Distributions (generalized functions)

• Motivations for need for concept of $\delta$-function: normalization of continuum states in quantum mechanics, charge distributions with point charges.

• Insight: Define distribution mathematically to implement an idealization of actual measurements of charge distributions (etc).

• Define space $D$ of test functions: infinitely differentiable, compact support.

• Define a distribution $\rho$ as a (continuous) linear functional on $D$. Notations: $\rho[t], \langle \rho, t \rangle, \int \rho(x) t(x) \, dx$.

• Particular distributions:
  – Distribution corresponding to function: $\bar{f}[t] = \int_{-\infty}^{\infty} f(x) t(x) \, dx$
  – Distribution $\delta_a$ implementing $\delta(x - a)$: $\delta_a[t] = t(a)$

• Function $f(x)$ can be reconstructed from its distribution $\bar{f}[t]$, if $f$ is continuous
Basic manipulations of distributions

- Rules of algebra and calculus (addition, multiplication, differentiation, limits, Fourier transforms):
  - Must agree with results for ordinary functions.
  - In particular find expression for, e.g., $\rho'[t]$ (derivative) in terms of $\rho$ that is correct for functions:
    $$\rho'[t] \overset{\text{def}}{=} -\rho[t']$$
- Hence we have derivative of all distributions, including non-differentiable functions. In particular $\theta'(x), \delta'(x)$.
- Consistency: Must prove theorems that we know are valid for ordinary functions, e.g., $(fg)' = f'g + fg'$. 
Distributions: limits and Fourier transforms

• Definition of limit of distributions: \( \lim_{\epsilon \to 0} \rho_\epsilon = \rho \) means that for each test function \( t(x) \), \( \lim_{\epsilon \to 0} \rho_\epsilon [t] = \rho [t] \).

• Example: \( \int_{-\infty}^{\infty} e^{i k x} \, dk = 2 \pi \delta(x) \), with proof by taking distributional limit of \( f_\Lambda(x) \equiv \int_{-\Lambda}^{\Lambda} e^{i k x} \, dk \), or of \( \int_{-\infty}^{\infty} e^{i k x} e^{-k^2/\Lambda^2} \, dk \), as \( \Lambda \to \infty \).

• Recall Fourier transform of ordinary function:

\[
\tilde{f}(k) = \int e^{-i k x} f(x) \, dx
\]

Then treat this as distribution, and show

\[
\tilde{f}[t] = \tilde{f}[\tilde{t}].
\]

Use this as definition of Fourier transform of distribution:

\[
\tilde{\rho}[t] = \rho[\tilde{t}].
\]

• Apply to \( e^{i k_0 x} \) and to delta function.