HW3 SOLUTIONS

2.3.7: Since \((1, 1), (2, 2), (3, 3), (4, 4) \in R\), we see that \(R\) is reflexive. Since \((1, 2), (2, 1) \in R\) and \((3, 4), (4, 3) \in R\) and there are no other pairs of distinct integers in \(R\), we see by exhaustion that \(R\) is symmetric. A similar consideration of cases implies that \(R\) is transitive.

The equivalence classes of \(R\) are:

\[
[(1, 1)] = \{(1, 1), (1, 2), (2, 1), (2, 2)\},
[(3, 3)] = \{(3, 3), (3, 4), (4, 3), (4, 4)\}.
\]

2.3.8: Since \(a + b = a + b\), we see that \((a, b)R(a, b)\); i.e. \(R\) is reflexive. Suppose that \((a, b)R(c, d)\). Then \(a + b = c + d\). This implies that \(c + d = a + b\), so \((c, d)R(a, b)\); i.e. \(R\) is symmetric. Finally, suppose that \((a, b)R(c, d)\) and \((c, d)R(e, f)\). Then \(a + b = c + d = e + f\), so \((a, b)R(e, f)\); i.e. \(R\) is transitive.

Geometrically, each equivalence class of \(R\) is the intersection of a line \(x + y = n\) for \(n \in \mathbb{N}\) with the grid of points \(S \times S\). Algebraically, the equivalence classes are:

\[
[(1, 1)] = \{(1, 1)\},
[(1, 2)] = \{(1, 2), (2, 1)\},
[(1, 3)] = \{(1, 3), (2, 2), (3, 1)\},
[(1, 4)] = \{(1, 4), (2, 3), (3, 2), (4, 1)\},
[(2, 4)] = \{(2, 4), (3, 3), (4, 2)\},
[(3, 4)] = \{(3, 4), (4, 3)\},
[(4, 4)] = \{(4, 4)\}.
\]

2.3.9: The relation is not transitive. For example, \((a, c) \in R\) and \((c, e) \in R\), but \((a, e) \notin R\). The digraph looks like a pentagram with arrows on both ends of the lines.

3.1.3: \(p \land (p \to q)\) is logically equivalent to \(p \land q\), since

\[
p \land (p \to q) \equiv p \land (\neg p \lor q) \equiv (p \land \neg p) \lor (p \land q) \equiv p \land q.
\]

\((p \land q) \iff p\) is logically equivalent to \(p \to q\), since

\[
(p \land q) \iff p \equiv ((p \land q) \land p) \lor (\neg (p \land q) \land \neg p)
\equiv (p \land q) \lor \neg ((p \land q) \lor p) \equiv (p \land q) \lor \neg p \equiv \neg p \lor q \equiv p \to q
\]

There are no other logically equivalent statements.

3.1.4:

(i) We compute that

\[
\neg p \iff q \equiv (\neg p \land q) \lor (p \land \neg q)
\equiv (\neg p \lor q) \land (\neg p \lor \neg q) \land (q \lor p) \land (q \lor \neg q) \equiv (\neg p \lor \neg q) \land (p \lor q).
\]

Since this is symmetric in \(p\) and \(q\), it follows that \((\neg p \iff q) \equiv (p \iff \neg q)\).
(ii) We compute that
\[(p \rightarrow \neg q) \land (p \rightarrow \neg r) \equiv (\neg p \lor \neg q) \land (\neg p \lor \neg r)\]
\[\equiv \neg p \lor (\neg q \land \neg r)\]
\[\equiv \neg (p \land (q \lor r)).\]

(iii) We compute that
\[(p \rightarrow (q \lor r)) \equiv \neg p \lor (q \lor r)\]
\[\equiv q \lor (\neg p \lor r)\]
\[\equiv \neg q \rightarrow (\neg p \lor r).\]

3.1.5: ‘\(x\) is an element of \(Y \setminus X\)’ is the statement \(\neg p \land q\).

In the remainder, let \(p\) be the statement ‘\(x\) is an element of \(A\)’, let \(q\) be the statement ‘\(x\) is an element of \(B\)’, and let \(r\) be the statement \(x\) is an element of \(C\).

(i) ‘\(x\) is an element of \(A \cap B\)’ is the statement \(p \land q\), while ‘\(x\) is an element of \(A \setminus B^c\)’ is the statement \(p \land \neg (\neg q)\). But
\[p \land \neg (\neg q) \equiv p \land q,\]
so \(A \cap B = A \setminus B^c\).

(ii) ‘\(x\) is an element of \(A \cup (B \setminus A)\)’ is the statement \(p \lor (q \land \neg p)\), while ‘\(x\) is an element of \(A \cup B\)’ is the statement \(p \lor q\). But
\[p \lor (q \land \neg p) \equiv (p \lor \neg p) \land (p \lor q) \equiv p \lor q,\]
so \(A \cup (B \setminus A) = A \cup B\).

(iii) ‘\(x\) is an element of \(A \setminus (B \cup C)\)’ is the statement \(p \land \neg (q \lor r)\), while ‘\(x\) is an element of \((A \setminus B) \cap (A \setminus C)\)’ is the statement \((p \land \neg q) \land (p \land \neg r)\). But
\[p \land \neg (q \lor r) \equiv p \land \neg q \land \neg r \equiv (p \land \neg q) \land (p \land \neg r),\]
so \(A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)\).

(iv) ‘\(x\) is an element of \(A \setminus (B \cap C)\)’ is the statement \(p \land \neg (q \land r)\), while ‘\(x\) is an element of \((A \setminus B) \cup (A \setminus C)\)’ is the statement \((p \land \neg q) \lor (p \land \neg r)\). But
\[p \land \neg (q \land r) \equiv p \land (\neg q \lor \neg r) \equiv (p \land \neg q) \lor (p \land \neg r),\]
so \(A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)\).

**Extra Problem #1:** Since \(R\) is reflexive, \(I_A \subseteq R\). Now let \((x, y) \in R\). Since \(R\) is an equivalence relation, it is symmetric, hence \((y, x) \in R\). Since \(R\) is a partial ordering, it is weakly antisymmetric. Thus, since \((x, y) \in R\) and \((y, x) \in R\), it holds that \(x = y\). In particular, \((x, y) \in I_R\). Thus \(R = I_A\), as desired.

**Extra Problem #2:** From the definition \(\leq_B = \leq \cap (B \times B)\), we see that \(\leq_B \subseteq B \times B\); i.e. \(\leq_B\) is a relation on \(B\). We now show that it is a partial ordering.

First, let \(x \in B\), so \((x, x) \in B \times B\). Since \(B \subseteq A\), \(x \in A\). Since \(\leq\) is reflexive, \((x, x) \in \leq\). Thus \((x, x) \in \leq_B\); i.e. \(\leq_B\) is reflexive.

Second, let \((x, y), (y, x) \in \leq_B\). Since \(\leq_B \subseteq \leq\), we have that \((x, y), (y, x) \in \leq\). Since \(\leq\) is weakly antisymmetric, \(x = y\); i.e. \(\leq_B\) is weakly antisymmetric.

Third, let \((x, y), (y, z) \in \leq_B\). Thus \(x, y, z \in B\), and hence \((x, z) \in B \times B\). Also, we have that \((x, y), (y, z) \in \leq\). Since \(\leq\) is transitive, \((x, z) \in \leq\). Hence \((x, z) \in \leq_B\); i.e. \(\leq_B\) is transitive. Put together, this shows that \(\leq_B\) is a partial ordering on \(B\).