Each path is closely we deform each into the other continuously, (differentiably), by
\[ \Gamma_a(s) \text{ with } \Gamma_a(0) = \Gamma(0) \text{ for all } a \in [0,1]. \]

The function \( \Phi(z) \) is analytic with \( z = \Gamma_a(s) \) for any \( a \in [0,1] \). Nothing needs to be modified until the end of eq. (3), which is
\[
\frac{d\Gamma_a}{ds} = \left[ \Phi(\Gamma_a(s)) \frac{\partial \Phi}{\partial a} \right] \Big|_{s=0}
\]

Now a change is needed. The derivation of eq. (3) is replaced by the observation that the above equation gives
\[
\frac{d\Gamma_a}{ds} = \Phi(\Gamma_a(s)) \frac{\partial \Phi}{\partial a} - \Phi(\Gamma_a(0)) \frac{\partial \Phi}{\partial a},
\]
which is zero since \( \Gamma_a(1) = \Gamma(0) \) and hence \( \frac{\partial \Phi}{\partial a} = 0 \).

So we get \( \frac{d\Gamma_a}{da} = 0 \), as in eq. (6) of the previous derivation, and now it follows that \( r = 1 \), as before, the desired result.
Suppose the paths had opposite directions, e.g.

\[ C \rightarrow C_1 \rightarrow \cdot \cdot \cdot \rightarrow C \]

Let \( t_0 \) be a point in the middle of both.

If the proof were to work, we'd need to make a continuous function \( z(s, t) \), \( z(0) \) gives path \( C_0 \) and \( z(1) \) gives \( C_1 \).

Along \( C_0 \), the angle of \( z(s, t) \) with respect to \( z_0 \) decreases by \( 2\pi \). Along \( C_1 \), the angle of \( z(s, t) \) increases by \( 2\pi \).

At intermediate \( t \), the path is closed & \( z(s, t) \) changes by \( \pm 2\pi \) as \( s \) goes from 0 to 1. This is not possible for a continuous function.

Instead, we notice that the theorem does apply when \( C_1 \) is reversed. So if \( C_0 \) & \( C_1 \) are oppositely oriented, then

\[
\oint_{C_0} f(z) \, dz = -\oint_{C_1} f(z) \, dz
\]