CONGRUENCE PROPERTIES MODULO 5 AND 7 FOR THE POD FUNCTION

SILVIU RADU AND JAMES A. SELLERS

Abstract. In this paper, we prove arithmetic properties modulo 5 and 7 satisfied by the function pod(n) which denotes the number of partitions of n wherein odd parts must be distinct (and even parts are unrestricted). In particular, we prove the following: For all n ≥ 0,

\[ \text{pod}(135n + 8) \equiv 0 \pmod{5}, \]
\[ \text{pod}(135n + 107) \equiv 0 \pmod{5}, \]
\[ \text{pod}(135n + 116) \equiv 0 \pmod{5}, \]
\[ \text{pod}(675n + 647) \equiv 0 \pmod{25}, \]
\[ \text{pod}(3375n + 1997) \equiv 0 \pmod{125}, \]
\[ \text{pod}(3375n + 3347) \equiv 0 \pmod{125}, \]
\[ \text{pod}(567n + 260) \equiv 0 \pmod{7}, \]
\[ \text{pod}(567n + 449) \equiv 0 \pmod{7}. \]

1. Introduction

The focus of this paper is the function pod(n) which denotes the number of partitions of n in which odd parts are distinct (and even parts are unrestricted). This function pod(n) has been considered by many from a product-series point of view as well as from other directions. For example, pod(n) appears in the works of Andrews [2, 3] and Berkovich and Garvan [6]. Moreover, Berkovich and Garvan note that Andrews [5] considered a restricted version of pod(n) wherein each part was required to be larger than 1. In very recent work, Alladi [1] obtained a series expansion for the product generating function for pod(n). It is significant to note that Hirschhorn and Sellers [7] appear to be the first to consider pod(n) from an arithmetic viewpoint.

In contrast to the work of Hirschhorn and Sellers [7], in which pod(n) was extensively studied modulo 3, we now wish to prove Ramanujan-like properties modulo 5 and 7 which are satisfied by pod(n). In particular, we prove the following theorem:

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Theorem 1.1. For all \( n \geq 0 \),

(1) \( \text{pod}(135n + 8) \equiv \text{pod}(135n + 107) \equiv \text{pod}(135n + 116) \equiv 0 \pmod{5} \),

(2) \( \text{pod}(675n + 647) \equiv 0 \pmod{25} \),

(3) \( \text{pod}(3375n + 1997) \equiv \text{pod}(3375n + 3347) \equiv 0 \pmod{125} \),

and

(4) \( \text{pod}(567n + 260) \equiv \text{pod}(567n + 449) \equiv 0 \pmod{7} \).

For the proof of our congruences we need the following lemma.

Lemma 1.2. Let \( p \) be a prime and \( \alpha \) a positive integer. Then

(5) \( \prod_{n=1}^{\infty} \frac{(1 - q^n)^p^\alpha}{(1 - q^n p^\alpha)^{p^\alpha - 1}} \equiv 1 \pmod{p^\alpha} \).

Proof. We note that for all primes \( p \) and \( X \) an indeterminate we have

(6) \( X \equiv 1 \pmod{p^\alpha} \Rightarrow X^p \equiv 1 \pmod{p^{\alpha+1}} \).

We see that (5) is true for \( \alpha = 1 \) because of the relation \( (1 - q^n p) \equiv (1 - q^n)^p \pmod{p} \). Next we prove that if (5) is true for \( \alpha = N \) with \( N \geq 1 \), then (5) is true for \( \alpha = N + 1 \). This follows by applying (6) with \( X = \prod_{n=1}^{\infty} \frac{(1 - q^n)^{p^N}}{(1 - q^n p^{N-1})} \).

By elementary partition theory we see that

(7) \( \sum_{m=0}^{\infty} \text{pod}(m)q^m = \prod_{n=1}^{\infty} \frac{1 + q^{2n-1}}{1 - q^{2n}} \).

From here, we can prove some additional elementary generating function results which are critical to our proof of these congruences.

Lemma 1.3.

(8) \( \sum_{m=0}^{\infty} \text{pod}(m)(-q)^m = \prod_{n=1}^{\infty} \frac{1 - q^n}{(1 - q^{2n})^2} \).

Proof. By (7) we find

\[
\sum_{m=0}^{\infty} \text{pod}(m)(-q)^m = \prod_{n=1}^{\infty} \frac{1 - q^{2n-1}}{1 - q^{2n}} = \prod_{n=1}^{\infty} \frac{1 - q^n}{1 - q^{2n} - 1 + q^{2n}} = \prod_{n=1}^{\infty} \frac{1 - q^n}{(1 - q^{2n})^2}. \]

\( \square \)
In order to prove the congruences (1)-(4) we could use Lemma 2.4 (below) directly. However, experiments show that a simple pre-processing of the congruences before the application of Lemma 2.4 gives us a proof where fewer computations are required. For this purpose we use the following related generating function and lemma to rewrite (1)-(4) in a form more convenient for us.

**Definition 1.4.** For all positive integers \( \alpha \) and primes \( p \) we define
\[
\sum_{m=0}^{\infty} \text{pod}_{\alpha,p}(m)q^m := \prod_{n=1}^{\infty} \frac{(1-q^n)^{p^{\alpha-1}}}{(1-q^{p^n})^{p^{\alpha-1}}},
\]

**Lemma 1.5.** The congruences (1)-(4) are true iff, for all \( n \geq 0 \),
\[
\text{pod}_{1,5}(135n+8) \equiv \text{pod}_{1,5}(135n+107) \equiv \text{pod}_{1,5}(135n+116) \equiv 0 \pmod{5},
\]
\[
\text{pod}_{2,5}(675n+647) \equiv 0 \pmod{25},
\]
\[
\text{pod}_{3,5}(3375n+1997) \equiv \text{pod}_{3,5}(3375n+3347) \equiv 0 \pmod{125},
\]
and
\[
\text{pod}_{1,7}(567n+260) \equiv \text{pod}_{1,7}(567n+449) \equiv 0 \pmod{7}.
\]

**Proof.** The lemma follows immediately by observing that
\[
\text{pod}_{\alpha,p}(n) \equiv (-1)^n \text{pod}(n) \pmod{p^\alpha},
\]
which follows from Lemma 1.2, Lemma 1.3 and Definition 1.4.

2. The Main Proof Machinery – Modular Forms

For \( M \) a positive integer let \( R(M) \) be the set of integer sequences indexed by the positive divisors \( \delta \) of \( M \). Let \( 1 = \delta_1 < \cdots < \delta_k = M \) be the positive divisors of \( M \) and \( r \in R(M) \). Then we will write \( r = (r_1, \ldots, r_k) \).

For \( s \) an integer and \( m \) a positive integer we denote by \([s]_m\) the set of all elements congruent to \( s \) modulo \( m \), in other words \([s]_m \in \mathbb{Z}_m \). Let \( \mathbb{Z}_m^* \) be the set of all invertible elements in \( \mathbb{Z}_m \). Let \( S_m \subset \mathbb{Z}_m^* \) be the set of all squares in \( \mathbb{Z}_m \).

**Definition 2.1.** For \( m \in \mathbb{N}^* \), \( r = (r_k) \in R(M) \) and \( t \in \{0, \ldots, m-1\} \) we define the map \( \overline{\text{r}} : S_{24m} \times \{0, \ldots, m-1\} \rightarrow \{0, \ldots, m-1\} \) with \([s]_{24m} \mapsto [s]_{24m} \overline{r} t \) and the image is uniquely determined by the relation \([s]_{24m} \overline{r} t \equiv ts + \sum_{\delta | M} \delta r_\delta (\text{mod} \ m) \). We define the set
\[
P_{m,r}(t) := \{[s]_{24m} \overline{r} t | [s]_{24m} \in S_{24m}\}.
\]

Let \( a \in \mathbb{Z} \) and \( p \) an odd prime, then \( \left( \frac{a}{p} \right) \) is the Legendre symbol.

**Lemma 2.2.** Let \( p \geq 5 \) be a prime and \( \alpha \) a positive integer. Let
\[
r^{(\alpha,p)} := (r_1^{(\alpha,p)}, r_2^{(\alpha,p)}, r_p^{(\alpha,p)}) = (1, p^\alpha, -2, -p^{\alpha-1}) \in R(2p).
\]
Let \( a, b \) be positive integers, \( m := 3^a p^b \) and \( g := \text{gcd}(m, 8t - 1) \). Then if
\[
3^{a-1} p^{b-1} | (8t - 1)
\]
we have

\[ P_{m,r^{(\alpha,p)}}(t) = \left\{ t' \mid g((8t'-1), \frac{(8t'-1)/g}{p}) = \frac{(8t'-1)/g}{p}, \text{ for each } p \mid \frac{m}{g} \right\}. \]

**Proof.** By Definition 2.1 we have

\[ P_{m,r^{(\alpha,p)}}(t) = \left\{ t' \mid t' \equiv ts + \frac{s-1}{24} \sum_{\delta|m} \delta r^{(\alpha,p)} \pmod 1, 0 \leq t' \leq m-1, [s]_{24m} \in S_{24m} \right\} \]

\[ = \left\{ t' \mid t' \equiv ts + \frac{1-s}{8} \pmod 1, 0 \leq t' \leq m-1, [s]_{24m} \in S_{24m} \right\} \]

\[ = \left\{ t' \mid s(8t-1) \equiv 8t'-1 \pmod 1, 0 \leq t' \leq m-1, [s]_{24m} \in S_{24m} \right\} \]

\[ = \left\{ t' \mid g((8t'-1), s(8t-1)/g) = (8t'-1)/g, 0 \leq t' \leq m-1, [s]_{m/g} \in S_{m/g} \right\}. \]

The proof is finished by noting that the existence of \([s]_{m/g} \in S_{m/g} \) such that

\[ s(8t-1)/g \equiv (8t'-1)/g \pmod m/g \]

is, for the case \( \frac{m}{g} \) squarefree, equivalent to

\[ \left( \frac{8t-1}{g} \right) = \left( \frac{8t'-1}{g} \right), \]

for each \( p \mid \frac{m}{g} \). We also used the fact that the canonical homomorphism \( \phi : S_n \to S_{n/d} \) is surjective for any positive integers \( n, d \) such that \( d \mid n \). \( \square \)

We now use Lemma 2.2 to compute \( P_{m,r^{(\alpha,p)}}(t) \) for

\( (\alpha, p, m, t) = (1, 5, 135, 8), (1, 5, 135, 107), (2, 5, 675, 647), (3, 5, 3375, 1997), (1, 7, 567, 260) \) and \( (1, 7, 567, 449) \).

\( (\alpha, p, m, t) = (1, 5, 135, 8) : \) We see that \( g = \gcd(135, 8-1) = 3^2 \) and \( \left( \frac{8t-1}{g} \right) \)

is \( \left( \frac{2}{3} \right) = -1 \) for \( p = 5 \) and \( \left( \frac{2}{3} \right) = 1 \) for \( p = 3 \). By Lemma 2.2 we need to solve the following equations for \( t' \):

\[ \left( \frac{(8t'-1)/g}{5} \right) = \left( \frac{2}{5} \right) \quad \text{and} \quad \left( \frac{(8t'-1)/g}{3} \right) = \left( \frac{1}{3} \right). \]

We see that \( \left( \frac{2}{5} \right) = -1 \) has the solutions \( x = 2, 3 \pmod 5 \) and \( \left( \frac{1}{3} \right) = 1 \) has the solution \( x \equiv 1 \pmod 3 \). By the Chinese Remainder Theorem we obtain \( x \equiv 7, 13 \pmod {15} \). Consequently we need to solve the following congruences for \( t' \):

\[ (8t'-1)/g \equiv 7, 13 \pmod {15}, \]

which is equivalent to

\[ (8t'-1) \equiv 7g, 13g \pmod {15g}, \]

and hence

\[ t' \equiv (1 + 7g)/8, (13g + 1)/8 \pmod {15g}. \]

Finally using \( g = 9 \) we obtain \( t' \equiv 8, 116 \pmod {135} \). This shows that

\[ P_{135,r^{(1,5)}}(8) = \{8, 116\}. \]
Throughout when we say that $c \equiv r \pmod{p}$ first state it and then explain the terminology.

In order to prove the congruences (1)-(4) we need a lemma ([8, Lemma 4.5]). We need to solve the following equation for $t'$:

$$\left(\frac{(8t' - 1)/g}{3}\right) = \left(\frac{1}{3}\right)$$

This gives

$$(8t' - 1)/g \equiv 1 \pmod{3} \Rightarrow (8t' - 1) \equiv g \pmod{3} \Rightarrow t' \equiv (1 + g)/8 \pmod{3g}.$$  

Using $g = 45$ we obtain $t' \equiv 107 \pmod{135}$. We conclude

$$P_{135,r,t}(107) = \{107\}.$$  

Applying Lemma 2.2 in analogous fashion we obtain:

$$P_{675,r,t}(647) = \{647\},$$  

$$P_{3375,r,t}(1997) = \{1997, 3347\},$$  

$$P_{567,r,t}(260) = \{260\},$$  

$$P_{567,r,t}(449) = \{449\}.$$  

By using (8)-(13) and Lemma 1.5 we see that Theorem 1.1 can be rewritten as:

**Lemma 2.3.** The congruences in Theorem 1.1 are true iff, for all $n \geq 0$,

$$\begin{align*}
\text{pod}_{1,5}(135n + t) &\equiv 0 \pmod{5}, \quad t \in P_{135,r,t}(8), \\
\text{pod}_{1,5}(135n + t) &\equiv 0 \pmod{5}, \quad t \in P_{135,r,t}(107), \\
\text{pod}_{1,5}(675n + t) &\equiv 0 \pmod{5}, \quad t \in P_{675,r,t}(647), \\
\text{pod}_{3,5}(3375n + t) &\equiv 0 \pmod{5}, \quad t \in P_{3375,r,t}(1997), \\
\text{pod}_{1,7}(567n + t) &\equiv 0 \pmod{7}, \quad t \in P_{567,r,t}(260), \\
\text{pod}_{1,7}(567n + t) &\equiv 0 \pmod{7}, \quad t \in P_{567,r,t}(449).
\end{align*}$$

For each $r \in R(M)$ we assign a generating function

$$f_r(q) := \prod_{s \in M} \prod_{n=1}^{\infty} (1 - q^{hn})^{r_s} = \sum_{n=0}^{\infty} c_r(n)q^n.$$  

Given $p$ a prime, $m \in \mathbb{N}$ and $t \in \{0, \ldots, m - 1\}$ we are concerned with proving congruences of the type $c_r(mn + t) \equiv 0 \pmod{p}, n \in \mathbb{N}$. The congruences we are concerned with here have some additional structure; namely $c_r((mn + t') \equiv 0 \pmod{p}, n \geq 0, t' \in P_{m,r}(t)$. In other words a congruence is a tuple $(r, M, m, t, p)$ with $r \in R(M)$, $m \geq 1, t \in \{0, \ldots, m - 1\}$ and $p$ a prime such that

$$c_r(mn + t') \equiv 0 \pmod{p}, n \geq 0, t' \in P_{m,r}(t).$$  

Throughout when we say that $c_r(mn + t) \equiv 0 \pmod{p}$ we mean that $c_r(mn + t') \equiv 0 \pmod{p}$ for all $n \geq 0$ and all $t' \in P(t)$.

In order to prove the congruences (1)-(4) we need a lemma ([8, Lemma 4.5]). We first state it and then explain the terminology.
Lemma 2.4. Let $u$ be a positive integer, $(m, M, N, t, r, (r_s)) \in \Delta^*$, $a = (a_\delta) \in R(N)$, $n$ the number of double cosets in $\Gamma_0(N)\Gamma/\Gamma_\infty$ and $\{\gamma_1, \ldots, \gamma_n\} \subset \Gamma$ a complete set of representatives of the double coset $\Gamma_0(N)\Gamma/\Gamma_\infty$. Assume that $p_{m, r}(\gamma_i) + p_a^\ast(\gamma_i) \geq 0$, $i \in \{1, \ldots, n\}$. Let $t_{\min} := \min_{r \in P_{m, r}(t)} t'$ and

$$\nu := \frac{1}{24} \left( \sum_{\delta \mid N} a_\delta + \sum_{\delta \mid M} r_\delta \right) [\Gamma : \Gamma_0(N)] - \sum_{\delta \mid N} \delta a_\delta - \frac{1}{24m} \sum_{\delta \mid M} \delta r_\delta - \frac{t_{\min}}{m}.$$ 

Then if

$$\sum_{n=0}^{\lfloor \nu \rfloor} c_r(mn + t')q^n \equiv 0 \pmod{u}$$

for all $t' \in P_{m, r}(t)$ (mod $u$) for all $t' \in P_{m, r}(t)$.

The lemma reduces the proof of a congruence modulo $u$ to checking that finitely many values are divisible by $u$. We first define the set $\Delta^*$. Let $\kappa = \kappa(m) = \gcd(m^2 - 1, 24)$ and $\pi(M, (r_s)) := (s, j)$ where $s$ is a non-negative integer and $j$ an odd integer uniquely determined by $\prod_{\delta \mid M} \delta^{ir_s} = 2^s j$. Then a tuple $(m, M, N, (r_s), t)$ belongs to $\Delta^*$ iff

- $m, M, N$ are positive integers, $(r_s) \in R(M)$, $t \in \{0, \ldots, m - 1\}$;
- $p|m$ implies $p|N$ for every prime $p$;
- $\delta|M$ implies $\delta|mN$ for every $\delta \geq 1$ such that $r_\delta \neq 0$;
- $\kappa N \sum_{\delta \mid M} \frac{N}{r_\delta} \equiv 0 \pmod{24}$;
- $\kappa N \sum_{\delta \mid M} r_\delta \equiv 0 \pmod{8}$;
- $24m \gcd(\kappa(-24t - \sum_{\delta \mid M} \delta r_\delta), 24m) \mid N$;
- for $(s, j) = \pi(M, (r_s))$ we have $(4|\kappa N$ and $8|N s)$ or $(2|s$ and $8|N(1 - j)$ if $2|m$.

Remark 2.5. We note that the condition $2|m$ in the last line is not in the definition of $\Delta^*$ in [8]. However every result in [8] holds with no modification having this extra condition. In fact this condition was somehow missed in [8] when $\Delta^*$ was defined and although the results hold without it, in some cases we obtain less optimality.

Next we need to define the groups $\Gamma$, $\Gamma_0(N)$ and $\Gamma_\infty$:

$$\Gamma := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\},$$

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \mid N|c \right\}$$

for $N$ a positive integer, and

$$\Gamma_\infty := \left\{ \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} \mid h \in \mathbb{Z} \right\}.$$
For the index we have
\[(20) \quad [\Gamma : \Gamma_0(N)] := N \prod_{p \mid N} (1 + p^{-1})\]
(see, for example, [9]).

Finally for \(m, M, N\) positive integers, \(r \in R(M)\), \(a \in R(N)\) and \(\gamma = (a \ b \ c \ d)\) we define
\[p_{m,r}(\gamma) := \min_{\lambda \in \{0,\ldots,m-1\}} \frac{1}{24} \sum_{\delta \mid M} r_{\delta} g_{\delta(a + \kappa \lambda c), mc} \delta m\]
and
\[p^*_a(\gamma) := \frac{1}{24} \sum_{\delta \mid M} a_{\delta} g_{\delta, c} \delta m\].

**Lemma 2.6.** Let \(N\) be a squarefree integer. Then
\[\bigcup_{\delta \mid N} \Gamma_0(N) \left(\begin{array}{cc} 1 & 0 \\ \delta & 1 \end{array}\right) \Gamma_\infty = \Gamma.\]

**Proof.** Let \(\left(\begin{array}{cc} a & b \\ c & d \end{array}\right)\in \Gamma.\) Then if \(h \in \mathbb{Z}\) such that
\[(21) \quad c + (ch - d) \gcd(c, N) \equiv 0 \pmod{N}\]
we have
\[\left(\begin{array}{cc} a & b \\ c & d \end{array}\right) \left(\begin{array}{cc} 1 & -h \\ 0 & 1 \end{array}\right) \left(\begin{array}{cc} 1 & 0 \\ -\gcd(c, N) & 1 \end{array}\right) = \left(\begin{array}{cc} c + (ch - d) \gcd(c, N) & * \\ * & * \end{array}\right) \in \Gamma_0(N).\]
This implies that
\[\left(\begin{array}{cc} a & b \\ c & d \end{array}\right) \in \Gamma_0(N) \left(\begin{array}{cc} 1 & 0 \\ \gcd(c, N) & 1 \end{array}\right) \Gamma_\infty.\]
In particular we have proven that \(\gamma = \left(\begin{array}{cc} a & b \\ c & d \end{array}\right)\in \Gamma\) implies
\[\gamma \in \bigcup_{\delta \mid N} \Gamma_0(N) \left(\begin{array}{cc} 1 & 0 \\ \delta & 1 \end{array}\right) \Gamma_\infty,\]
if for any \(c, d \in \mathbb{Z}\) with \(\gcd(c, d) = 1\) there exists a \(h \in \mathbb{Z}\) such that (21) holds. Next observe that (21) is equivalent to
\[ch \equiv d \frac{c}{\gcd(c, N)} \pmod{N/\gcd(c, N)},\]
which has a solution if \(\gcd(c, N/\gcd(c, N)) = 1\). This is always true because \(N\) is squarefree. \(\square\)
3. The Congruences

We start by proving (14). We apply Lemma 2.4 with
\[(m, M, N, t, r) = (135, 10, 30, 8, (6, -2, -1)) \in \Delta^*,\]
and
\[a = (a_1, a_2, a_3, a_5, a_6, a_{10}, a_{15}, a_{30}) = (-7, 14, 2, 0, -4, 0, 0, 0).\]
For \(\delta \in \mathbb{Z}\) let \(\gamma_\delta := \left(\begin{array}{c} 1 \\ \delta \\ 0 \end{array}\right)\). Then by Lemma 2.6 a complete set of double coset representatives is contained in the set
\[\{\gamma_\delta : \delta|N\}.\]
Hence verifying the condition
\[p_{m,r}(\gamma_\delta) + p_\delta^*(\gamma_\delta) \geq 0\]
for each \(\delta|N\) is sufficient to fulfill the assumption of Lemma 2.4. This verification has been carried out using MAPLE. Next we obtain
\[
\nu = \frac{1}{24} \left( \sum_{\delta|N} a_\delta + \sum_{\delta|M} r_\delta \right) \left[ \Gamma : \Gamma_0(N) \right] - \sum_{\delta|N} \delta a_\delta - \frac{1}{24m} \sum_{\delta|M} \delta r_\delta - \frac{t_{\min}}{m}
\]
\[= \frac{1}{24} ((5 + 3) \cdot 72 - 3) + \frac{3}{24 \cdot 135} - \frac{8}{135}\]
\[= \frac{1429}{60}.\]
Here we have used (20) to compute \([\Gamma : \Gamma_0(30)] = (1 + 2)(1 + 3)(1 + 5) = 72\). This gives \([\nu] = 23\). By Lemma 2.4 we obtain that
\[(22) \quad pod_{1,5}(135n + 8) \equiv pod_{1,5}(135n + 116) \equiv 0 \quad (\text{mod} 5) \quad \text{for each} \quad 0 \leq n \leq 23\]
implies
\[pod_{1,5}(135n + 8) \equiv pod_{1,5}(135n + 116) \equiv 0 \quad (\text{mod} 5) \quad \text{for all} \quad n \geq 0.\]
We have verified (22) with MAPLE. This proves (14). In an analogous fashion, applying Lemma 2.4 we prove the congruences (15)-(19) below:

**Congruence (15).** We apply Lemma 2.4 with
\[(m, M, N, t, r) = (r_1, r_2, r_5) = (135, 10, 30, 107, (6, -2, -1))\]
and
\[a = (a_1, a_2, a_3, a_5, a_6, a_{10}, a_{15}, a_{30}) = (-7, 14, 2, 0, -4, 0, 0, 0).\]
We obtain \([\nu] = 23\) and \(P(t) = \{107\}\).

**Congruence (16).** We apply Lemma 2.4 with
\[(m, M, N, t, r) = (r_1, r_2, r_5) = (675, 10, 30, 647, (26, -2, -5))\]
and
\[a = (a_1, a_2, a_3, a_5, a_6, a_{10}, a_{15}, a_{30}) = (-32, 64, 10, 6, -20, -12, -2, -4).\]
We obtain \([\nu] = 109\) and \(P(t) = \{647\}\).

**Congruence (17).** We apply Lemma 2.4 with
\[(m, M, N, t, r) = (r_1, r_2, r_5) = (3375, 10, 30, 1997, (126, -2, -25))\]
and

$$a = (a_1, a_2, a_3, a_5, a_6, a_{10}, a_{15}, a_{30}) = (-159, 317, 54, 32, -106, -62, -11, 21).$$

We obtain $\lfloor \nu \rfloor = 554$ and $P(t) = \{1997, 3347\}.$

**Congruence (18).** We apply Lemma 2.4 with

$$(m, M, N, t, r) = (r_1, r_2, r_7)) = (567, 14, 42, 260, (8, -2, -1))$$

and

$$a = (a_1, a_2, a_3, a_7, a_6, a_{14}, a_{21}, a_{42}) = (-13, 26, 4, -8, 0, 0, 0).$$

We obtain $\lfloor \nu \rfloor = 55$ and $P(t) = \{260\}.$

**Congruence (19).** We apply Lemma 2.4 with

$$(m, M, N, t, r) = (r_1, r_2, r_7)) = (567, 14, 42, 449, (8, -2, -1))$$

and

$$a = (a_1, a_2, a_3, a_7, a_6, a_{14}, a_{21}, a_{42}) = (-13, 26, 4, -8, 0, 0, 0).$$

We obtain $\lfloor \nu \rfloor = 55$ and $P(t) = \{449\}.$

The above information is summarized in the following table:

<table>
<thead>
<tr>
<th>Cong.</th>
<th>$(m, M, N, t, r)$</th>
<th>$\lfloor \nu \rfloor$</th>
<th>$n$</th>
<th>$P(t)$</th>
</tr>
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<td>(14)</td>
<td>(135, 10, 30, 8, (6, -2, -1))</td>
<td>23</td>
<td>$(-7, 14, 2, 0, -4, 0, 0, 0)$</td>
<td>${8, 116}$</td>
</tr>
<tr>
<td>(15)</td>
<td>(115, 10, 30, 107, (6, -2, -1))</td>
<td>23</td>
<td>$(-7, 14, 2, 0, -4, 0, 0, 0)$</td>
<td>${107}$</td>
</tr>
<tr>
<td>(16)</td>
<td>(675, 10, 30, 647, (6, -2, -5))</td>
<td>109</td>
<td>$(-32, 64, 10, 6, -20, -12, -2, -4)$</td>
<td>${647}$</td>
</tr>
<tr>
<td>(18)</td>
<td>(567, 14, 42, 260, (8, -2, -1))</td>
<td>55</td>
<td>$(-13, 26, 4, -8, 0, 0, 0, 0)$</td>
<td>${260}$</td>
</tr>
<tr>
<td>(19)</td>
<td>(567, 14, 42, 449, (8, -2, -1))</td>
<td>55</td>
<td>$(-13, 26, 4, -8, 0, 0, 0, 0)$</td>
<td>${449}$</td>
</tr>
</tbody>
</table>

In each of the cases, we used MAPLE to verify that the congruences (14)-(19) hold up to the bound $\lfloor \nu \rfloor$. Thus, by Lemma 2.4 we have proven (14)-(19). Hence, by Lemma 1.5, we have proven Theorem 1.1.

### 4. Notes On Computations

In our proofs above, we needed to check the divisibility by $p^n$ of $pod_{\alpha, p}(n)$ for certain $\alpha, n \in \mathbb{N}$ and a prime $p$. However, we observe that

$$p^n|pod_{\alpha, p}(n) \iff p^n|pod(n) \iff p^n|(-1)^n\text{pod}(n).$$

These facts simplify the check of divisibility because we can build a nice recurrence for $(-1)^n\text{pod}(n)$ which we deduce in the following way. From Jacobi’s Triple Product Identity [4, Theorem 2.8], we see that

$$1 + \sum_{n=1}^{\infty} q^n(2n-1) + q^{n(2n+1)} = \sum_{n \in \mathbb{Z}} q^{n(2n+1)} = \prod_{n=1}^{\infty} \frac{(1 - q^{2n})^2}{1 - q^n}.$$

Together with Lemma 1.3, we have

$$\left(1 + \sum_{n=1}^{\infty} q^{n(2n-1)} + q^{n(2n+1)}\right) \left(\sum_{n=0}^{\infty} (-1)^n\text{pod}(n)q^n\right) = 1.$$
Therefore, by using the formula for the Cauchy product of two sequences and simplifying, one obtains the following for all positive integers $n$:

\[
(-1)^{n+1} \text{pod}(n) = \sum_{k \geq 1, k(2k-1) \leq n} \text{pod}(n - k(2k - 1))(-1)^{n-k} + \sum_{k \geq 1, k(2k+1) \leq n} \text{pod}(n - k(2k + 1))(-1)^{n-k}.
\]

This provides an extremely efficient method for calculating the values of $\text{pod}(n)$ which are needed to complete our proofs.

5. Closing Thoughts

It is truly satisfying to prove these congruences modulo 5 and 7 for the $\text{pod}$ function. However, our ultimate goal was to identify an infinite family of congruences modulo arbitrarily large powers of 5 or 7 satisfied by $\text{pod}(n)$. Unfortunately, we were unable to find such a family. We may return to this theme in the future.

References


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