Properties of Dirac matrices, etc. Metric \((-,+,+,-+)\)

## Dirac matrices, etc

- Conventions for Dirac matrices, etc:
  - The metric is \(g^{\mu\nu} = \text{diag}(-,+,+,+)\).
  - The totally antisymmetric tensor \(\epsilon_{\kappa\lambda\mu\nu}\) is defined to obey \(\epsilon^{0123} = 1\). Thus \(\epsilon^{0123} = -1, \epsilon^{0012} = 0, \epsilon_{0123} = -1\), etc. Take particular note of the last case. This definition is the same as in Burgess & Moore.
  - The anticommutation relations of the Dirac matrices are \(\{\gamma^\mu, \gamma^\nu\} = -2g^{\mu\nu}I\). This is reversed compared with Burgess & Moore.
  - \(\gamma_5 = -i\gamma^0\gamma^1\gamma^2\gamma^3\). This is the same as in Burgess & Moore.

- Spin sums:
  \[
  \sum_{\text{spins}} u\bar{u} = -p + M, \\
  \sum_{\text{spins}} v\bar{v} = -p - M.
  \]

- Trace formulae in 4 dimensional space-time that generalize to \(n\) dimensions:
  \[
  \text{tr}\ 1 = 4, \\
  \text{tr} \gamma^\mu\gamma^\nu = -4g^{\mu\nu}, \\
  \text{tr} \gamma^\kappa\gamma^\lambda\gamma^\mu\gamma^\nu = 4\left(g^{\kappa\lambda}g^{\mu\nu} - g^{\kappa\mu}g^{\lambda\nu} + g^{\kappa\nu}g^{\lambda\mu}\right),
  \]
  \[
  \text{tr}\ \text{Odd number of } \gamma^\mu\text{s} = 0.
  \]

Generalizations can be derived readily.

- In a general (non-integer) space-time dimension \(n\), it can be shown (e.g., Collins, “Renormalization”, Sec. 4.5) that the above formulae remain correct, except that the value of the trace of the unit Dirac matrix need not be 4 in a general space-time dimension. The value of \(\text{tr}\ 1\) for non-integer \(n\) is, in fact, a choosable convention.

- A standard convention is indeed that \(\text{tr}\ 1 = 4\). One alternative possibility is for the trace to be \(2^{n/2}\), which agrees with the dimension of a standard representation in ordinary space-times of even integer dimension. If a different convention like this is used, then in Eqs. (3)–(5) all the occurrences of ‘4’ should be replaced by the chosen value of the trace. The change amounts to a renormalization-group transformation, as regards 4-dimensional physics. (But the proof takes a little work.)
• The treatment of $\gamma_5$ in $n$ dimensions is rather harder. The following formulae are strictly in 4 space-time dimensions, as is sufficient for almost all of this course:

$$\text{tr} \gamma_5 = \text{tr} \gamma_5 \gamma^\mu = \text{tr} \gamma_5 \gamma^\lambda \gamma^\mu = 0,$$

$$\text{tr} \gamma_5 \gamma^\lambda \gamma^\mu \gamma^\nu = 4i\epsilon^{\lambda\mu\nu}.$$  

(7)

(8)

Here $\epsilon^{\lambda\mu\nu}$ is the totally antisymmetric tensor defined above, with $\epsilon^{0123} = 1$.

• The Burgess & Moore Dirac matrices in terms of the ones above are

$$\gamma_\mu^{\text{BM}} = -i\gamma_\mu$$  

(9)

**Cross sections, etc**

The Lorentz-invariant integral over final-state “phase-space” for a system of $N$ particles of momenta $q_1, \ldots, q_N$ is

$$d\text{PS} = \prod_{j=1}^{N} \frac{d^3q_j}{(2\pi)^32E_{q_j}}(2\pi)^4\delta^{(4)}(q_f - p_i),$$  

(10)

where $p_i$ is the total incoming 4-momentum, and $q_f = \sum_{j=1}^{N}q_j$ is the total outgoing 4-momentum.

A cross section involves two incoming particles of momenta $p_1$ and $p_2$ and $N$ outgoing particles as above. The formula for the fully differential cross section is

$$d\sigma = \frac{1}{2E_12E_2|v_1 - v_2|}d\text{PS}|A|^2$$  

$$= \frac{1}{4\sqrt{p_1 \cdot p_2^2 - m_1^2m_2^2}}d\text{PS}|A|^2,$$  

(11a)

(11b)

where $A$ is the matrix element with an overall $(2\pi)^4\delta^{(4)}(p_f - p_i)$ removed. The incoming particles have energies $E_1$ and $E_2$ and velocities $v_1$ and $v_2$. The velocities are assumed to be in the same or opposite directions for Eq. (11a) to apply as written.

The differential contribution to the rest-frame decay rate of an unstable particle of mass $M$ is given by

$$d\Gamma = \frac{1}{2M}d\text{PS}|A|^2.$$  

(12)

The total decay rate (or width) $\Gamma$ is given by summing and integrating over all possible final states.