SOME INPLACE IDENTITIES FOR INTEGER COMPOSITIONS

AUGUSTINE O. MUNAGI AND JAMES A. SELLERS

ABSTRACT. In this paper, we give two new identities for compositions, or ordered partitions, of integers. These two identities are based on closely-related integer partition functions which have recently been studied. Thanks to the structure inherent in integer compositions, we are also able to extensively generalize both of these identities. Bijective proofs are given and generating functions are provided for each of the types of compositions which arise. A number of arithmetic properties satisfied by the functions which count such compositions are also highlighted.

1. INTRODUCTION

In recent years, a pair of partition functions, usually denoted by $pod(n)$ and $ped(n)$, has been the focus of study by a number of mathematicians. These functions denote the number of partitions of n where the odd parts (respectively, the even parts) must be distinct. Arithmetic properties satisfied by $pod(n)$ and $ped(n)$ were recently proved by Hirschhorn and Sellers [7] and Andrews, Hirschhorn, and Sellers [2], respectively. Since then, numerous other authors have worked on these functions or close relatives thereof; the interested reader is directed to [3, 4, 8, 10, 12].

In particular Andrews, Hirschhorn, and Sellers established the following result.

Theorem 1.1. *The number of partitions of n in which each even part occurs with even multiplicity equals the number of partitions of n where no part is congruent to* $2 \pmod 4$.

It can be shown that the two classes of partitions in Theorem 1.1 are also enumerated by the function $pod(n)$.

The first goal of this paper is to find an analogue to Theorem 1.1 from the perspective of compositions (also known as ordered partitions). This work continues to explore the theme of a recent paper, by the first author [9], of discovering composition analogues of classical partition identities.

In order to state such an analogue in the present setting, we place a restriction on the compositions in question by speaking of certain parts being *inplace.* A part appears j times *inplace* in a composition if it appears in j **consecutive** positions in the composition. For example, in the composition $(2, 2, 2, 2, 3, 4, 4, 5, 6, 6, 2, 2, 3, 1)$, even parts appear inplace with even multiplicity while odd parts are inplace distinct. (Note that in the combinatorics of words, inplace parts correspond to "runs of identical letters", or "levels" as contrasted with rises and falls. See, for example, [6]). The term *inplace* is a more precise terminology adapted from the library of certain computer algebra systems.

Our first identity is the following seamless analogue of Theorem 1.1 for compositions.

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Theorem 1.2. *For all $n \geq 0$, the number of compositions of n when each even part occurs inplace with even multiplicity equals the number of compositions of n in which no part is congruent to* $2 \pmod 4$.

In contrast to Theorem 1.2, our second identity focuses on odd parts but relies on color compositions (see, for example [1]).

Theorem 1.3. *For all $n \geq 0$, the number of compositions of $2n$ such that each odd part appears inplace with even multiplicity equals the number of compositions of n where each odd part can be of two kinds.*

Example 1.4. *As an illustration of Theorem 1.3 let $n = 3$. Then the first set of compositions contains the following 14 objects:*
$(6), (3,3), (4,2), (4,1,1), (2,4), (1,1,4), (2,2,2), (2,2,1,1), (2,1,1,2), (1,1,2,2),$
$(2,1,1,1,1), (1,1,2,1,1), (1,1,1,1,2), (1,1,1,1,1,1)$
The second set of compositions contains these 14 objects:
$(3), (3^*), (2,1), (2,1^*), (1,2), (1^*,2), (1,1,1), (1,1,1^*), (1,1^*,1), (1^*,1,1),$
$(1,1^*,1^*), (1^*,1,1^*), (1^*,1^*,1), (1^*,1^*,1^*)$
Note that we designate the second "kind" of odd part with the use of an asterisk.

Theorem 1.2 is the special case of $k = \ell = 2$ of the following two-parameter generalization.

Theorem 1.5. *Let $k \geq 2$ and $\ell \geq 2$ be fixed integers. For all $n \geq 0$, the number of compositions of n when each part divisible by k occurs inplace with multiplicity a multiple of ℓ equals the number of compositions of n in which no part is congruent to $ik \pmod{\ell k}$, where $1 \leq i \leq \ell - 1$.*

Theorem 1.3 has the following natural generalization (for which the case $k = 2$ yields Theorem 1.3).

Theorem 1.6. *Let $k \geq 2$ be a fixed integer. For all $n \geq 0$, the number of compositions of kn such that each part not divisible by k appears inplace with multiplicity divisible by k equals the number of compositions of n when each part not divisible by k can be of two kinds.*

In proving these theorems we concentrate primarily on bijective reasoning. We also highlight the enumerating generating functions.

2. A Proof of Theorem 1.5

For clarity we first give a proof of Theorem 1.5 when $k = \ell = 2$. Consider a composition of n in which no part is congruent to 2 modulo 4. This means all the parts are either odd or are multiples of 4. To construct the image of this composition of n, we do the following:

- Any odd part is simply mapped to itself.
- Any part which is a multiple of 4, say $4r$ for some positive integer r, is mapped to $2r, 2r$ in the new composition.

Then by construction, the weight of the new composition is still n. Moreover, the even parts in the image composition are always next to one another in pairs; this preserves the need for each part to occur inplace with even multiplicity.

The inverse operation should be clear. Again, map the odd parts to themselves. And each pair $2r, 2r$ encountered in the composition is converted to a single part $4r$. This yields a composition in which no part is congruent to 2 modulo 4.

We now extend the foregoing bijection to the general case. Begin with a composition of n in which no part is congruent to $ik \pmod{\ell k}$, where $1 \leq i \leq \ell - 1$. This means all the parts are either not divisible by k or, if divisible by k, then those parts are divisible by ℓk. To construct the image of this composition, we do the following:

- Any part not divisible by k is simply mapped to itself.
- Any part which is a multiple of ℓk, say $(\ell k)r$ for some positive integer r, is mapped to $\underbrace{kr, kr, kr, \ldots, kr}_{\ell \text{ times}}$ in the new composition.

The rest of the proof follows as above. This completes the proof.

For example, when $k = \ell = 2$, the composition $(8, 1, 3, 3, 5, 4, 4, 12, 7)$ which has the property that no parts are congruent to 2 modulo 4 is mapped to the composition $(4, 4, 1, 3, 3, 5, 2, 2, 2, 2, 6, 6, 7)$ and vice versa under the bijection.

We can obtain a rational generating function which gives $C_{k,\ell}(n)$, the common number of compositions of n enumerated in Theorem 1.5 for any k and ℓ. Such a generating function, using the second class of compositions asserted in the theorem, is given by:

$$\begin{aligned}\sum_{n\geq 1} C_{k,\ell}(n)x^n &= \sum_{j\geq 1}((x+x^2+\cdots)-(x^k+x^{k+\ell k}+\cdots)-\cdots-(x^{(\ell-1)k}+x^{(\ell-1)k+\ell k}+\cdots))^j \\ &= \sum_{j\geq 1}\left(\frac{x}{1-x}-\frac{x^k(1-x^{(\ell-1)k})}{(1-x^{\ell k})(1-x^k)}\right)^j\end{aligned}$$

This geometric series can be written in closed form as

$$\frac{\dfrac{x-2x^{\ell k+1}+x^{(\ell+1)k+1}-x^k+x^{\ell k}}{(1-x)(1-x^k)(1-x^{\ell k})}}{1-\dfrac{x-2x^{\ell k+1}+x^{(\ell+1)k+1}-x^k+x^{\ell k}}{(1-x)(1-x^k)(1-x^{\ell k})}}$$

which, when simplified becomes

$$\frac{x-2x^{\ell k+1}+x^{(\ell+1)k+1}-x^k+x^{\ell k}}{1-2x+x^{k+1}-2x^{\ell k}+3x^{\ell k+1}+x^{(\ell+1)k}-2x^{(\ell+1)k+1}}. \tag{1}$$

When $k = \ell = 2$, (1) reduces to

$$\sum_{n\geq 1} C_{2,2}(n)x^n = \frac{x+x^3+x^4}{1-x-x^3-2x^4}. \tag{2}$$

The function (1) gives an effective way to count the sets of compositions of n in Theorem 1.5, for any $k \geq 2$ and $\ell \geq 2$, using a computer algebra package such as `MAPLE`.

Corresponding recurrence information can also be obtained for this sequence of values based on the above generating function.

Before closing this section, we prove an interesting parity property satisfied by $C_{2,2}(n)$.

Theorem 2.1. *For all $n \geq 2$,*

$$C_{2,2}(n) \equiv \begin{cases} 0 \pmod 2, & \text{if } n \equiv 1, 3, 4 \pmod 7 \\ 1 \pmod 2, & \text{otherwise.}\end{cases}$$

Proof. Thanks to (2), the generating function equivalent of the statement of this theorem is that

$$\frac{x+x^3+x^4}{1-x-x^3-2x^4} \equiv \frac{x^2+x^5+x^6+x^7}{1-x^7} + x \pmod 2$$

which is the same as

$$\frac{x+x^3+x^4}{1-x-x^3} \equiv \frac{x+x^2+x^5+x^6+x^7+x^8}{1-x^7} \pmod 2. \tag{3}$$

But (3) is easily checked by noting that $(x+x^3+x^4)(1-x^7)$ and $(1-x-x^3)(x+x^2+x^5+x^6+x^7+x^8)$ are each congruent to $x+x^3+x^4+x^8+x^{10}+x^{11}$ modulo 2. ∎

Similar proof techniques can be employed to prove related results for other functions $C_{k,\ell}(n)$ for small values of k and ℓ. For example, one can prove that

$$C_{3,3}(n) \equiv \begin{cases} 0 \pmod 2, & \text{if } n \equiv 0 \pmod 2 \text{ or } n \equiv 1,2,4 \pmod 7 \\ 1 \pmod 2, & \text{otherwise.} \end{cases}$$

3. A Proof of Theorem 1.6

We now transition to a bijective proof of Theorem 1.6 in the spirit of the proof of Theorem 1.5. Namely, begin with a composition of n where each part not divisible by k can be of two kinds. Then construct the corresponding composition of kn where each part not divisible by k appears inplace with multiplicity divisible by k, to which our original composition is mapped, in the following way:

- Any part divisible by k is mapped to a new part which is k times that part.
- Any part not divisible by k which is not marked with an asterisk is mapped to a new part which is k times that part.
- Any part not divisible by k, say p, which is marked with an asterisk is mapped to $\underbrace{p,p,p,\ldots,p}_{k \text{ times}}$ in the new composition.

It is clear that the new composition has weight kn given that the original composition has weight n. It should also be clear that the new composition has the property that all of the parts not divisible by k are "inplace" with multiplicity divisible by k. Lastly, this mapping is obviously invertible. This completes the proof.

This bijection is demonstrated in the case $k=2$, when $n=3$, in Example 1.4; members of the two lists of compositions correspond one-to-one according to the bijection.

The generating function for the function $\mathfrak{C}_k(n)$ which enumerates the number of compositions of n when each part not divisible by k can be of 2 kinds is given by

$$\begin{aligned}\sum_{n\geq 0}\mathfrak{C}_k(n)x^n &= \sum_{j\geq 0}\left(2(x+x^2+\cdots)-(x^k+x^{2k}+\cdots)\right)^j \\ &= \frac{(1-x)(1-x^k)}{1-3x+2x^{k+1}} \\ &= \frac{1-x^k}{1-2x-2x^2-2x^3-\cdots-2x^k}.\end{aligned}$$

As we close, we share a few remarks about the arithmetic behavior of $\mathfrak{C}_k(n)$. First, it is clear from the generating function that $\mathfrak{C}_k(n)$ satisfies the recurrence

$$\mathfrak{C}_k(n) = 2(\mathfrak{C}_k(n-1)+\mathfrak{C}_k(n-2)+\cdots+\mathfrak{C}_k(n-k))$$

for $n \geq k$. Owing to the factor of 2 on the right–hand side of this recurrence, we know that, for fixed $j \geq 1$, $\mathfrak{C}_k(n) \equiv 0 \pmod{2^j}$ for all $n > jk$. Moreover, in the special case $k=2$, our recurrence reduces to

$$\mathfrak{C}_2(n) = 2(\mathfrak{C}_2(n-1)+\mathfrak{C}_2(n-2)).$$

This second–order recurrence is quite reminiscent of the recurrence satisfied by the Fibonacci numbers [11, A000045]. Indeed, it is the case that $\mathfrak{C}_2(n)$ satisfies a number of Fibonacci–like arithmetic properties [5]. As an example, one can show that, for fixed $m \geq 1$ and all $n \geq 0$, $\mathfrak{C}_2(m) \mid \mathfrak{C}_2(2(m+1)n+m)$.

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The John Knopfmacher Centre for Applicable Analysis and Number Theory, University of the Witwatersrand, Johannesburg, South Africa, Augustine.Munagi@wits.ac.za

Department of Mathematics, Penn State University, University Park, PA 16802, USA, sellersj@psu.edu