HW2 SOLUTIONS

Problem 2.2.5:
(i) \( f : \mathbb{R}^+ \to \mathbb{R} \) defined by \( f(x) = \ln x \) is a bijection from \( \mathbb{R}^+ = \{ x \in \mathbb{R} \mid x > 0 \} \) to \( \mathbb{R} \).
(ii) \( f : (-\pi/2, \pi/2) \to \mathbb{R} \) defined by \( f(x) = \tan x \) is a bijection.
(iii) \( f : \mathbb{N} \to \mathbb{Z} \) defined by
\[
f(n) = \begin{cases} 
m, & \text{if } n = 2m + 1, \\
-m, & \text{if } n = 2m
\end{cases}
\]
is a bijection (note that I am using \( \mathbb{N} = \{1, 2, 3, \ldots \} \)).

Problem 2.2.7:
(i) \( g : \mathbb{R} \to \mathbb{R} \) defined by \( g(x) = 4 - 3x \) is the inverse of \( f \).
(ii) Note that \( x^3 - 3x^2 + 3x - 1 = (x - 1)^3 \). It follows that \( g : \mathbb{R} \to \mathbb{R} \) defined by \( g(x) = x^{1/3} + 1 \) is the inverse of \( f \).

Problem 2.3.1:
(a) Reflexive
(b) Reflexive, symmetric, transitive
(c) Symmetric
(d) Symmetric
(e) Reflexive, transitive
(f) Reflexive, symmetric, transitive
(g) Reflexive, weakly antisymmetric, transitive

Problem 2.3.2:
(a) Suppose to the contrary that \( R \neq \emptyset \). Let \( (x, y) \in R \). Since \( R \) is symmetric, \( (y, x) \in R \). Since \( R \) is antisymmetric, \( (y, x) \notin R \). This is a contradiction. Hence \( R = \emptyset \).
(b) Since \( R \) is antisymmetric, it is never true that \( (x, y) \in R \) and \( (y, x) \in R \). Thus \( R \) is trivially weakly antisymmetric (the conditional in the definition of “weakly antisymmetric” is never satisfied).
(c) The identity relation \( I_\mathbb{R} \) on the set \( \mathbb{R} \) is symmetric and weakly antisymmetric.
(d) Suppose \( (x, y) \in R^c \). Then \( (x, y) \notin R \). Either \( (y, x) \in R \) or \( (y, x) \notin R \). Since \( R \) is symmetric, if \( (y, x) \in R \), then \( (x, y) \in R \), which is impossible. Hence \( (y, x) \notin R \). Thus the complement \( R^c \) is symmetric.
(e) Suppose \( (x, y) \in R^{rev} \) and \( (y, z) \in R^{rev} \). Then \( (y, x) \in R \) and \( (z, y) \in R \). Since \( R \) is transitive, \( (z, x) \in R \). Hence \( (x, z) \in R^{rev} \). Thus \( R^{rev} \) is transitive.
(f) Let \( X = \{1, 2\} \) and let \( R = \{(1, 1)\} \subseteq X \times X \). Clearly \( R \) is symmetric and transitive. Since \( (2, 2) \notin R \), \( R \) is not reflexive.

Problem 2.3.3: The problem is that there need not be a \( y \in X \) such that \( (x, y) \in R \). For example, in our example for Problem 2.3.2(f), there is no \( y \in \{1, 2\} \) such that \( (1, y) \in R \).
Extra Problem #1: Suppose that $|A| = m$ and $|B| = n$. That is, there are bijections $f: A \to \{0, 1, \ldots, m - 1\}$ and $g: B \to \{0, 1, \ldots, n - 1\}$. Let $h: A \times B \to \{0, 1, \ldots, mn - 1\}$ be the function defined by

$$h(a, b) = nf(a) + g(b)$$

for all $a \in A$ and $b \in B$. We claim that $h$ is bijective. Let $x \in \{0, 1, \ldots, mn - 1\}$. As we will discuss when we cover Section 1, there are unique integers $p, r$ such that $r \in \{0, 1, \ldots, n - 1\}$ and $x = np + r$; roughly, $p$ is the integer one gets by dividing $x$ by $n$ and $r$ is the remainder of that division (think elementary school long division). Since $0 \leq x \leq mn - 1$, we see that $0 \leq p \leq m - 1$. Since $f$ and $g$ are bijections, there are unique $a \in A$ and $b \in B$ such that $f(a) = p$ and $g(b) = r$. Thus $(a, b) \in A \times B$ is the unique element of $A \times B$ such that $h(a, b) = x$. This shows that $h$ is bijective. Since $\{0, 1, \ldots, mn - 1\}$ has $mn$ elements, we see that $|A \times B| = |A| \cdot |B|$.

Extra Problem #2: Define $f: A \times B \to B \times A$ by $f(a, b) = (b, a)$ for all $(a, b) \in A \times B$. Suppose $f(a, b) = f(c, d)$. Then $(b, a) = (d, c)$. This means that $b = d$ and $a = c$. Hence $(a, b) = (c, d)$, so $f$ is injective. Now let $(b, a) \in B \times A$. Then $f(a, b) = (b, a)$, so $f$ is surjective. Therefore $f$ is bijective and so, by definition, $|A \times B| = |B \times A|$.