Properties of Fourier transform

• Examples of standard cases with easy calculation:

<table>
<thead>
<tr>
<th>( f(x) )</th>
<th>( \hat{f}(k) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( e^{-(x-x_0)^2/2\epsilon} e^{ik_0x} )</td>
<td>( \sqrt{2\pi\epsilon} e^{-(k-k_0)^2\epsilon/2} e^{-i(k-k_0)x_0} )</td>
</tr>
<tr>
<td>( \theta(t-t_0)e^{-\alpha(t-t_0)} )</td>
<td>( -ie^{-i\omega t_0/\omega - i\alpha} )</td>
</tr>
</tbody>
</table>

• Shifts and exponential factors:
  - Transform of \( f(x-x_0) \) is \( \hat{f}(k)e^{-ikx_0} \).
  - Transform of \( e^{ik_0x}f(x) \) is \( \hat{f}(k-k_0) \).

• Derivative of function gives factor of \( ik \) in transform (and vice versa with sign change).

• Discontinuity in \( f(x) \) or a derivative \( \implies \) characteristic large \( k \) behavior of \( \hat{f}(k) \). (Derive by integration by parts.)

• More generally singularity in \( f \) or \( \hat{f} \implies \) large argument behavior in other.

• Analyticity and asymptotic behavior:
  - \( \hat{f}(k) \) decreases faster than any power at \( |k| \to \infty \)
  \( \iff \) \( f(x) \) is infinitely differentiable.
  - Hence transform of a good function\(^1\) is a good function.
  - \( f(x) \) decreases exponentially as \( x \to \pm\infty \implies \hat{f}(k) \) analytic near real axis.
  - \( f(x) \) vanishes for all negative enough \( x \implies \hat{f}(k) \) analytic in lower half plane

• Convolution theorem: \( \hat{f}(k)\hat{g}(k) \) is transform of \( \int_{-\infty}^{\infty} d\xi \ f(x-\xi)g(\xi) \).

• Results for \( \langle f|g \rangle \):

\[
\frac{1}{2\pi} \int_{-\infty}^{\infty} dk \ \hat{f}(k)^* \hat{g}(k) = \int_{-\infty}^{\infty} dx \ \hat{f}(x)^* \hat{g}(x),
\]

\[
\frac{1}{2\pi} \int_{-\infty}^{\infty} dk \ |\hat{f}(k)|^2 = \int_{-\infty}^{\infty} dx \ |f(x)|^2.
\]

\(^1\) A “good function” of a real variable is one which is infinitely differentiable and decreases faster than any power at infinity.
Fourier transforms of distributions

- After examination of distribution corresponding to ordinary function, define Fourier transform of distribution $\rho$ by

$$\tilde{\rho}[t] = \rho[\tilde{t}]$$

- To make this work, we use tempered distributions. These are a subset of general distributions that are defined when the test functions are in the “Schwartz space” of “good functions”.

- (General distributions are required only to be defined on test functions of compact support.)

- Use this to construct Fourier transform of functions whose ordinary Fourier transform is given by a divergent integral.

- Useful calculational tool: Use appropriate limits with exponential factors to give convergence.