Let \( f(z) = \begin{cases} \frac{x^2(i) - y^2(i-i)}{x^2+y^2} & \text{if } z = x+iy \neq 0 \\ 0 & \text{if } z = 0 \end{cases} \)

So \( u = \frac{x^2-y^2}{x^2+y^2} \) and \( v = \frac{x^2+y^2}{x^2+y^2} \).

Now we compute the partial derivatives at \( z = 0 \):

\[
\frac{du}{dx} = \lim_{x \to 0} \frac{u(x,0) - u(0,0)}{x} = \lim_{x \to 0} \frac{x^2}{x^2} = 1
\]

\[
\frac{du}{dy} = \lim_{y \to 0} \frac{u(0,y) - u(0,0)}{y} = \lim_{y \to 0} \frac{-y^2}{y} = -1
\]

\[
\frac{dv}{dx} = \lim_{x \to 0} \frac{v(x,0) - v(0,0)}{x} = \lim_{x \to 0} \frac{x^2}{x^2} = 1
\]

\[
\frac{dv}{dy} = \lim_{y \to 0} \frac{v(0,y) - v(0,0)}{y} = \lim_{y \to 0} \frac{y^2}{y} = 1
\]

So \( \frac{du}{dx} = \frac{dv}{dy} + \frac{du}{dy} \frac{dv}{dx} \), i.e. the Cauchy-Riemann equations are obeyed.
\[ \frac{f(dz) - f(0)}{dz} = \frac{(dx)^2 (1+i) + (dy)^3 (1-i)}{(dy)^2 + (dy)^2} \]

Let \( dx = r \cos \theta \), \( dy = r \sin \theta \)

Then \[ \frac{f(dz) - f(0)}{dz} = \frac{r^3 \left[ \cos^3 \theta (1+i) - \sin^3 \theta (1-i) \right]}{r^2 (\cos^2 \theta + \sin^2 \theta) e^{1+i \theta}} \]

\[ = \frac{\cos^3 \theta (1+i) - \sin^3 \theta (1-i)}{\cos \theta + i \sin \theta} \]

This depends on \( \theta \), so it does not have a definite limit as \( dz \to 0 \). To see this, consider:

\( \theta = 0 \); \( \text{rhs} = \frac{1+i}{1} = 1+i \)

\( \theta = \frac{\pi}{4} \), so \( \cos \theta = \sin \theta = \frac{\sqrt{2}}{2} \):

\[ \text{rhs} = \left( \frac{1}{2} \right)^2 \left[ (1+i) - (1-i) \right] \]

\[ = \frac{1}{2} \left( \frac{2i}{1+i} \right) \]

\[ = \frac{i}{1+i} \]

\[ = \frac{i(1-i)}{(1+i)(1-i)} = \frac{1}{2} (1+i) \]
Extra material

To see in more detail what the problem is, consider
\[ \frac{\partial z}{\partial x} \text{ at a general value of } x, y. \]

\[ \frac{\partial z}{\partial x} = \frac{1}{(x^2 + y^2)^2} \left( (x^2 + y^2)^2 x^2 - (x^3 - y^3) 2x \right) \]

\[ = \frac{1}{(x^2 + y^2)^2} \left( x^4 + 3x^2y^2 + 2xy^3 \right). \]

The limit as \((x, y) \to (0, 0)\) depends on how the limit is taken, e.g., on which direction \((x, y) \to (0, 0)\) is approached. So the derivatives are discontinuous at the origin.

One cannot write
\[ f(dx + dy) - f(0) = \text{coefficient } dx + \text{coefficient } dy + o(dx, dy). \]

as is needed for \( \frac{\partial f}{\partial x} \) to exist.