\[ S(k) = (k^2 + m^2)^{1/4}. \]

The singularities are where
\[ k^2 + m^2 = 0 \text{ i.e. } k = \pm \text{i}m, \]
and are clearly branch points,

given its fractional power

There are 4 possible values of \((k^2 + m^2)^{1/4}\). Given one value, the others differ by \(2\pi \text{i}\) \(-1, -\text{i}, \text{i}, 1\). (These factors together with 1 are all the fourth roots of \(1\).)

Let \(k\) be on the negative imaginary axis below \(-\text{i}m\):
\[ k = -\text{i}k \text{ with } k > m. \]

Then \[ S(k) = (\text{-i}k^2 + m^2)^{1/4}. \]

The possible values (on 4 sheets of the function) are:
\[ (k^2 - m^2)^{1/4} \times (\text{e}^{\text{i}\pi/4}, \text{e}^{3\text{i}\pi/4}, \text{e}^{5\text{i}\pi/4}, \text{e}^{7\text{i}\pi/4}) \]

with \((k^2 - m^2)^{1/4}\) defined to be real & positive.
To determine the one we need take \( k \) large: i.e. \( k \to 0 \).

Continue from \( \theta = 0 \) to \( \theta = -\pi/2 \) to come to the right of the lower cut.

\[
S(k) = \left( r^2 e^{2i\theta} + m^2 \right)^{\frac{1}{4}}
\]

\[
= \left[ r^2 e^{i\theta} \left( 1 + \frac{m^2}{r^2 e^{2i\theta}} \right) \right]^{\frac{1}{4}}
\]

\[
= r^{1/4} e^{i\theta/2} \left[ 1 + \frac{m^2}{r^2 e^{2i\theta}} \right]^{\frac{1}{4}} \quad \text{which is } r^{1/4} \text{ e}^{i\theta/2} \text{ large}
\]

where the choice of overall factor is to get the defined value of \( S(k) \) on the positive real axis.

At \( \theta = -\pi/2 \) we get

\[
r^{1/4} e^{-i\pi/4} \left( 1 + O(1/r) \right)
\]

So the value of \( S(k) \) on the right of the lower cut is

\[
(k^2 - m^2)^{1/4} e^{-i\pi/4},
\]

which is the one case in (1) that matches (2). (Note that

\[
e^{-i\pi/4} = e^{7i\pi/4} \text{ of course.}
\]
Alternative derivation

Start on the real axis where \( f(k) = \) positive real value \( (k^2 + m^2)^{1/4} \).

Go to \( k = 0 \): \( f(0) = m^{1/2} \).

Now go down the negative real axis: \( f(\pm ik) = (m^2 - k^2)^{1/4} \) is still real & positive. No other value of the fourth root of \( k^2 + m^2 \) can be used; otherwise the function would be discontinuous.

Near \( k = -im \) we let \( k = -im + re^{i\theta} \).

\[
 f(k) = (k^2 + m^2)^{1/4} = \left( -2imr e^{i\theta} + o(r) \right)^{1/4} \\
 = \left( 2mr \right)^{1/4} e^{i\theta/4} e^{-im\theta/2} (1 + o(r)),
\]

where we used \(-i = e^{-i\pi/2}\) so the overall factor was chosen to make \( f(k) \) real at \( \theta = \pi/2 \).

Go to \( \theta = \pi/2 \) to get to the right side of the cut.

Then \( f(k) = (2mr)^{1/4} e^{-im\pi/4} (1 + o(r)) \) below \( k = -im \) in the right.
This matches with

\[ f(-i\kappa) = (\kappa^2 - m^2)^{1/2} e^{-\kappa/\lambda}, \]

the same as before.