Summary of definition of a tensor

• Here we consider tensors corresponding to ordinary space, but the idea generalizes easily.

• A tensor is an object specified by an indexed array with respect to a coordinate system, with the transformation properties as follows:
  – Let the tensor’s coordinates in one coordinate frame be $T_{j_1 \ldots}^{i_1 \ldots}$
  – Let transformation of position coordinates to a new system be
    \[ r'^\alpha = R^\alpha_i r^i \]
    (Einstein summation convention assumed throughout.)
  – Tensor’s coordinates transform by
    \[ T'^{\alpha_1 \ldots}_{\beta_1 \ldots} = \prod R_{i_p}^{\alpha_p} \prod (R^{-1})^{j_q}_{\beta_q} T_{j_1 \ldots}^{i_1 \ldots} \]

• Names: Upper index: “contravariant”. Lower index: “covariant”.

• Exact order of indices depends on object and conventions.

• Symbols for indices, etc: Choose for clarity
Elementary properties, supplementary, definitions, etc

- Rank of tensor is number of indices
- Scalar is rank-0 tensor
- Vector is rank-1 tensor
- Paradigm of contravariant vector: Position vector
- Paradigm of covariant vector: Derivative of scalar function of position
- Contraction of upper and lower indices; use Einstein summation convention, and get tensor corresponding to free indices
- $\delta^i_j$ is mixed tensor invariant under all linear coordinate transformations
- Coordinate transformations preserve symmetry and antisymmetry properties for same type of index
- Metric tensor defined by $x \cdot y = g_{ij}x^i y^j$
- It can be used to convert between covariant and contravariant components
Restriction to RH Cartesian coordinates:

- Most common elementary situation

- Transformation matrix restricted to orthogonal matrices \((R^T R = I)\) of determinant +1.

- Then
  - same transformation law applies to covariant and contravariant indices;
  - therefore most people (including me) don’t make distinction between contravariant and covariant indices, when only Cartesian coordinates are used;
  - metric tensor \(g_{ij} = \delta_{ij}\) is invariant
$\epsilon$ tensor — summary

In 3 dimensions

- Definition of $\epsilon_{ijk}$, with:
  - Right-handed Cartesian coordinates
  - General coordinates

- Vector product

- Surface integrals, with example.

- Example of change of coordinate system

- Volume integral

Extension to $N$ dimensions