Randomness for Ergodic Measures

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www.personal.psu.edu/jmr71/

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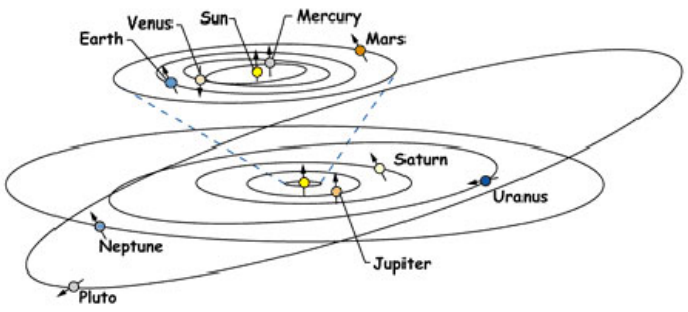
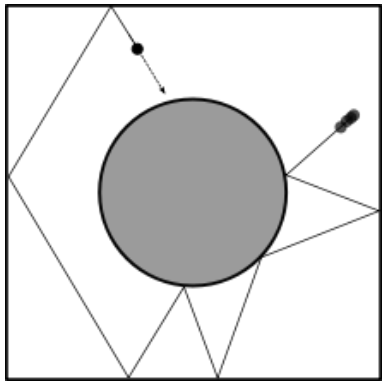
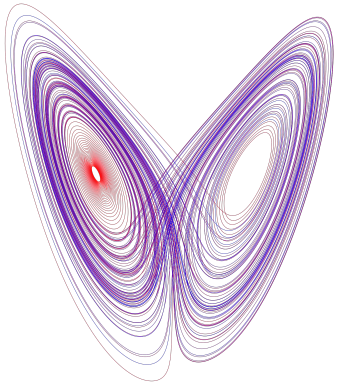
Ergodic theory

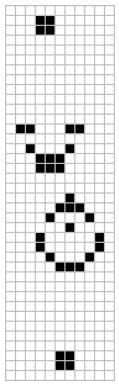
Dynamical systems

- Dynamical systems is the study of "motion".









- Concerns a **space** equipped with an **action**.
  - Topological space with continuous action vs. **probability space with a measure-preserving action.**
  - Continuous time dynamics vs. **discrete time dynamics.**

Symbolic dynamics

- **Space:** Cantor space $\{0,1\}^{\mathbb{N}}$ (space of infinite binary sequences)
- **Transformation:** Left shift map $S\colon \{0,1\}^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}}$

$$0x, 1x \mapsto x.$$

- **Measure:** A Borel probability measure μ on $\{0,1\}^{\mathbb{N}}$ given by

$$\mu(\sigma) = \text{probability that a sequence starts with } \sigma \quad (\sigma \in \{0,1\}^*)$$
$$\mu(\sigma) \geqslant 0, \quad \mu(\sigma 0) + \mu(\sigma 1) = \mu(\sigma), \quad \mu(\ ) = 1.$$

- A measure is computable if $\mu(\sigma)$ is computable from σ.
- We also require that the measure be invariant under the shift action.

Shift-invariant measures

- The fair-coin measure $\lambda(\sigma) = (1/2)^{|\sigma|}$ satisfies the following:

  If one removes the first bit, one still has the same probability measure.

- Such a measure is called **shift-invariant**.
- More formally, μ is shift invariant if
  - $\mu(S^{-1}A) = \mu(A)$ for all measurable $A \subseteq \{0,1\}^{\mathbb{N}}$ (for left shift map S).
  - Equivalently, $\mu(0\sigma) + \mu(1\sigma) = \mu(\sigma)$ for all finite strings $\sigma \in \{0,1\}^*$.
- There are a wide and complex variety of shift invariant measures.
  - Bernoulli processes (i.i.d. weighted coin flips)
  - Markov processes (under certain initial conditions)
  - and many many more

Combining shift-invariant measures

- If μ_1 and μ_2 are shift-invariant measures, then the convex combination

$$\frac{1}{3}\mu_1 + \frac{2}{3}\mu_2$$

  is shift-invariant.
- The same is true of any convex combination of shift-invariant measures.

- However, the fair-coin measure satisfies this property.

  The fair-coin measure is not a (nontrivial) convex combination of two or more different shift-invariant measures.

- A shift-invariant measure that cannot be decomposed is called a **ergodic measure**.
- Also, μ is ergodic iff for all sets A, if $T^{-1}(A) = A$ then $\mu(A)$ is 0 or 1.

Intermission: Algorithmic randomness

Algorithmic randomness

- Informally, a point is random if it is satisfies every "effective probability one property".
- For a *computable* probability measure μ on $\{0,1\}^{\mathbb{N}}$:
  - A **μ-Martin-Löf test** is a sequence (U_n) of effectively open (Σ^0_1) sets in $\{0,1\}^{\mathbb{N}}$ such that $\mu(U_n) \leqslant 2^{-n}$.
  - A **μ-Martin-Löf random** is a point not in $\bigcap_n U_n$ for any Martin-Löf tests (U_n).
- For a *non-computable* probability measure μ on $\{0,1\}^{\mathbb{N}}$:
  - A **μ-Martin-Löf random** is a point not in $\bigcap_n U_n$ for any Martin-Löf tests (U_n) *relativized* to the measure μ.
  - A **blind-μ-Martin-Löf random** is a point not in $\bigcap_n U_n$ for any Martin-Löf tests (U_n) (*not relativized*).

Back to our program: Decomposing measures

Substring frequencies / empirical ergodic measures

- The frequency of times that the string σ appears in the sequence x is

$$\mathbf{freq}(\sigma, x) = \lim_n \frac{\#\{k \leqslant n : x \textbf{ contains substring } \sigma \textbf{ at position } k\}}{n}.$$

- E.g., x is normal iff $\mathrm{freq}(\sigma, x) = (1/2)^{|\sigma|} = \lambda(\sigma)$ (fair-coin measure).

- If μ is shift-invariant,
  - $\mathrm{freq}(\sigma, x)$ exists for μ-almost-every x (pointwise ergodic theorem).
  - $\mathrm{freq}(\cdot, x)$ is μ-almost-surely an ergodic measure.
  - If μ is ergodic, then μ-almost-surely $\mathrm{freq}(\sigma, x) = \mu(\sigma)$.
  - $\mathrm{freq}(\cdot, x)$ is also called the **empirical ergodic measure**.
  - μ is a unique convex combination of ergodic measures:

$$\mu(\sigma) = \int \mathrm{freq}(\sigma, x)\, d\mu(x).$$

Bucket of weighted-coins

- Bucket of weighted-coins process:
  - Start with a bucket of weighted coins (under some distribution).
  - Step -1: Take a coin at random from the bucket.
  - Steps 0,1,2,...: Flip coin and record outcome. (Same coin every time.)
- The resulting measure μ is
  - shift-invariant, hence $\mu(\sigma) = \int \mathrm{freq}(\sigma, x)\, d\mu(x)$.
  - **exchangeable**, i.e. invariant under finite permutations of bits.
  - $\mathrm{freq}(\cdot, x)$ is μ-almost-surely a Bernoulli (weighted coin) measure.
  - μ is a convex combination of Bernoulli measures
- de Finetti's theorem: Every exchangeable measure comes from "a bucket of weighted coins".
- If μ is computable and exchangeable and x is μ-random,
  - $\mathrm{freq}(\cdot, x)$ exists and is a Bernoulli measure.
  - $\mathrm{freq}(\cdot, x)$ is computable from x.
  - x is $\mathrm{freq}(\cdot, x)$-random.
  - For each Bernoulli measure ν, blind-ν-randomness = ν-randomness.

Main Questions and Answers

- We have the analogy

$$\frac{\text{ergodic}}{\text{shift-invarient}} = \frac{\text{Bernoulli}}{\text{exchangable}}.$$

- Assume μ is computable and shift-invariant and x is μ-random.
  - Does $\mathrm{freq}(\cdot, x)$ exist, and is it a measure?
    - Yes and yes. (V'yugin)
  - Is $\mathrm{freq}(\cdot, x)$ ergodic?
    - **Still open.**
  - Is $\mathrm{freq}(\cdot, x)$ computable from x?
    - No. (Hoyrup)
  - Is x $\mathrm{freq}(\cdot, x)$-random (or ν-random for some ergodic ν)?
    - **No (and No). (Reimann and R.) (also Monin).**
  - For ergodic ν, does ν-random = blind-ν-random.
    - **No. (Reimann and R.) (also Monin).**
- (The answers are all yes if the "ergodic decomposition is computable.")

Proof Method (Cutting and Stacking)

Proof sketch

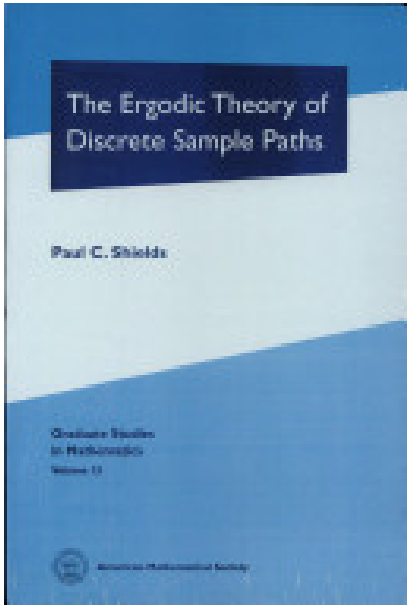
- Construct:
  - A computable shift-invariant measure μ
  - a μ-random x
- Such that:
  - $\mathrm{freq}(\cdot, x)$ is an ergodic measure
  - x is blind-$\mathrm{freq}(\cdot, x)$-random
  - $\mathrm{freq}(\cdot, x)$ computes x
- Hence:
  - x is μ-random and blind-$\mathrm{freq}(\cdot, x)$-random, but
  - x is not $\mathrm{freq}(\cdot, x)$-random (nor ν-random for any ergodic ν).
- Method:
  - Cutting and stacking

Cutting and stacking

- Originated in ergodic theory.
- Has been used by V'yugin and Franklin/Towsner in randomness.
- Well-suited to computability theorists.

- See pictures on the board.

- For more information see:
  - Paul C. Shields. *The Ergodic Theory of Discrete Sample Paths.*



Closing Thoughts

Thank You!

These slides will be available on my webpage:

http://www.personal.psu.edu/jmr71/

Or just Google™ me, "Jason Rute".

P.S. I am on the job market.