Alternative proof of Cauchy theorem

Let $\Gamma_0$ and $\Gamma_1$ be two contours in the complex plane with the same endpoints $z_1$ and $z_2$. Let $f(z)$ be a function that is analytic in the whole of the region bounded by the contours, and on the contours themselves. We aim to prove Cauchy’s theorem

$$\int_{\Gamma_0} f(z) \, dz = \int_{\Gamma_1} f(z) \, dz. \tag{1}$$

Let us choose a continuous series of contours $\Gamma_a$ that is parameterized by a real variable $a$, with $0 \leq a \leq 1$, and that continuously changes from $\Gamma_0$ at $a = 0$ to $\Gamma_1$ at $a = 1$, with the endpoints fixed. The contour $\Gamma_a$ is to stay inside the region bounded by $\Gamma_0$ and $\Gamma_1$. Let us parameterize each contour by a variable $s$ that runs between 0 and 1. Then we write the points on the contours as $\Gamma_a(s)$, a function of $a$ and $s$, such that $f(z)$ is analytic at $z = \Gamma_a(s)$. The fixed endpoints give $\Gamma_a(0) = z_1$ and $\Gamma_a(1) = z_2$:

where the dashed line is a possible contour $\Gamma_a$ for some intermediate value of $a$

Let $I_a$ be the integral of $f(z)$ along contour $\Gamma_a$: 

$$I_a = \int_{\Gamma_a} f(z) \, dz = \int_0^1 f(\Gamma_a(s)) \frac{\partial \Gamma_a(s)}{\partial s} \, ds. \tag{2}$$

Then

$$\frac{dI_a}{da} = \int_0^1 \left[ f(\Gamma_a(s)) \frac{\partial^2 \Gamma_a(s)}{\partial a \partial s} + f'(\Gamma_a(s)) \frac{\partial \Gamma_a(s)}{\partial a} \frac{\partial \Gamma_a(s)}{\partial s} \right] \, ds$$

$$= \int_0^1 \frac{\partial}{\partial s} \left[ f(\Gamma_a(s)) \frac{\partial \Gamma_a(s)}{\partial a} \right] \, ds$$

$$= \left[ f(\Gamma_a(s)) \frac{\partial \Gamma_a(s)}{\partial a} \right]_{s=0} \tag{3}$$

where, as usual, $f'(z) = df/\, dz$. Since the endpoints are fixed, $\partial \Gamma_a(s)/\partial a = 0$ when $s = 0$ or $s = 1$. Hence

$$\frac{dI_a}{da} = 0, \tag{4}$$

so that $I_a$ is a constant, and hence $I_0 = I_1$, which is Eq. (1).