Differentiation of (complex) function of complex variable

• Let $f$ be function $\mathbb{C} \rightarrow \mathbb{C}$. Can also treat it as a function of two real variables, $f(x, y)$, with $z = x + iy$.

• Let its real and imaginary parts be: $f(x, y) = u(x, y) + iv(x, y)$

• General concept of differentiation w.r.t. vector in real vector space:

Assume existence of linearized change in $f$ due to change in $(x, y)$:

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$

$$= dz \frac{1}{2} \left( \frac{\partial f}{\partial x} - i \frac{\partial f}{\partial y} \right) + dz^* \frac{1}{2} \left( \frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y} \right)$$

• Therefore define

$$\frac{\partial f}{\partial z} \overset{\text{def}}{=} \frac{1}{2} \left( \frac{\partial f}{\partial x} - i \frac{\partial f}{\partial y} \right) = \frac{1}{2} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) + \frac{i}{2} \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right)$$

$$\frac{\partial f}{\partial z^*} \overset{\text{def}}{=} \frac{1}{2} \left( \frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y} \right) = \frac{1}{2} \left( \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) + \frac{i}{2} \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right)$$
• Hence, it is as if $z$ and $z^*$ are independent:

$$df = \frac{\partial f}{\partial z} \, dz + \frac{\partial f}{\partial z^*} \, dz^*$$

• General function $f$:

– 2 real functions of 2 real variables $\implies$ 4 real derivative functions total
– Each of $\partial f/\partial z$, $\partial f/\partial z^*$ is 2 real functions.

• Defn.: A function $f(z)$ is **differentiable** at a point $z$ if $\frac{\partial f}{\partial z^*} = 0$

• A function differentiable at $z$ has

$$\frac{df}{dz} = \frac{\partial f}{\partial z} = \lim_{\delta z \to 0} \frac{f(z + \delta z) - f(z)}{\delta z}$$

like formula for derivative of function of real variable.

• It obeys Cauchy-Riemann equations:

$$\begin{align*}
\frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y}, \\
\frac{\partial v}{\partial x} &= -\frac{\partial u}{\partial y}
\end{align*}$$

i.e., $\text{div} \cdot (v, u) = 0$, $\text{curl} \times (v, u) = 0$.

• Hence $\nabla^2 u = \nabla^2 v = 0$, $\nabla u \cdot \nabla v = 0$. 

Aug. 27, . . . , 2014 (corrected)
Analytic functions: further notes

- \( f(z) \) is **differentiable** at \( z \) \iff \( \lim_{\delta z \to 0} \frac{f(z + \delta z) - f(z)}{\delta z} \)

- We define a function \( f(z) \) to be **analytic** at \( z \) if it is differentiable in a **neighborhood** of \( z \).

- This is (much) more stringent than ordinary differentiation for functions of real variables.

- It applies to functions constructed by any combination addition, multiplication, and division applied to \( z \), with limits, etc.

- For such cases, the standard formulae for derivatives of standard cases (e.g., \( \frac{d}{dz} z^n = nz^{n-1} \)) work exactly the same as for the corresponding functions of real variables.

- But we do not get an analytic function when \( z^* \) or \( |z| \) is used.

- I’ll give examples to illustrate these ideas and some further definitions given on the next slide.
Analytic functions: some basic definitions

- A function $f(z)$ is **analytic** at $z$ if in a neighborhood of $z$, $\frac{\partial f}{\partial z^*} = 0$ and $\frac{\partial f}{\partial z}$ exists.

- A **regular point** of $f(z)$ is a point where it is analytic.

- A **singular point** or **singularity** of $f(z)$ is a point where it is not analytic.

- An **isolated singular point** or **isolated singularity** of $f(z)$ is a point where it is not analytic, but is analytic in the rest of a neighborhood of the point.

- An **entire function** is one which is analytic in the whole of $\mathbb{C}$.

- We'll mostly restrict attention to functions with isolated singularities (and later some simple generalizations), and we'll often call these analytic functions.

- There are concepts of “**multi-valued**” functions and **branch points** that we’ll get to later in detail. (E.g., for $\sqrt{z}$ and $\ln z$.)

- Similarly for the idea of **analytic continuation**.