Algorithmic Randomness

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Outline of the Lectures

Lecture 1: Martin-Löf tests and martingales.
Lecture 2: Kolmogorov complexity.
Lecture 3: The computational power of randomness.
Lecture 4: Randomness for non-uniform distributions.
Lecture 5: Randomness in Fractal Geometry and Dynamical Systems.
Lecture 6: Lowness and Triviality
# Some Basics

## Books

- Downey and Hirschfeldt, *Algorithmic Randomness and End Complexity* (Springer)
- Li and Vitanyi, *An Introduction to Kolmogorov Complexity and its Applications* (Springer)
- Nies, *Randomness and Computability* (Oxford UP)

## Prerequisites

- Basic computability theory – Turing machines, computable sets, halting problem, Turing jump, r.e. sets
- Basic mathematical logic, some set theory later on
- Basic measure theory (helpful)
Initial Clarifications

Logic

- We are particularly concerned with **definability**, i.e. the relation between a language and mathematical objects we can describe.

- Our language will mostly be that of (second order) **arithmetic**, $+, \cdot, =, \in$.

- By the well-known correspondence between definability in arithmetic and (relative) computability, our study of randomness is also referred to as **algorithmic randomness**.
**Initial Clarifications**

**Randomness**

Our goal: define what an *individual random object is*. Think of an outcome of an infinite sequence of coin tosses.

**Three Paradigms**

- **Typicalness:** A random object is the typical outcome of a random variable.
- **Unpredictability:** A random object should be impossible to predict.
- **Incompressibility:** A random object should not have a shorter description than itself.

In the following, we want to give a sound meaning to these paradigms.
Initial Clarifications

Do earthquakes occur randomly?

Hauksson-Shear-Yang catalog of southern CA earthquakes 1981-2011
Initial Clarifications

If we do not restrict the methods allowed for betting, compressing, etc., we easily end up in paradox:

- A typical outcome should satisfy all probabilistic laws, such as the Law of Large Numbers.
- A probabilistic law is essentially a set of measure 1.
- However, the intersection of all sets of measure 1 is empty!
A different example:

- A random sequence $X$ of coin tosses should be unpredictable – there should be no prediction function that, given as input a finite sequence $X(1)\ldots X(n)$ of outcomes so far, it predicts the next bit of $X$ correctly.

- However, there clearly is such a prediction function – one that, on any input of length $n$, simply outputs the $n+1$-st bit of $X$. 

Initial Clarifications

Remedy:

Admit only
   *definable* laws, betting strategies, compression methods.