Inference on Three-parameter Gamma Distribution Based on Progressively Censored Data

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Outline

- Three-parameter Gamma Distribution
- Progressive Censoring
- Estimation Methods
  - Maximum Likelihood Estimator
  - Expected Moment Estimator
- Illustrative Example
- Simulation Studies
- Concluding Remarks

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Three-parameter Gamma Distribution

- Lognormal distribution used for lifetimes or reaction-times and in particular for long-tailed and positively skewed data.
- It has been studied by Cohen and Norgaard (1977) and Cohen and Whitten (1988).
- The pdf of the three-parameter gamma distribution is

$$f(y;\boldsymbol{\theta}) = \begin{cases} \frac{1}{\Gamma(\alpha)\beta^{\alpha}}(y-\gamma)^{\alpha-1}e^{-\frac{x-\gamma}{\beta}}, \ \gamma < y < \infty, \ \beta > 0, \alpha > 0 \\ \\ 0 \ \text{ otherwise} \end{cases} \tag{1}$$

in which $\boldsymbol{\theta} = (\gamma, \beta, \alpha)$.

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Three-parameter Gamma Distribution

- The mean, variance, and skewness for this distribution are

$$\left.\begin{aligned} \mu_y &= \gamma + \alpha\beta \\ \sigma_y^2 &= \alpha\beta^2 \\ s_y &= \frac{2}{\sqrt{\alpha}}. \end{aligned}\right\}$$

- β is the scale parameter and α is the shape parameter of Y.
  - $\alpha \leq 1$: the distribution is "J" shaped.
  - $\alpha = 1$: exponential distribution.
  - $\alpha > 1$: the distribution is bell-shaped.
- γ is the threshold (location) parameter.
- Estimation methods become more complex when γ is unknown.
- It becomes even more complex when lifetimes data are censored.
- It becomes some more complex when progressively censored.

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Progressive Type II Right Censoring

- n units placed on an experiment
- m completely observed until failure
- censoring occurs progressively in m stages
  - First failure (the first stage): r_1 of the $n-1$ surviving units randomly withdrawn,
  - Second failure (the second stage): r_2 of the $n-2-r_1$ surviving units are withdrawn,
  - and so on.
  - Finally, the m-th failure (the m-th stage): remaining $r_m = n - m - r_1 - \cdots - r_{m-1}$ are withdrawn.

Type-II right censoring : $r_1 = r_2 = \cdots = r_{m-1} = 0$ and $r_m = n - m$.
Complete sampling scheme: $n = m$ and $r_1 = r_2 = \cdots = r_m = 0$.

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Maximum Likelihood Estimation Methods

- n independent gamma distributed units placed on a life-testing experiment.
- We observe only $\boldsymbol{Y} = (Y_1, \cdots, Y_m)$ where $Y_1 \leq \cdots \leq Y_m$ are m progressively censored order statistics.
- In the past, some work has been done on the three-parameter logormal distribution estimation methods (see, for example, Cohen and Norgaard (1977).
- Purpose of this article: to discuss different estimation procedures of $\boldsymbol{\theta}$ based on $\boldsymbol{Y}$.

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Maximum Likelihood Estimation Methods

- The log-likelihood function $\log L(\boldsymbol{\theta})$

$$
\begin{aligned}
= \quad & const. - m \log \Gamma(\alpha) - m\alpha \log \beta - \frac{1}{\beta} \sum_{j=1}^{m} (y_j - \gamma) \\
& + (\alpha - 1) \sum_{j=1}^{m} \log(y_j - \gamma) + \sum_{j=1}^{m} R_j \log \left[1 - F(y_j) \right],
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- Three equations required to solve simultaneously for $\hat{\theta}$:

$$\frac{m}{\beta} - (\alpha - 1)\sum_{j=1}^{m}\frac{1}{y_j - \gamma} + \frac{1}{\sqrt{\alpha}\beta}\sum_{j=1}^{m}R_j L_j = 0$$

$$-\frac{m\alpha}{\beta} + \frac{1}{\beta^2}\sum_{j=1}^{m}(y_j - \gamma) + \frac{1}{\sqrt{\alpha}\beta^2}\sum_{j=1}^{m}R_j(y_j - \gamma)L_j = 0$$

$$-m\psi(\alpha) - m\log\beta + \sum_{j=1}^{m}\log(y_j - \gamma) + \frac{1}{2\alpha^{\frac{3}{2}}\beta}\sum_{j=1}^{m}R_j(y_j - \gamma + \alpha\beta)L_j \tag{2}$$

in which $\psi(\alpha) = \frac{\partial \log\Gamma(\alpha)}{\partial\alpha}$, $L_j = \frac{g(z_j)}{1-G(z_j)}$

$z_j = \frac{Y_j - \gamma - \mu_y}{\sigma_y} = \frac{Y_j - \gamma - \alpha\beta}{\sqrt{\alpha}\beta}$

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Trial and error version

- 1st equation in (2) serves as a "test equation."
- Start with $\gamma_{(0)} < y_1$, get conditional estimates $\mu(\gamma_{(0)}), \sigma(\gamma_{(0)})$ by 2nd and 3rd equation of (2) *by a suggested iterative procedure*.
- Substitute $(\gamma_{(0)}, \mu(\gamma_{(0)}), \sigma(\gamma_{(0)}))$ in the "test equation."
- If RHS of the "test equation" is 0, stop. Otherwise, choose a second approximation γ_1 and repeat.
- Continue until two $\gamma_{(i-1)}$ and $\gamma_{(i)}$ are found such that $\frac{\partial \log L}{\partial \gamma_{(i-1)}} < (>) 0 < (>) \frac{\partial \log L}{\partial \gamma_{(i)}}$.
- Final estimate of γ is obtained through interpolation. Then final estimates of β and α are obtained from that.

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- Start with $\gamma_{(0)} < y_1$, get conditional estimates $\mu(\gamma_{(0)}), \sigma(\gamma_{(0)})$ by 2nd and 3rd equation of (2) *by a suggested iterative procedure*.
- Substitute $(\gamma_{(0)}, \mu(\gamma_{(0)}), \sigma(\gamma_{(0)}))$ in the "test equation."
- If RHS of the "test equation" is 0, stop. Otherwise, choose a second approximation γ_1 and repeat.
- Continue until two $\gamma_{(i-1)}$ and $\gamma_{(i)}$ are found such that $\frac{\partial \log L}{\partial \gamma_{(i-1)}} < (>) 0 < (>) \frac{\partial \log L}{\partial \gamma_{(i)}}$.
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Suggested Iterative Procedure

- For fixed value $\gamma_{(i)}$, first get of $\beta_{(i)}$ going through stages.
- The iterative scheme for getting the estimate $\beta_{(i)}$:

$$\beta^{(n+1)} = \frac{\sum_{j=1}^{m}(y_j - \gamma)}{g(\beta^{(n)})}. \tag{3}$$

$\beta_{(i)}^{(n+1)}$: the (n+1)-th stage value of $\beta_{(i)}$, depends on a fixed value of $\gamma_{(i)}$, $\alpha_{(i-1)}$ and the n-th stage value $\beta_{(i)}^{(n)}$ and we suppressed the subscripts (i) and $(i-1)$ for simplicity.

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$g(\beta^{(n)})$ is given by

$$g(\beta^{(n)}) = m\alpha - \frac{1}{\sqrt{\alpha}\beta^{(n)}} \sum_{j=1}^{m} R_j (y_j - \gamma) L_j|_{(n)}$$

where

$$L_j|_{(n)} = \frac{g(z_j|_{(n)})}{1 - G(z_j|_{(n)})},$$

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- We proved that the iterative scheme in (3) converges monotonically to the solution of the second equation of (2).

  Lemma 2.1 $\frac{\partial \log L}{\partial \beta}|_{(n)}$ has the same sign as $\Delta\beta = \beta^{(n+1)} - \beta^{(n)}$.

  Lemma 2.2 $\frac{\partial \log L}{\partial \beta}|_{(n+1)}$ has the same sign as $\Delta\beta = \beta^{(n+1)} - \beta^{(n)}$

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- Once the estimate $\beta_{(i)}$ is obtained through the iterative scheme (3), $\alpha_{(i)}$ is obtained, based on $\gamma_{(i)}$ and $\beta_{(i}$, going again through some stages. The iterative scheme for getting the estimate $\alpha_{(i)}$:

$$\alpha^{(n+1)} = \frac{m}{h(\alpha^{(n)}) + \sum_{j=1}^{m} \log(y_j - \gamma) - m \log \beta} + 1. \qquad (4)$$

$\alpha_{(i)}^{(n+1)}$, the (n+1)-th stage value of $\alpha_{(i)}$, depends on a fixed value of $\gamma_{(i)}$, $\beta_{(i)}$ and the n-th stage value $\alpha_{(i)}^{(n)}$ and we suppressed the subscripts (i) for simplicity.

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Outline

1. Three-parameter Gamma Distribution
2. Progressive Type II Right Censoring
3. Maximum Likelihood Estimation Methods
4. Expected Moment Estimator
5. Illustrative Example
6. Simulation Studies
7. Concluding Remarks

Expected Moment Estimator

- Progressive censored sampling is highly effective in saving time and money. Scheme is not very popular because of complications in estimation methods;
- Progressive censoring model can be viewed as missing data problem.
- Another option to explore is the use of EM algorithm for numerically finding the MLEs.
  - E-step: replace any missing data by its expected value.
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Expected Moment Estimator

- we propose an alternative to EM estimates where M-step is replaced by the moment estimators (Moment-step). Moment estimates have been traditionally used for Gamma distribution since these are given by explicit expressions.
- The censored data vector as $\boldsymbol{Z} = (\boldsymbol{Z}_1, \boldsymbol{Z}_2, \cdots, \boldsymbol{Z}_m)$ where $\boldsymbol{Z}_j = (Z_{j1}, Z_{j2}, \cdots, Z_{jR_j}); j = 1, 2, \cdots, m.$
- The complete data set is then obtained combining the observed data $\boldsymbol{Y}$ and the censored data $\boldsymbol{Z}$.
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Expected Moment Estimator – Moment-step

- Starting value $\hat{\boldsymbol{\theta}}_{(0)}$:
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$$\left.\begin{aligned}
\hat{\alpha}_{(0)} &= \frac{1}{n} \cdot \frac{\left[\sum_{j=1}^{m}(R_j+1)y_j - \gamma_{(0)}\right]^2}{\sum_{j=1}^{m}(R_j+1)\left[y_j - \frac{1}{n}\sum_{j=1}^{m}(R_j+1)y_j\right]^2} \\
\hat{\beta}_{(0)} &= \frac{\sum_{j=1}^{m}(R_j+1)\left[y_j - \frac{1}{n}\sum_{j=1}^{m}(R_j+1)y_j\right]^2}{\sum_{j=1}^{m}(R_j+1)y_j - \gamma_{(0)}}
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- $\hat{\boldsymbol{\theta}}_{(h+1)}$: $(h+1)$-th iteration updated estimates of $\boldsymbol{\theta}$. It is obtained using $\hat{\boldsymbol{\theta}}_{(h)}$ as follows:

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$$\left.\begin{aligned}
\bar{Y}_{(h)} &= \frac{1}{n}\left[\sum_{j=1}^{m}(y_j-\gamma_{(h)})+\right. \\
&\quad \left.\sum_{j=1}^{m}R_jE\left[(z-\gamma_{(h)})|z>y_j;\boldsymbol{\theta}_{(h)}\right]\right] \\
V_{(h)} &= \frac{1}{n}\left[\sum_{j=1}^{m}(y_j-\gamma_{(h)})^2+\right. \\
&\quad \left.\sum_{j=1}^{m}R_jE\left[(z-\gamma_{(h)})^2|z>y_j;\boldsymbol{\theta}_{(h)}\right]\right]-\bar{Y}_{(h)}^2.
\end{aligned}\right\} \quad (7)$$

Expected Moment Estimator

- In the above expressions

$$\left.\begin{aligned} E\left[Z-\gamma_{(h)}|Z>y_j;\boldsymbol{\theta}_{(h)}\right] &= \beta_{(h)}\cdot P_{1j(h)} \\ E\left[(Z-\gamma_{(h)})^2|Z>y_j;\boldsymbol{\theta}_{(h)}\right] &= \beta_{(h)}^2\cdot P_{2j(h)} \end{aligned}\right\}$$

where

$$z_{j(h)}=\frac{y_j-\gamma_{(h)}-\alpha_{(h)}\beta_{(h)}}{\sqrt{\alpha_{(h)}}\beta_{(h)}},$$

$$P_{1j(h)}=P_{1j}(\theta_{(h)})=\frac{\int_{\sqrt{\alpha_{(h)}}(\sqrt{\alpha_{(h)}}+z_{j(h)})}^{\infty} m^{\alpha_{(h)}}e^{-m}dm}{\int_{\sqrt{\alpha_{(h)}}(\sqrt{\alpha_{(h)}}+z_{j(h)})}^{\infty} m^{\alpha_{(h)}-1}e^{-m}dm},$$

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- $\hat{\gamma}_{(h+1)}$ obtained by solving the following equation for γ (which can also serve as an test equation):

$$\frac{m}{\beta_{(h+1)}} - (\alpha_{(h+1)} - 1)\sum_{j=1}^{m}\left[\frac{1}{y_j - \gamma} + R_j E\left[\frac{1}{Z-\gamma}|Z > y_j; \gamma, \boldsymbol{\theta}^*_{(h+1)}\right]\right] = 0, (8)$$

where $\boldsymbol{\theta}^*_{(h+1)} = (\beta_{(h+1)}, \alpha_{(h+1)})$.

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Trial and Error Version

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- (8) serves as a “test equation."
- Start with $\gamma_{(0)} < y_1$, get conditional estimates $\beta(\gamma_{(0)})$, $\alpha(\gamma_{(0)})$ by (5).
- Update β and α by (7) to get $\beta^*_{(0)}$, $\alpha^*_{(0)}$.
- Substitute $(\gamma_{(0)}, \beta^*_{(0)}, \alpha^*_{(0)})$ in the “test equation."
- If RHS of the “test equation" is 0, stop. Otherwise, choose a second approximation {(1)} and repeat.
- Continue until two $\gamma_{(i-1)}$ and $\gamma_{(i)}$ are found such that “test equation" based on $\gamma_{(i-1)} < (>) 0 < (>)$ “test equation" based on $\gamma_{(i)}$.
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2. Progressive Type II Right Censoring
3. Maximum Likelihood Estimation Methods
4. Expected Moment Estimator
5. Illustrative Example
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7. Concluding Remarks

Illustrative Example

- To illustrate the computational methods presented in this article, we use an example given in Cohen and Norgaard (1977).
- The data is the life span of 100 randomly selected units of a certain electronic device which is distributed as gamma with $\gamma = 50$, $\beta = 15$, and $\alpha = 10$.
- Out of these 100 units, 80 complete life spans were observed and recorded and 20 observations were censored in two stages.
- In terms of the notations used in the article $n = 100$ and $m = 80$, $R_{50} = 10$ and $R_{80} = 10$ with $R_i = 0$ for $i \neq 50, 100$.
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- We use the same progressively censored data in the example to carry out the estimation procedures in order to see whether they produce similar results or not.

Table 1. Progressively censored data for the Example

107.65	137.72	154.55	163.69	184.02	193.36	213.51	231.65
110.10	138.39	155.32	165.35	184.52	195.73	217.01	234.21
112.11	142.56	155.39	166.18	184.93	196.21	217.20	234.42
114.52	145.55	157.21	167.75	185.08	199.03	219.23	237.16
117.45	146.15	158.26	174.76	187.42	201.68	219.64	239.80
126.49	146.51	158.41	175.40	189.14	202.09	220.02	253.46
128.27	147.15	158.44	176.17	189.63	206.69	220.46	256.35
133.79	149.61	159.70	177.91	189.87	209.33	222.73	256.57
134.09	149.83	160.55	182.41	190.32	210.94	226.72	257.74
136.81	151.98	160.57	182.50	192.66	213.06	230.29	260.22

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We will denote the expected momemt estimate by the notation EMOMENT. The MLEs and EMOMENTs are presented in Table 2.

Table 2. Estimates for γ, β and α based on data in Table 1

Estimator	γ	β	α
MLE	$\hat{\gamma} = 89.29$	$\hat{\beta} = 29.68$	$\hat{\alpha} = 3.73$
EMOMENT	$\tilde{\gamma} = 63.28$	$\tilde{\beta} = 18.56$	$\tilde{\alpha} = 5.96$

- Cohen and Norgaard (1977) Estimates of γ, β and α to be 91.75, 31.96 and 3.42 respectively by using a different numerical procedure. One can observe the similarities of these estimates to our MLEs given in Table 1.
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Simulation Studies

Simulation Study 1:

- We constructed 95% confidence intervals based on 10,000 randomly generated progressively censored samples from the three-parameter gamma distribution with $\gamma = 50, \beta = 15$, and $\alpha = 10$ corresponding to the two proposed estimation procedures.
- We used the same progressive censoring schemes ($n = 100, m = 80, R_{50} = 10$ and $R_{80} = 10$ with $R_i = 0$ for $i \neq 50, 100$) as given in the above example.
- Throughout the simulation study, it was observed that the number of iterative steps required for obtaining MLEs were cosiderably lower than those for obtaining EMOMENTs.
- We provide the confidence intervals from this simulation study in Table 3 for different estimators.

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Simulation Studies

Table 3. 95% confidence intervals for the parameters for the proposed estimation methods

Estimator	γ	β	α
MLE	(80.88, 91.62)	(26.17, 34.87)	(3.56, 6.06)
EMOMENT	(60.83, 69.47)	(13.93, 28.72)	(4.06, 8.36)

Simulation Studies

Simulation Study 2:

- To compare the performance of MLE and EMOMENT in terms of the probability coverages of 95% confidence intervals for the parameters for different sample sizes and different degrees of censoring.
- 10,000 samples simulated from the distribution used in example 3 for each of sample size $n = 20, 40, 80, 100$.
- For $n = 20$, we considered $m = 12, 16$ stages.
- For $n = 40$, we considered $m = 24, 32$ stages.
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- For $n = 100$, we considered $m = 60, 80$ stages.

Simulation Studies

Simulation Study 2:

- To compare the performance of MLE and EMOMENT in terms of the probability coverages of 95% confidence intervals for the parameters for different sample sizes and different degrees of censoring.
- 10,000 samples simulated from the distribution used in example 3 for each of sample size $n = 20, 40, 80, 100$.
- For $n = 20$, we considered $m = 12, 16$ stages.
- For $n = 40$, we considered $m = 24, 32$ stages.
- For $n = 80$, we considered $m = 48, 64$ stages.
- For $n = 100$, we considered $m = 60, 80$ stages.

Simulation Studies

- Also, for fixed values of n and m and hence for fixed proportion (P) of uncensored data, two levels of censoring (one reflecting comparatively delayed censoring than other) are considered.
- The probabilty coverages (PC) of 95% confidence intervals from this simulative study for the parameters γ, β and α are reported in Tables 4, 5 and 6 respectively.

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Table 4. Coverage probabilities of 95% confidence intervals for γ

n	m	R	MLE	EMOMENT
20	12	(0,0,0, $\cdots$, 4, $\cdots$,4)	92.68	93.72
20	12	(6,2,0, $\cdots$, 0)	90.82	93.14
20	12	(0,0,0, $\cdots$,2,6)	91.25	94.21
20	16	(0,0,0, $\cdots$, 2, $\cdots$,2)	92.13	93.45
20	16	(3,1,0, $\cdots$, 0)	91.59	93.27
20	16	(0,0,0, $\cdots$,1,3)	92.82	93.95
40	24	(0,0,0, $\cdots$, 8, $\cdots$,8)	92.95	93.92
40	24	(12,4,0, $\cdots$, 0)	93.72	93.02
40	24	(0,0,0, $\cdots$,4,12)	94.51	93.52
40	32	(0,0,0, $\cdots$, 4, $\cdots$,4)	93.16	94.17
40	32	(6,2,0, $\cdots$, 0)	91.08	94.15
40	32	(0,0,0, $\cdots$,2,6)	93.72	94.04

Table 4. Coverage probabilities of 95% confidence intervals for γ (Contd)

n	m	R	MLE	EMOMENT
80	48	(0,0,0, $\cdots$, 16, $\cdots$,16)	93.02	94.03
80	48	(24,8,0, $\cdots$, 0)	92.48	94.21
80	48	(0,0,0, $\cdots$,8,24)	93.75	94.98
80	64	(0,0,0, $\cdots$, 8, $\cdots$,8)	94.34	95.33
80	64	(10,6,0, $\cdots$, 0)	92.18	95.75
80	64	(0,0,0, $\cdots$,6,10)	94.59	95.81
100	60	(0,0,0, $\cdots$, 20, $\cdots$,20)	94.87	94.11
100	60	(30,10,0, $\cdots$, 0)	95.21	95.02
100	60	(0,0,0, $\cdots$,10,30)	95.15	95.27
100	80	(0,0,0, $\cdots$, 10, $\cdots$,10)	94.83	95.25
100	80	(15,5,0, $\cdots$, 0)	95.02	95.11
100	80	(0,0,0, $\cdots$,5,15)	95.62	95.21

Table 5. Coverage probabilities of 95% confidence intervals for β

n	m	R	MLE	EMOMENT
20	12	(0,0,0, $\cdots$, 4, $\cdots$,4)	94.12	95.36
20	12	(6,2,0, $\cdots$, 0)	93.78	94.06
20	12	(0,0,0, $\cdots$,2,6)	94.25	94.76
20	16	(0,0,0, $\cdots$, 2, $\cdots$,2)	94.11	95.23
20	16	(3,1,0, $\cdots$, 0)	94.05	95.34
20	16	(0,0,0, $\cdots$,1,3)	94.51	95.47
40	24	(0,0,0, $\cdots$, 8, $\cdots$,8)	93.76	94.37
40	24	(12,4,0, $\cdots$, 0)	94.63	94.09
40	24	(0,0,0, $\cdots$,4,12)	94.75	95.02
40	32	(0,0,0, $\cdots$, 4, $\cdots$,4)	94.21	94.74
40	32	(6,2,0, $\cdots$, 0)	94.01	95.48
40	32	(0,0,0, $\cdots$,2,6)	94.87	95.89

Table 5. Coverage probabilities of 95% confidence intervals for β(Contd)

n	m	R	MLE	EMOMENT
80	48	(0,0,0, $\cdots$, 16, $\cdots$,16)	94.64	94.58
80	48	(24,8,0, $\cdots$, 0)	94.16	95.06
80	48	(0,0,0, $\cdots$,8,24)	95.04	94.98
80	64	(0,0,0, $\cdots$, 8, $\cdots$,8)	95.63	95.21
80	64	(10,6,0, $\cdots$, 0)	95.87	95.02
80	64	(0,0,0, $\cdots$,6,10)	94.95	95.23
100	60	(0,0,0, $\cdots$, 20, $\cdots$,20)	95.23	94.93
100	60	(30,10,0, $\cdots$, 0)	94.99	95.23
100	60	(0,0,0, $\cdots$,10,30)	95.34	95.55
100	80	(0,0,0, $\cdots$, 10, $\cdots$,10)	94.89	94.93
100	80	(15,5,0, $\cdots$, 0)	96.01	95.38
100	80	(0,0,0, $\cdots$,5,15)	95.06	95.22

Table 6. Coverage probabilities of 95% confidence intervals for α

n	m	R	MLE	EMOMENT
20	12	(0,0,0, $\cdots$, 4, $\cdots$,4)	93.35	94.44
20	12	(6,2,0, $\cdots$, 0)	94.06	94.41
20	12	(0,0,0, $\cdots$,2,6)	93.98	94.46
20	16	(0,0,0, $\cdots$, 2, $\cdots$,2)	93.77	94.18
20	16	(3,1,0, $\cdots$, 0)	94.22	93.12
20	16	(0,0,0, $\cdots$,1,3)	94.35	94.57
40	24	(0,0,0, $\cdots$, 8, $\cdots$,8)	93.77	94.05
40	24	(12,4,0, $\cdots$, 0)	94.14	95.55
40	24	(0,0,0, $\cdots$,4,12)	95.21	95.08
40	32	(0,0,0, $\cdots$, 4, $\cdots$,4)	94.13	95.02
40	32	(6,2,0, $\cdots$, 0)	93.88	94.17
40	32	(0,0,0, $\cdots$,2,6)	94.41	94.65

Table 6. Coverage probabilities of 95% confidence intervals for α(contd)

n	m	R	MLE	EMOMENT
80	48	(0,0,0, $\cdots$, 16, $\cdots$,16)	95.15	95.03
80	48	(24,8,0, $\cdots$, 0)	94.99	95.17
80	48	(0,0,0, $\cdots$,8,24)	94.38	95.24
80	64	(0,0,0, $\cdots$, 8, $\cdots$,8)	96.13	95.32
80	64	(10,6,0, $\cdots$, 0)	95.45	95.21
80	64	(0,0,0, $\cdots$,6,10)	95.32	94.89
100	60	(0,0,0, $\cdots$, 20, $\cdots$,20)	95.21	94.
100	60	(30,10,0, $\cdots$, 0)	95.37	95.02
100	60	(0,0,0, $\cdots$,10,30)	95.22	95.15
100	80	(0,0,0, $\cdots$, 10, $\cdots$,10)	94.96	95.31
100	80	(15,5,0, $\cdots$, 0)	95.27	94.74
100	80	(0,0,0, $\cdots$,5,15)	95.02	94.83

Simulation Studies

It has to be noted that :

- probability coverages are better when the proportion P of uncensored data is larger.
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Outline

1. Three-parameter Gamma Distribution
2. Progressive Type II Right Censoring
3. Maximum Likelihood Estimation Methods
4. Expected Moment Estimator
5. Illustrative Example
6. Simulation Studies
7. Concluding Remarks

Concluding Remarks

- In this article, we proposed an iterative scheme for maximum likelihood estimation and expected moment estimation based on missing value principle for Type II progressively censored samples from a three-parameter gamma distribution.
- The results of Monte Carlo simulation study indicates that the EMOMENT procedure performs little better thsn MLE procedure in terms of the coverage probabilty.
- The results of Monte Carlo simulation study indicates that the MLE procedure took less number of iterative steps,in general, as compared to EMOMENT procedure.
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