complex-valued

For functions $f(x)$ defined on $0 \leq x \leq L$, we use the inner product

$$\langle f | g \rangle = \int_0^L dx \ f^*(x) g(x).$$  \hfill (1)

Define functions $|e_m\rangle$ by $e_m(x) = e^{ik_m x}$ with $k_m = \frac{2\pi m}{L}$.

We know these are basis functions and obey

$$\langle e_m | e_{m'} \rangle = L \delta_{m,m'}.$$  \hfill (2)

Define

$$|e_m\rangle = \frac{|e_m\rangle + |e_{-m}\rangle}{\sqrt{2}}, \quad \text{and} \quad e_m(x) = \cos k_m x$$

$$|s_m\rangle = \frac{|e_m\rangle - |e_{-m}\rangle}{\sqrt{2i}}, \quad \text{and} \quad s_m(x) = \sin k_m x.$$  \hfill (3)

For the $|e_m\rangle$s we'll restrict to $m > 0$, since $|e_{-m}\rangle = |e_m\rangle$
and the $|e_m\rangle$s with negative $m$ aren't independent.

Similarly, for the $|s_m\rangle$s we'll restrict to $m > 0$, now excluding $m = 0$ since
$$|s_0\rangle = \frac{|e_0\rangle - |e_{-0}\rangle}{\sqrt{2i}} = \frac{|e_0\rangle - |e_0\rangle}{\sqrt{2i}} = 0.$$  \hfill (4)

Thus $\{e_m\}$ and $\{s_m\}$s form a basis.

We prove this by showing that every $|e_m\rangle$ can be expressed in terms of the $|e_m\rangle$s and $|s_m\rangle$s.
\( m = 0 \):

\[ |e_0\> = |c_0\> \]

\( m > 0 \):

\[ |e_m\> = |c_m\> + i |s_m\> \]

\( m < 0 \):

\[ |e_{-m}\> = |c_m\> - i |s_m\> \]

(From eqs. (2) & (3)).

To show orthogonality, the following cases need to be considered:

1. \( \langle c_0 | c_m \rangle \) with \( m > 0 \)

\[ \langle c_0 | c_m \rangle = \frac{1}{2} \langle c_0 | (1 |e_m\> + |e_{-m}\>) \rangle = 0 \]

2. \( \langle c_0 | s_m \rangle \) with \( m > 0 \).

\[ \langle c_0 | s_m \rangle = \frac{1}{2i} \langle c_0 | (|e_m\> - |e_{-m}\>) \rangle = 0 \]

3. \( \langle c_m | c_{m_1} \rangle \) with \( m_1 = m_2 = m > 0, m > 0 \)

\[ \langle c_m | c_{m_1} \rangle = \frac{1}{4} \left( \langle c_m | + \langle c_{-m} | \right) (|e_{m_1}\> + |e_{-m_1}\>) \]

\[ = 0 \]

since \( m_1 = m_2, m_1 = -m_2, -m_1 + m_2 > 0, -m_1 - m_2 > 0 \).

4. Similarly

\[ \langle s_m | s_{m_1} \rangle = 0 \text{ when } m_1 = m \]

5. \( \langle e_m | s_{m_2} \rangle \) when \( m_1, m_2 > 0 \).
\[ \langle c_m | s_{m_i} \rangle = \frac{1}{4i} \left( \langle c_m | + \langle c_{-m} | \right) \langle 1_{m_i} - 1_{-m_i} \rangle \]

\[ = \frac{1}{4i} \left( \delta_{m_i, m_i} - \delta_{m_i, -m_i} + \delta_{-m_i, m_i} - \delta_{-m_i, -m_i} \right) \]

\[ = \frac{1}{4i} \left( \delta_{m_i, m_i} - 0 + 0 - \delta_{m_i, m_i} \right) \]

\[ = 0. \]