac{\partial u_1}{\partial x} = H_d + \xi_s H_s + \xi_c H_c + S_1.$$  

(13)
The moisture equation in (2) remains unchanged except that now we remove the scale factor \( \frac{2 \sqrt{2}}{\pi} \) in front of \( P \) since it is already included in (12).

The new congestus parameterization uses exactly the same switch function \( \Lambda \) in (5) with middle-troposphere equivalent potential temperature approximation in (4). The precipitation, \( P \), in (6) is replaced with

\[
H_d = (1 - \Lambda)Q_d, \quad (14)
\]

with bulk energy available for deep convection given by

\[
Q_d = \left\{ \bar{Q} + \frac{1}{\tau_{conv}} [a_1 \theta_{eb} + a_2 q - a_0 (\theta_1 + \gamma_2 \theta_2)] \right\}^+. \quad (15)
\]

In (15), parameter \( \bar{Q} \) is the bulk convective heating determined at the RCE state. The downdraft in (6) is also replaced with

\[
D = \frac{m_a}{Q} \left[ \bar{Q} + \mu_2(H_s - H_c) \right]^+ (\theta_{eb} - \theta_{em}). \quad (16)
\]

Compared to (6), this new parameterization assigns \( \Lambda^* = 0 \) for the deep convection heating rate and ignores the factor \( \Lambda \) in the original downdraft equation. The corresponding dynamical equations for the stratiform and congestus heating are

\[
\frac{\partial H_s}{\partial t} = \frac{1}{\tau_s} (\alpha_s H_d - H_s), \quad (17)
\]

\[
\frac{\partial H_c}{\partial t} = \frac{1}{\tau_c} (\alpha_c \Lambda Q_c - H_c), \quad (18)
\]

where

\[
Q_c = \left\{ \bar{Q} + \frac{1}{\tau_{conv}} [\theta_{eb} - a_0' (\theta_1 + \gamma_2 \theta_2)] \right\}^+. \quad (19)
\]

denotes a “bulk energy” for congestus heating.

In our numerical experiment, we use the same parameter values as in Khouider and Majda (2008a). The bulk constants in Table I are not changed. The convective parameters in Table II are used with \( \Lambda^* = 0, \mu_2 = 0.25, \alpha_c = 0.1, \tau_s = 3 \) h, \( \tau_c = 1 \) h, \( a_0 = 5, a_1 = a_2 = 0.5, \tau_{conv} = 2 \) h, and \( \theta_{eb} - \theta_{em} = 14 \) K. The additional new parameters for the enhanced congestus parameterization include the coefficients representing contributions of stratiform and congestus clouds to the first baroclinic heating, \( \xi_s = 0.5 \) and \( \xi_c = 1.25 \), respectively; inverse convective buoyancy time scale associated with congestus clouds, \( a_0' = 2 \); the bulk convective heating \( \bar{Q} \) that is determined at RCE. Interested readers should consult Khouider and Majda (2008a) for the details of the linear stability analysis.

Here, we are interested in the initiation of a convectively coupled wave train to mimic the high resolution two-dimensional explicit Cloud Resolving Model solutions in Grabowski and Moncrieff (2001). In particular, we integrate the model with a localized piece of a single unstable linear waves of small amplitude centered at 20,000 km as the initial condition (see the space-time plot of the first two-baroclinic velocities, potential temperatures, congestus and deep heating rates, moisture, and precipitation in Figure 2). Note that this setup is exactly the regime analyzed in Frenkel et al. (2011a) in which they focused on understanding the effect of diurnal cycle and we neglect the diurnal cycle here. Notice there are fast moving waves (see \( g, H_d, \) and \( P \) in Figure 2) during the first 2 days moving away from the 20,000 km mark. After about 8-10 days, additional waves appear; this wave initiation is partly due to the convectively coupled wave interactions with faster moving gravity waves. After about 100 days, these waves mature to a wave train of six individual eastward moving waves with a wave speed of approximately 14.5 ms\(^{-1}\) (see Figure 3). Such wave structure and wave train organization resembles to the structure found in the explicit simulations.
with a cloud resolving model by Grabowski and Moncrieff (2001). Moreover, the mature waves have a total convective heating pattern (with backward and upward tilt in the wind and temperature fields, upper-tropospheric warm temperature anomalies slightly leading the region of the upward motion, which is in phase with the heating anomalies, with low level convergence) that is very similar to convectively coupled Kelvin waves observed in nature (Wheeler and Kiladis 1999; Wheeler et al. 2000; Straub and Kiladis 2002).

4. Algorithms for filtering moisture coupled waves from sparse observations

In this section, we first describe the sparse observation networks and then discuss in details the reduced stochastic filtering algorithms.

4.1. Sparse observation networks

In the present paper, we consider horizontally sparse observations at every 2,000 km. This means we only have $M = 20$ observations at $x_j = jh, h = 2\pi/40,000$ km in a non-dimensionalized unit assuming that the equatorial belt circumference is 40,000 km. For compact notation, we define $\Psi_{j,m} = (u_1, u_2, \theta_1, \theta_2, \theta_{eb}, q, H_s, H_c)^T$; we use subscripts $j$ and $m$ to specify that each component in $\Psi$ is
evaluated at grid point $x_j$ and discrete time $t_m$, respectively.

We define a general observation model

$$G\Psi_j^o = G\Psi_{j,m} + G\sigma_{j,m}, \quad \sigma_{j,m} \sim \mathcal{N}(0, \mathbf{R}^o), \quad (20)$$

where $G$ is an observation operator that maps the model state to the observation state space and $\sigma_{j,m}$ are eight-dimensional independent Gaussian white noises with mean zero and diagonal covariance matrix $\mathbf{R}^o$. Vertically, we consider four observation networks with specific $G$:

**SO (Surface Observations):** Here, we consider observing the wind, potential temperature at a surface height $z_s = 100$ m, and the equivalent boundary layer potential temperature $\theta_{eb}$. The corresponding observation operator is a $3 \times 8$ matrix $G$ with nonzero components

$$G_{1,1} = G(z_s), \quad G_{1,2} = G(2z_s),$$
$$G_{2,3} = G'(z_s), \quad G_{2,4} = 2G'(2z_s),$$
$$G_{3,5} = 1, \quad (21)$$

where $G, G'$ are the vertical baroclinic profiles defined in Section 2.

**SO+MT (Surface Observations + Middle Troposphere Temperature):** This observation network includes temperature at middle-troposphere height $z_m = 8$ km in addition to SO. The corresponding observation operator is a $4 \times 8$ matrix $G$ with nonzero components

$$G_{4,3} = G'(z_m), \quad G_{4,4} = 2G'(2z_m), \quad (22)$$

in addition to (21).

**SO+MTV (Surface Observations + Middle Troposphere Temperature & Velocity):** This observation network includes velocity at middle-troposphere height $z_m = 8$ km in addition to SO+MT. The corresponding observation operator is a $5 \times 8$ matrix $G$ with nonzero components

$$G_{5,1} = G(z_m), \quad G_{5,2} = 2G(2z_m). \quad (23)$$

in addition to (21) and (22).

**CO (Complete Observations):** This vertically complete observation network is defined with $G = \mathbf{I}$ for diagnostic purposes.

### 4.2. Filtering Algorithms

In this paper, we consider the simplest version of our reduced stochastic filters, the Mean Stochastic Model (MSM, Harlim and Majda 2008a, 2010a,b; Majda and Harlim 2012). The new feature in the present context is that we have multiple variables $\Psi_j$ as opposed to a scalar field and therefore we need to design the MSM in an appropriate coordinate expansion to avoid parameterizing various coupling terms.

As in Harlim and Majda (2008a), our design of the filter prior model is based on the standard approach for modeling turbulent fluctuations (Majda et al. 1999; Majda and Timofeyev 2004; DelSole 2004; Majda et al. 2008), that is, we introduce model errors through linearizing the nonlinear models about a frozen constant state and replacing the truncated nonlinearity with a dissipation and spatially correlated noise (white in time) to mimic rapid energy transfer between different scales. In the present context, we consider the linearized multicloud model about the RCE,

$$\frac{d\Psi'}{dt} = \mathcal{P}(\partial_x)\Psi', \quad (24)$$

where $\Psi'$ denotes the perturbation field about the RCE and $\mathcal{P}$ denotes the linearized differential operator of the multicloud model at RCE. A comprehensive study of the
linear stability analysis of (24) involves solving eigenvalues of an $8 \times 8$ dispersion matrix, $\omega(k)$, and was reported in Majda et al. (2007) for the MJO-like wave and in Khouider and Majda (2008a); Frenkel et al. (2011a) for the multicloud model with enhanced congestus heating.

Consider a numerical discretization for (24) with spatial mesh size of $\Delta x = 2000$ km such that the model state space is essentially similar to the observation state space. With this approximation, the PDE in (24) becomes

$$\frac{d\Psi_k}{dt} = i\omega(k)\Psi_k, \quad |k| \leq M/2 = 10, \quad (25)$$

where \{ $\hat{\Psi}_k$ \}$_{|k| \leq M/2}$ are the discrete Fourier components of \{ $\Psi_j$ \}$_{j=1,...,M}$. Now consider an eigenvalue decomposition, $i\omega(k)Z_k = Z_k \Lambda_k$, where $\Lambda_k$ is a diagonal matrix of the eigenvalues and $Z_k$ is a matrix whose columns are the corresponding eigenvectors. Then we can write (25) as a diagonal system,

$$\frac{d\Phi_k}{dt} = \Lambda_k \Phi_k, \quad |k| \leq M/2 = 10, \quad (26)$$

with the following transformation

$$\hat{\Phi}_k = Z_k^{-1} \hat{\Psi}_k. \quad (27)$$

4.2.1. The MSM-Filter

The Mean Stochastic Model is defined through the following stochastic differential system,

$$d\Phi_k = \left[ (-\Gamma_k + i\Omega_k)\Phi_k + f_k \right] dt + \Sigma_k dW_k, \quad (28)$$

for $|k| \leq M/2$. In (28), $\Gamma_k, \Omega_k,$ and $\Sigma_k$ are diagonal matrices with diagonal components obtained through regression fitting to the climatological statistics while the forcing term is proportional to the climatological mean field, $f_k = (\Gamma_k - i\Omega_k)\langle \Phi_k \rangle$; here, the angle bracket $\langle \cdot \rangle$ denotes an average. Notice that the realizability of this stochastic model (referred as MSM-1 Majda et al. 2010; Harlim and Majda 2010b; Majda and Harlim 2012) is guaranteed since $\Gamma_k$ is always positive definite as opposed to the alternative approach which sets $\Omega_k = -i\Lambda_k$ (Penland 1989; DelSole 2000). Throughout this paper, the climatological statistics are computed from solutions of full multicloud model resolved at 40 km grid points with different temporal resolutions for the two cases: the MJO-like traveling wave and the initiation of a convectively coupled wave train (see Sections 5.1 and 5.2).

The discrete-time Kalman filtering problem with the MSM as the prior model is defined for each horizontal wavenumber $k$ as follows

$$\hat{\Psi}_{k,m} = \mathcal{F}_k(\Delta t)\hat{\Psi}_{k,m-1} + g_{k,m} + \eta_{k,m}, \quad (29)$$

$$G\hat{\Psi}_{k,m}^o = G\hat{\Psi}_{k,m} + G\hat{\sigma}_{k,m}, \quad (30)$$

where the observation model in (30) is the discrete Fourier component of the canonical observation model in (20) with Gaussian noises, $\hat{\sigma}_{k,m} \sim N(0, R^o/M)$. The discrete filter model in (29) has coefficients

$$\mathcal{F}_k(t) = Z_k \exp \left( (-\Gamma_k + i\Omega_k)t \right) Z_k^{-1}, \quad (31)$$

$$g_{k,m} = -(I - \mathcal{F}_k(t_m))( -\Gamma_k + i\Omega_k)^{-1}f_k, \quad (32)$$

and unbiased Gaussian noises $\eta_{k,m}$ with covariance matrix

$$Q_k = \frac{1}{2} Z_k \Sigma_k^2 \Gamma_k^{-1} (I - |\mathcal{F}_k(\Delta t)|^2) Z_k^*. \quad (33)$$

These coefficients are obtained by evaluating the analytical solutions of the stochastic differential system in (28) at observation time interval $\Delta t = t_{m+1} - t_m$ and applying the transformation in (27).
The MSM-filter in (29)-(30) is computationally very cheap since it only involves $M/2 + 1$ independent $8 \times 8$ Kalman filtering problems, ignoring cross-correlations between different horizontal wavenumbers. Such a diagonal approximation may seem to be counterintuitive since it generates severe model errors but we have shown that it provides high filtering skill beyond the perfect model simulations in various contexts including the regularly spaced sparse observations (Harlim and Majda 2008b), irregularly spaced sparse observations (Harlim 2011), strongly chaotic nonlinear dynamical systems (Harlim and Majda 2008a, 2010a), and midlatitude baroclinic wave dynamics (Harlim and Majda 2010b).

Applying the Kalman filter formula on each wavenumber in (29)-(30) provides the following background (or prior) mean and error covariance estimates,

$$
\hat{\Psi}^b_{k,m} = \mathcal{F}_k(\Delta t) \hat{\Psi}^a_{k,m-1} + g_{k,m},
$$

and analysis (or posterior) mean and error covariance estimates

$$
\hat{\Psi}^a_{k,m} = \hat{\Psi}^b_{k,m} + K_{k,m} (G \hat{\Psi}^a_{k,m} - G \hat{\Psi}^b_{k,m}),
$$

$$
R^b_{k,m} = (I - K_{k,m} G) R^a_{k,m},
$$

$$
K_{k,m} = R^b_{k,m} G^* (G (R^b_{k,m} + R^a/M) G^*)^{-1},
$$

where $K_{k,m}$ is the Kalman gain matrix.

### 4.2.2. The complete 3D-VAR

For diagnostic purposes, we also consider a 3D-VAR version in the MSM framework above. That is, we simply set the background error covariance matrix to be independent of time,

$$
B_k \equiv \lim_{\Delta t \to \infty} R^b_{k,m} = \frac{1}{2} Z_k \Sigma_k^2 G_k^{-1} Z_k^*,
$$

and repeat the mean prior and posterior updates in (34), (36) with a constant Kalman gain matrix,

$$
K_k = B_k G^* (B_k + R^a/M) G^*)^{-1}.
$$

We called this approach the complete 3D-VAR because the forward model parameters in (31), (32), (33) and the background covariance matrix in (37) are determined from complete solutions of the multicloud model in (1), including the moisture and heating variables from (2), (7), (8). This formulation is significantly different from an earlier approach with variational techniques (Zagar et al. 2004b) in which the background covariance matrix is parameterized in an eigenmode basis constructed from the dry equatorial waveguide.

### 4.2.3. The “dry and cold” 3D-VAR

To mimic the approach in (Zagar et al. 2004b), we consider only using the wind and temperature data, $u_1, u_2, \theta_1, \theta_2, \theta_{eb}$, to construct the “dry and cold” eigenmode basis and background covariance matrix $B_k$. Technically, we still use the MSM model in (28) but replace the transformation in (27) with

$$
\hat{\Phi}^{dc}_{k} = Z_k^{-1} \begin{bmatrix} I_{5 \times 5} & 0 \\ 0 & 0 \end{bmatrix} \hat{\Psi}_k.
$$

In this sense, the parameters $\Gamma_k, \Omega_k,$ and $\Sigma_k$ in (28) are fitted to climatological statistics of $\hat{\Phi}^{dc}_{k}$ based on only the wind and temperature variables. Repeating the 3D-VAR algorithm described above in this setup provides an honest
“dry and cold” version analogous to the earlier approach in Zagar et al. (2004b,a).

Besides the eigenmode basis difference, we should note that the “dry and cold” 3D-VAR here is computationally much cheaper than that in Zagar et al. (2004b,a) since we perform both the prior and posterior updates in the diagonalized Fourier basis with reduced stochastic filters through (34)-(36) as opposed to their approach that propagates the nonlinear dry shallow water equations in physical space and applies the analysis step in the spectral diagonal basis. On each data assimilation step, their approach requires back-and-forth transformations in between the physical and spectral spaces with a rotational transformation matrix that is quite often ill-conditioned as reported in Zagar et al. (2004b).

For diagnostic purposes, we will also consider the “moist and cold” 3D-VAR in the numerical simulations in Section 5.1; this model is constructed exactly like the “dry and cold” model described above with moisture $q$ in addition to the wind and temperatures, $u_1, u_2, \theta_1, \theta_2, \theta_{eb}, q, H_s, H_c, P$ obtained from the true solutions of the test model in Section 3.1 and the posterior mean estimates in (36). The moving average is taken in a reference frame at 6.1 ms$^{-1}$ from time period of 750-1000 days. In Figures 4-8, we show the moving average from assimilations with observation time interval of 24 hours for complete observations (CO) with $R^o = 0$, and for all observation networks discussed in Section 4.1, CO, SO+MTV, SO+MT, SO with small observation noise covariance, $R^o > 0$. For observation network CO without observation errors, $R^o = 0$ (see Figure 4), the three schemes, MSM-filter, Complete 3D-VAR and “dry and cold” 3D-VAR, are identical and they perfectly recover the averaged MJO structure except for slight overestimation on the stratiform heating and precipitation.

In the presence of observation noise, we include results with “moist and cold” 3D-VAR (see the end of Section 4.2.3 for detailed discussion). We find that all the four schemes are able to recover $u_1$ and $\theta_{eb}$ with any observation network. When middle-troposphere wind observation is absent (see SO+MT and SO in Figures 7, 8), the estimate for $u_1$ slightly degrades but is completely wrong for $u_2$. The MSM-filter overestimates $\theta_2$ roughly by 0.1 K even with surface and middle-troposphere potential temperature observations; we find that this poor estimation is attributed to an inaccurate mean estimate (on the zeroth horizontal mode) of $\theta_2$. The MSM-filter, the Complete and “moist and cold” 3D-VAR are able to recover the oscillating structure of the moisture $q$ with any observation network (with slight errors for the
MSM-filter with SO) reflecting the active and suppressed convective phases of the MJO-like wave. On the other hand, the “dry and cold” 3D-VAR cannot produce $q$ accurately even with observation network CO and simply predicts a dry atmosphere (with zero moisture profile) when the moisture is unobserved. All the four filters are not able to reproduce the stratiform and congestus heating profiles when they are not observed.

Except for the surface observation (SO) network, both the Complete and “moist and cold” 3D-VAR are able to reasonably recover the precipitation rate (P) which in this model is exactly the deep convection heating rate; here, the “cold and dry” 3D-VAR precipitation estimate is very inaccurate (see Figures 5-7). On the other hand, the MSM-filter captures the peak of the precipitation on all the three observation networks: CO, SO+MTV, and SO+MT, but overestimates the profile on the last two observation networks. This overestimation on the precipitation (as well as those observed when we only assimilate the surface observation (SO) network (see Figure 8) can be explained as follows. From the precipitation budget in (9), it is obvious that the contributions of $\theta_{eb}$, $q$, and $\theta_2$ to the convective parameterization are small (with scale factors $a_1 = 0.1$, $a_2 = 0.5$, $a_0 \gamma_2 = 1.2$, respectively) relative to $\theta_1$ (with scale factor $a_0 = 12$). Therefore, the wet filtered state (with large precipitation estimates as seen in Figure 8) is attributed to the slight underestimation of the first baroclinic potential temperature, $\theta_1$. The Complete 3D-VAR underestimates $\theta_1$ by as much as 0.5 K; this yields spatially uniform precipitation rate of about 2.3 K day$^{-1}$. The MSM-filter underestimates $\theta_1$ by as much as 1.5 K and its corresponding precipitation estimate is about 20 K day$^{-1}$.

In Figures 9-12, we show the detailed vertical structure of the total potential temperature $\Theta$, the velocity vector field $(V, w)$, the total convective heating, and horizontal velocity from the MJO-like wave in Section 3.1 and the Complete 3D-VAR estimates with observation networks SO+MTV, SO+MT, and SO, respectively. In particular, the vertical tilted structure in the potential temperature is recovered with any of these three observation networks; similar recovery (not shown) is also obtained with the MSM-filter and the “moist and cold” 3D-VAR; the “dry and cold” 3D-VAR also recovers this tilted structure except with observation network SO. On the other hand, the tilted structure in the horizontal velocity with low level convergence that is in phase with the deep convective heating is not recovered whenever the middle-troposphere wind observation is absent. Notice also that the deep convective heating is recovered except with observation network SO; similar recovery (not shown) is also attained.
with the MSM-filter and the “moist and cold” 3D-VAR but not with the “dry and cold” 3D-VAR.

We also find that both the Complete and “moist and cold” 3D-VAR are able to reconstruct the detail precipitation structure in Figure 1 except when assimilated with observation network SO (results are not shown). The MSM-filter is able to capture the peak but overestimates the detail profile. The “moist and cold” 3D-VAR reproduces the eastward MJO-like signal but fails to capture the westward intermittent moist fluctuations within the MJO envelope as shown in Figure 1.

We also repeated the numerical experiments above with different observation time intervals ranging from 6 hours to 8 days with the Complete 3D-VAR and MSM-filter (see Figure 13 for the average RMS errors on the MSM-filter case). Particularly noteworthy is that the posterior estimates have roughly similar RMS errors for the observed variables independent of the observation times; for the unobserved variables, the RMS errors for the shorter observation times are larger than those for the longer observation times! This latter result can be understood as follows. The dynamical operator \( F_k \) in (31) is essentially marginally stable (with largest eigenvalue 0.9899) for \( \Delta t = 6 \) hours and is strictly stable (with largest eigenvalue 0.8836) for longer \( \Delta t = 72 \) hours. The observability condition, which is a necessary condition for accurate filtered solutions when the dynamical operator is marginally stable (Anderson and Moore 1979; Majda and Harlim 2012), is practically violated here; our test with SO+MT observation network suggests...
that the observability matrix is ill-conditioned with 
\[ \det \left( [G^T (G F_k)^T] \right) \approx 10^{-20}. \] This explains why the longer observation times produce more accurate filtered solutions. Thus, with the crude spatial observation network and the inefficient behavior of MSM at short times, this simple filtering strategy necessarily cannot capture sub-grid scale features of the wave with high skill; by design this is also true for 3D-VAR. We encounter similar behavior of filtered solutions in the next example in Section 5.2.

In Figure 13, we include the climatological errors (dash-dotted line) and observation errors (thin dashes) for diagnostic purposes. Recall that the observation error covariance \( R^o \) in our experiments is 10\% of the climatological variances and the observation errors are only relevant for diagnostic purposes when the corresponding variable is observed. So, in real-time, the MSM-filter with sparse observation networks SO+MT, SO+MT has reasonable skill as long as its RMS errors are below the climatological errors. In this sense, we observe that the MSM-filter is very skillful for variables any observation network as well as for \( \theta_1 \) and \( q \) for any observation network as well as for \( \theta_1 \) for observation networks other than SO. Our conjecture is that on these variables, the RMS errors will increase as the observation time interval is near its slowest decaying time (70 days for this model). For the other variables, the filtering skill is not better than the climatological variability and further improvement will be addressed in the future work.
5.2. Initiation of a convectively coupled wave train

Here, our goal is to check the filtering skill in recovering the transient behavior of initiation of a convectively coupled wave train (Section 3.2) with the MSM forward model in (28) where parameters, (31)-(33), are specified from a time series at the climatological state for the period of time 500-1000 days with temporal resolution of 3 hours.

In Figures 14-17, we report the space-time plot of the filtered estimates at the initial period of time 0-50 days from the Complete 3D-V AR with observation time $\Delta t = 24$ hours, observation noise variance $R^o > 0$, and observation networks SOMTV, SOMT, and SO. By eye-sight, we can see that the emerging pattern in Figure 2 is recovered for all variables except for the deep convection heating rate with the complete observation network! This poor estimate is attributed to an overestimation of $\theta_1$ (which sets the available convective heating $Q_d$ in (15) to zero). In this case, the precipitation budget in the filtered solution is dominated by the stratiform and congestus heating rates. On the other hand, even if the pattern of $H_d$ is always captured with networks SO+MTV, SO+MT, SO, its accuracy is questionable as we will see below.

To be more precise, we quantify the filter skill with the average RMS error and pattern correlation (between the posterior mean estimate and the truth) at the initiation period of time 0-75 days before these waves lock into a wave train of six waves as shown in Figure 3. In Figures 18-25, we plot these two performance measures as functions of observation times for observation networks CO, SO+MTV, SO+MT, and SO, respectively. In each panel, we compare

Figure 9. The true vertical profile of the MJO-like waves computed with moving average is in a reference frame at 6.1 ms$^{-1}$. The contour intervals are 0.07 K for the potential temperature, 0.29 K day$^{-1}$ for the total convective heating, and 1 ms$^{-1}$ for the horizontal velocity. Solid (dashes) contours denote positive (negative) values.

Figure 10. The vertical profile from Complete 3D-V AR estimate with observation network SO+MTV and $R^o > 0$, and $\Delta t = 24$ hours. The contour details are similar to those in Fig 9.
Figure 11. The vertical profile from Complete 3D-V AR estimate with observation network SO+MT and $R^o > 0$, and $\Delta t = 24$ hours. The contour details are similar to those in Fig 9.

Figure 12. The vertical profile from Complete 3D-V AR estimate with observation network SO and $R^o > 0$, and $\Delta t = 24$ hours. The contour details are similar to those in Fig 9.

The average RMS errors for simulations with $R^o > 0$ (dashes with markers) for variables $\theta_1, \theta_2, q, H_c$ decay as functions of observation time even with complete observation network (see Figure 18). We find that the larger errors with shorter observation times here are attributed to the violation of practical controllability (Anderson and Moore 1979; Harlim and Majda 2008b; Majda and Harlim 2012) which is also a necessary condition for optimal filtering when the system is marginally stable (here $\mathcal{F}_k$ has maximum eigenvalue close to 1). Additionally, we observe a similar decaying pattern for the error as function of observation time with SO+MTV, SO+MT, and SO for the unobserved variables $q, H_c, H_d$, and $P$ even when the observed wind and potential temperatures have no errors ($R^o = 0$). Here, the larger errors in the unobserved variables for shorter observation
times are attributed to the violation of practical observability as explained in Section 5.1.

When $R^o > 0$, the RMS errors of the deep convection heating rate $H_d$ are roughly 1 K day$^{-1}$ (see Figure 18) with observation network CO but the pattern correlations (PC) are roughly zero (the PC curves are below 0.5 in Figure 19). The PC confirms the inability to recover $H_d$ as shown in Figure 14. Both filtering schemes with the other observation networks (SO+MTV, SO+MT, SO) recover the structure of $H_d$ (with PC of roughly 0.6 from Figures 21, 23, 25) but their errors are very large (as much as 10 K from Figures 20, 22, 24). The failure to even capture the deep convection pattern with CO is attributed to overestimation of $\theta_1$ as explained before by contrasting the detailed space-time structure of $\theta_1$ in Figures 2 and 14. The average RMS errors show such a tendency for failure with larger error with CO compared to those with the other networks, but they don’t inform us whether the potential temperature estimates are warmer or colder than the truth which is important for accurate precipitation estimation.

Finally, notice that with networks SO+MTV, SO+MT, and SO, the RMS errors of the unobserved variables for assimilation with $R^o > 0$ are larger than those with $R^o = 0$; for example, see the errors for variables $\theta_1$, $q$, $H_c$, $H_d$, $P$ in Figure 24. We find that these larger errors with $R^o = 0$ are due to an ill-conditioned Kalman gain matrix in (36) with sparse observation networks with operator, $G \in \mathbb{R}^{S \times 8}$, $S < 8$. 

Figure 13. Average RMS errors as functions of observation time interval (in days). Observation error (thin dashes), climatological errors (dash-dotted line), CO (thick solid line), SO+MTV (thick dashes), SO+MT (circles) and SO(squares).

Figure 14. Space-time plot from the Complete 3D-VAR estimate with observation network CO and $R^o > 0$, and $\Delta t = 24$ hours. The contour intervals are 0.25 ms$^{-1}$ for the zonal wind, the temperature, 0.025 K for the potential temperature and humidity, and 0.05 K day$^{-1}$ for the heating rates and precipitation. Solid black (dash grey) contours denote positive (negative) values for $u_1, u_2, \theta_1, \theta_2, q$. Solid black (dash grey) contours denote heating rates greater (smaller) than 1 K day$^{-1}$ for $H_c, H_d, P$. 

Copyright © 0000 Royal Meteorological Society
6. Summary and concluding discussion

In this paper, we use multicloud models (Khouider and Majda 2006a, 2007; Majda et al. 2007; Khouider and Majda 2008a) as the test models for filtering moist tropical convection. In particular, we aim to establish guidelines for the future design of filtering schemes in assimilating and predicting tropical atmospheric dynamics. We view the multicloud model, with convective parameterization which includes three cloud types, congestus, deep, and stratiform, above the boundary layer, as a candidate for the simplest toy model for moisture-coupled tropical waves (analogous to the Lorenz 96 for the midlatitude weather dynamics) for the following reasons:

It is very successful in capturing most of the spectrum of the convectively coupled waves (Kiladis et al. 2009; Khouider and Majda 2008a,b) as well as the nonlinear organization of large scale envelopes mimicking across scale interactions of the MJO (our first example in Section 3.1) and convectively coupled waves (our second example in Section 3.2). More importantly, this model also captures the vertical profile with front and rear tilting, the phase speed, and dispersion relations that match the observational record (Kiladis et al. 2005, 2009).

Here, we demonstrate the filtering skill with a suite of reduced stochastic filters with model errors, based on linear stochastic models (Harlim and Majda 2008a, 2010a; Majda and Harlim 2012) in capturing the intraseasonal MJO-like wave (Majda et al. 2007) and the transient initiation of a convectively coupled wave train that
resembles the results from simulations with a Cloud Resolving Model (Grabowski and Moncrieff 2001). From these numerical experiments, we find the following facts:

1. The key factor for accurate precipitation estimates is an accurate estimation of the first baroclinic potential temperature. Our test problems suggests that slight overestimation in $\theta_1$ produces a dry atmosphere with no rain at all and slight underestimation in $\theta_1$ produces a wet atmosphere with unrealistic high precipitation rate;

2. Our simple reduced stochastic filters are able to recover moisture and precipitation field profile (even when online observations of these variables are not available) provided that the filter forward prior model is designed in a moisture coupled eigenmode basis. This result suggests that the future design of tropical data assimilation algorithms should account for a moisture coupled eigenmode basis.

Figure 17. Space-time plot from the Complete 3D-V AR estimate with observation network SO and $R^O > 0$, and $\Delta t = 24$ hours. The contour details are similar to those in Fig 15.

Figure 18. RMS errors as functions of observation time interval for observation network CO. MSM-filter (grey), Complete 3D-V AR (black), $R^O = 0$ (solid line and dashes without any markers), $R^O > 0$ (dashes with square/circle markers).

Figure 19. Pattern correlations as functions of observation time interval for observation network CO. MSM-filter (grey), Complete 3D-V AR (black), $R^O = 0$ (solid line and dashes without any markers), $R^O > 0$ (dashes with square/circle markers).
Figure 20. RMS errors as functions of observation time interval for observation network SO+MTV. MSM-filter (grey), Complete 3D-VAR (black), $R^o = 0$ (solid line and dashes without any markers), $R^o > 0$ (dashes with square/circle markers).

Figure 21. Pattern correlations as functions of observation time interval for observation network SO+MTV. MSM-filter (grey), Complete 3D-VAR (black), $R^o = 0$ (solid line and dashes without any markers), $R^o > 0$ (dashes with square/circle markers).

Figure 22. RMS errors as functions of observation time interval for observation network SO+MT. MSM-filter (grey), Complete 3D-VAR (black), $R^o = 0$ (solid line and dashes without any markers), $R^o > 0$ (dashes with square/circle markers).

Figure 23. Pattern correlations as functions of observation time interval for observation network SO+MT. MSM-filter (grey), Complete 3D-VAR (black), $R^o = 0$ (solid line and dashes without any markers), $R^o > 0$ (dashes with square/circle markers).
instead of dry eigenmode basis as in Zagar et al. (2004b,a); (3) A better estimate for the tropical convection wave patterns requires more than surface wind and potential temperature observations; (4) The skill of the reduced filtering methods with horizontally and vertically sparse observations suggests that more accurate filtered solutions are achieved with less frequent observation times. Such a counterintuitive finding is justified through an analysis of the classical observability and controllability conditions which are necessary for optimal filtering especially when the observation timescale is too short relative to the timescale of the true signal.

We hope that the encouraging results in this paper can convince researchers who are interested in tropical data assimilation to investigate: (1) The potential of improving the estimate accuracy with more sophisticated filtering schemes through the test models here; this includes testing with the ensemble Kalman filters (Evensen 1994; Anderson 2001; Bishop et al. 2001; Hunt et al. 2007), particle filters with small ensemble sizes (van Leeuwen 2010; Anderson 2010), and other reduced stochastic filters (Gershgorin et al. 2010b,a; Majda and Harlim 2012) or the Gaussian closure filter (Branicki et al. 2012) with stochastic parameterizations that account for model errors “on-the-fly”; (2) Exploring the filtering skill for other parameter regimes in the multicloud models with more realistic sea surface temperature profile, active barotropic dynamics and coupled nonlinear advection which allows for tropical-extratropical wave interactions (Lin et al. 2000; Majda and Biello 2003; Biello and Majda 2004); (3) Using the test models to design appropriate filters to cope with various observation networks with irregularly spaced sparse observations (Harlim 2011) and satellite measurements.

**Acknowledgement**

The authors thank Boualem Khouider, Samuel Stechmann, and Yevgeniy Frenkel for sharing the relevant multicloud

![Figure 24. RMS errors as functions of observation time interval for observation network SO. MSM-filter (grey), Complete 3D-VAR (black), \( R^2 = 0 \) (solid line and dashes without any markers), \( R^2 > 0 \) (dashes with square/circle markers).](image1)

![Figure 25. Pattern correlations as functions of observation time interval for observation network SO. MSM-filter (grey), Complete 3D-VAR (black), \( R^2 = 0 \) (solid line and dashes without any markers), \( R^2 > 0 \) (dashes with square/circle markers).](image2)
model source codes. The research of J.H. is partially supported by the Office of Naval Research Grant N00014-11-1-0310, the NC State startup fund, and the NC State Faculty Research and Professional Development fund. The research of A.J.M. is partially supported by the National Science Foundation Grant DMS-0456713 and the Office of Naval Research Grants ONR DRI N00014-10-1-0554 and N00014-11-1-0306.

References


