Let $V$ be a vector space over a field $F$. Then $W$ is a subspace of $V$ means that $W$ is a subset of $V$ that is itself a vector space over $F$ with the same addition and multiplication operations as $V$.

Given a linear operator $A: V \rightarrow W$,

$$\ker A = \{ v \in V : Av = 0 \}$$

To show $\ker A$ is a subspace of $V$, we need to show addition of elements of $\ker A$ and multiplication by numbers (elements of $F$) are closed. That is, any vector of $V$ is in $\ker A$ or that the addition inverse of a vector of $\ker A$ is in $\ker A$.

If $v, v' \in \ker A$, then $A(v + v') = A(v) + A(v') = 0 + 0 = 0$.

If $v \in \ker A$ and $\lambda \in F$, then $A(\lambda v) = \lambda A(v) = \lambda 0 = 0$.

So addition and multiplication are closed in $\ker A$.

Let $0_v = \text{zero of } V$. Then $0_v = 0_v + 0_v$.

$$A(0_v) = A(0_v + 0_v) = A(0_v) + A(0_v) = 0_v + 0_v = 0_v$$

Thus $0_v \in \ker A$.

Let $v \in \ker A$. Then $-v = (-1) \cdot v$.

$$A(-v) = A((-1) \cdot v) = (-1) A(v) = -A(v)$$

Hence $\ker A$ is a subspace of $V$. 

M.1.