Summary of definition of a tensor

• Here we consider tensors corresponding to ordinary space, but the idea generalizes easily.

• A tensor is an object specified by an indexed array with respect to a coordinate system, with the transformation properties as follows:
  – Let the tensor’s coordinates in one coordinate frame be $T_{j_1...}^{i_1...}$
  – Let transformation of position coordinates to a new system be $r'^{\alpha} = R_i^{\alpha} r^i$

  (Einstein summation convention assumed throughout.)
  – Tensor’s coordinates transform by
    $$T_{\beta_1...}^{\alpha_1...} = \prod R_{ip}^{\alpha_p} \prod (R^{-1})_{jq}^{j_q} T_{j_1...}^{i_1...}$$

• Names: Upper index: “contravariant”. Lower index: “covariant”.

• Exact order of indices depends on object and conventions.

• Symbols for indices, etc: Choose for clarity

Elementary properties, supplementary, definitions, etc

• Rank of tensor is number of indices

• Scalar is rank-0 tensor

• Vector is rank-1 tensor

• Paradigm of contravariant vector: Position vector

• Paradigm of covariant vector: Derivative of scalar function of position

• Contraction of upper and lower indices; use Einstein summation convention, and get tensor corresponding to free indices

• $\delta^i_j$ is mixed tensor invariant under all linear coordinate transformations

• Coordinate transformations preserve symmetry and antisymmetry properties for same type of index

• Metric tensor defined by $\mathbf{x} \cdot \mathbf{y} = g_{ij} x^i y^j$

• It can be used to convert between covariant and contravariant components
Restriction to RH Cartesian coordinates:

• Most common elementary situation

• Transformation matrix restricted to orthogonal matrices \( R^T R = I \) of determinant +1.

• Then
  – same transformation law applies to covariant and contravariant indices;
  – therefore most people (including me) don’t make distinction between contravariant and covariant indices, when only Cartesian coordinates are used;
  – metric tensor \( g_{ij} = \delta_{ij} \) is invariant

\( \epsilon \) tensor — summary

In 3 dimensions

• Definition of \( \epsilon_{ijk} \), with:
  – Right-handed Cartesian coordinates
  – General coordinates

• Vector product

• Surface integrals, with example.

• Example of change of coordinate system

• Volume integral

Extension to \( N \) dimensions