\[ S(x) = \int_{-\infty}^{\infty} \frac{e^{ikx}}{\sqrt{k^2 + m^2}} \, dk = \int_{-\infty}^{\infty} h(k, x) \, dk \]

On the real axis, \( |h(k, x)| = \frac{1}{\sqrt{k^2 + m^2}} \).

This is of order \( \frac{1}{x} \) as \( k \to \pm \infty \), so \( \int_{-\infty}^{\infty} |h| \, dk \) diverges, and the integral for \( S \) is not absolutely convergent.

For convergence, consider one cycle of \( e^{ikx} \) from \( k = 2\pi n / x \) to \( 2\pi (n+1) / x \). Split into \( \frac{1}{2} \) cycles with opposite values for \( e^{ikx} \). Let

\[ f_n(x) = \int_{\frac{2\pi n}{x}}^{\frac{2\pi (n+1)}{x}} h(k, x) \, dk = \frac{1}{x} \int_{0}^{2\pi} d\theta \frac{e^{i\theta}}{\sqrt{\frac{(2\pi n + \theta)^2}{x^2} + m^2}} \]

(\text{where } k = \frac{2\pi n}{x} + \frac{\theta}{x})

\[ = \frac{1}{x} \left\{ \int_{0}^{\pi} d\theta \frac{e^{i\theta}}{\sqrt{\frac{(2\pi n + \theta)^2}{x^2} + m^2}} + \int_{\pi}^{2\pi} d\theta \frac{e^{i\theta}}{\sqrt{\frac{(2\pi n + \theta)^2}{x^2} + m^2}} \right\} \]

\[ = \frac{1}{x} \int_{0}^{\pi} d\theta e^{i\theta} \left[ \frac{1}{\sqrt{\frac{(2\pi n + \theta)^2}{x^2} + m^2}} - \frac{1}{\sqrt{\frac{(2\pi n + (-\theta))^2}{x^2} + m^2}} \right] \]

(since \( e^{i\theta} = -e^{-\theta} \)).
\[ \int_0^{\pi} d\theta \, e^{i \theta} \left\{ \frac{1}{\sqrt{1 + \left( \frac{\partial x}{2 \pi n} \right)^2 + \left( \frac{mx}{2 \pi n} \right)^2}} - \right\} \]

\[ = \frac{1}{2\pi n} \int_0^{\pi} d\theta \, e^{i \theta} \sim O \left( \frac{1}{n} \right) \]

\[ = O \left( \frac{1}{n^2} \right) \]

when \( n \to +\infty \)

\[ f(x) = \sum_{n=-\infty}^{\infty} f_n(x) \]

This converges since \( f_n = O \left( \frac{1}{n^2} \right) \) as \( n \to +\infty \)

(by comparison with \( \int dx / se^x \))

So the integral converges, but not absolutely

Analyticity of integral: \( L_k \)

Branch points at \( k = \pm \text{i}m \)

Choose cuts to go out along

\(-\text{i}m, \text{i}m, \text{positive imaginary axis}, \text{negative imaginary axis}, \text{real axis}\)
Since \( x > 0 \), the exponential gets a suppression in the \( V/K \). So we can deform the contour to

\[
\Gamma = \begin{array}{c}
\end{array}
\]

with \( R \to \infty \). By Jordan's lemma, the quarter circles go to zero.

Hence

\[
S(x) = \int_{-\infty}^{\infty} \text{id}k \, \frac{e^{-Kx}}{\sqrt{(k)^2 + m^2}} \bigg|_{\text{right half \, cut}}^{\text{left of cut}}
\]

with \( k = iK \).

\[
\sqrt{(k)^2 + m^2} = \sqrt{-k^2 + m^2} = \pm ik \sqrt{k^2 - m^2}.
\]

To find the correct sign, consider large \( k = Re^{i\theta} \) with \( R > 0 \) or \( \theta \) from \( 0 \) to \( \pi/2 \). Then \( \sqrt{k^2 + m^2} = k + 0(\sqrt{k^2}) \approx Re^{i\theta} \).
So we need $\sqrt{(k^2 - m^2)} = i \sqrt{k^2 - m^2}$ on a branch cut, to give continuity to $\sqrt{k^2 - m^2}$.

We need $-i \sqrt{k^2 - m^2}$ to be left of the cut.

So \[
\tag{11.4.9} \int f(x) = 2 \int dm \frac{e^{-kx}}{\sqrt{k^2 - m^2}}
\]