Let \( I = \int_0^\infty \frac{\sin x}{x} \, dx \).

When \( x \to 0 \) the integrand is \( \frac{x - \frac{x^3}{3!} + \cdots}{x} = 1 + O(x^2) \).

So we have convergence at \( x = 0 \). The integrand is bounded and doesn't diverge anywhere. So the only issue is convergence at \( x \to \infty \).

(a) Absolute convergence means \( \int_0^\infty \left| \frac{\sin x}{x} \right| \, dx \) converges.

Split integral into ranges: \( 0 \) to \( \pi \), \( \pi \) to \( 2\pi \), etc.

From \( n\pi \) to \( (n+1)\pi \): \( \left| \frac{\sin x}{x} \right| < \frac{1}{n\pi} \left| \frac{\sin x}{x} \right| \)

So \( \int_{n\pi}^{(n+1)\pi} \frac{\sin x}{x} \, dx < \frac{1}{n\pi} \int_{n\pi}^{(n+1)\pi} \left| \frac{\sin x}{x} \right| \, dx = \frac{1}{n\pi} \int_{n\pi}^{(n+1)\pi} \frac{\sin x}{x} \, dx \)

\[ = \frac{1}{n\pi} \left[ -\cos x \right]_0^{(n+1)\pi} = \frac{2}{\pi} \frac{1}{n} = O(1/n) \]

\( \int_0^\infty \frac{\left| \sin x \right|}{x} \, dx < \int_0^\pi \frac{\left| \sin x \right|}{x} \, dx + \sum_{n=0}^{\infty} \frac{2}{\pi} \frac{1}{n} \)

The sum is divergent — by comparison with \( \int_0^\infty \frac{1}{x} \) at large \( x \).

Hence the integral \( \int_0^\infty \frac{\sin x}{x} \, dx \) is not absolutely convergent.
(b) Convergence.

Split the integral into ranges $2n\pi$ to $2(n+1/2)\pi$ that include neighboring positive and negative sections.

\[ I = \sum_{n=0}^{\infty} \int_{2n\pi}^{2(n+1/2)\pi} \frac{\sin x}{x} \, dx \quad (1) \]

\[ \int_{2n\pi}^{2(n+1/2)\pi} \frac{\sin x}{x} \, dx = \int_{0}^{2\pi} \frac{\sin x}{(2n\pi+x)} \, dx \]

\[ = \int_{0}^{\pi} \frac{\sin x}{2n\pi+x} \, dx + \int_{\pi}^{2\pi} \frac{\sin x}{2n\pi+x} \, dx \]

\[ = -\frac{\pi}{2} + \int_{0}^{\pi} \frac{\sin(x+y)}{2n\pi+y} \, dy \]

\[ = \int_{0}^{\pi} \left( \frac{1}{2n\pi+x} - \frac{1}{2n\pi+y} \right) \, dx \]

\[ = \int_{0}^{\pi} \frac{\sin x}{(2n\pi+y)(2n+y+x)} \, dx \]

As $n \to \infty$, the integrand is $O(1/n^2)$. Since $\sum_{n=1}^{\infty}$ converges, the integral $\int_{0}^{\infty} \frac{\sin x}{x} \, dx$ converges.

In more detail,

\[ \cdots \]
\[0 \leq \int_0^\pi \frac{\sin x}{(2\pi^2 + x)(2\pi^2 + \pi x)} \, dx \leq \int_0^\pi \frac{\sin x}{2\pi^2} \, dx\]
\[= \frac{1}{4\pi^2} \int_0^\pi \sin x \, dx\]
\[= \frac{\text{constant}}{\pi^2}\]

By comparison with the convergent series \(\sum \frac{1}{n^2}\),
the sum in (1) and hence the integral \(\int_0^\pi \frac{\sin x}{x} \, dx\) converge.