Formal definitions: limits

Let $f_n$ be defined for all integer $n$ (or large enough $n$).

- **Limit**: $\lim_{n \to \infty} f_n = l$ (or $f_n$ converges to $l$ as $n \to \infty$) means that for every $\delta > 0$, there’s an $n_0(\delta)$ such that for all $n > n_0(\delta)$, $|f_n - l| < \delta$.

- $f_n$ is a **Cauchy sequence** means that for every $\delta > 0$, there’s an $n_0(\delta)$ such that for all $n, m > n_0(\delta)$, $|f_n - f_m| < \delta$.

- There’s a theorem that the spaces of real and complex numbers (and . . . ) are topologically complete: All Cauchy sequences have limits.

- Obvious generalizations for other kinds of limit.

- (Further generalizations exist when we deal with spaces that don’t have a natural “metric” defined.)

Definitions for convergence of sums

Given a sequence $t_n$, let partial sum be

$$S_N = \sum_{n=1}^{N} t_n.$$ 

Overall question: Does $S_\infty$ exist?

- “The sum $\sum_{n=1}^{\infty} t_n$ converges” means that the limit $\lim_{N \to \infty} S_N$ exists.

- “The sum is absolutely convergent” means that $\sum_{n=1}^{\infty} |t_n|$ converges.

- “The sum is conditionally convergent” means that it is convergent but not absolutely convergent.

- The same ideas apply to convergence of integrals, notably where the range goes to infinity, or when the integrand has divergence(s).
Tests of convergence

Two workhorse tests:

- **Overall aim:** Compare target series (or integral) with one with an easy solution.

- **Comparison test:** If $\sum_{n=1}^{\infty} u_n$ is absolutely convergent and $|v_n| < (\text{constant})|u_n|$, then $\sum_{n=1}^{\infty} v_n$ converges.

- **Ratio test(s):** If $\lim_{n \to \infty} |u_{n+1}/u_n| = r < 1$, then $\sum_{n=1}^{\infty} u_n$ is absolutely convergent. If $r > 1$, the series diverges. If $r = 1$, we have to find a better method.

- The idea of the ratio test also works if we have a number $r < 1$ for which $|u_{n+1}/u_n| \leq r$ for all large enough $n$. 

Sep. 24, 2014 (correction of misprint in ratio test) 3/3