A note on different Predictors of Lifetimes of Censored Items in Progressively Censored Samples from Normal Distribution

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Outline

- Progressive Censoring

- Prediction Problem

- Different Predictors
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Progressive Type II Right Censoring

- $n$ units placed on an experiment
- $m$ completely observed until failure
- Censoring occurs progressively in $m$ stages
  - First failure (the first stage): $r_1$ of the $n - 1$ surviving units randomly withdrawn,
  - Second failure (the second stage): $r_2$ of the $n - 2 - r_1$ surviving units are withdrawn,
  - and so on.
  - Finally, the $m$-th failure (the $m$-th stage): remaining $r_m = n - m - r_1 - \cdots - r_{m-1}$ are withdrawn.

Type-II right censoring: $r_1 = r_2 = \cdots = r_{m-1} = 0$ and $r_m = n - m$.
Complete sampling scheme: $n = m$ and $r_1 = r_2 = \cdots = r_m = 0$. 
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Progressive Type II Right Censoring

- *n* units placed on an experiment
- *m* completely observed until failure
- Censoring occurs progressively in *m* stages
  - First failure (the first stage): *r*₁ of the *n* − 1 surviving units randomly withdrawn,
  - Second failure (the second stage): *r*₂ of the *n* − 2 − *r*₁ surviving units are withdrawn,
  - and so on.
  - Finally, the *m*-th failure (the *m*-th stage): remaining *r*ₘ = *n* − *m* − *r*₁ − · · · − *r*ₘ−₁ are withdrawn.

Type-II right censoring: *r*₁ = *r*₂ = · · · = *r*ₘ−₁ = 0 and *r*ₘ = *n* − *m*.

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Prediction

- $X_1, \ldots, X_n$: failure times of $n$ independent units from cdf $F(x, \theta)$ and pdf $f(x, \theta)$ with parameter $\theta$.
- We observe only $Y = (Y_1, \ldots, Y_m)$ where $Y_1 \leq \cdots \leq Y_m$: $m$ progressively censored order statistics.
- Purpose of this article: to predict life-lengths $Y_{j:r_i}; j = 1, 2, \ldots, r_i; i = 1, 2, \ldots, m$ of all censored units in all $m$ stages of censoring.
- Prediction of times to failure of only the last $r_m$ units still surviving at the observation $Y_m$ has been considered by Balakrishnan and Rao (1997) for Exponential Distribution.
- BLUP, MLP, MMLP, AMLP and CMP are derived for normal distribution for all $Y_{j:r_i}; j = 1, 2, \ldots, r_i; i = 1, 2, \ldots, m$. 
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**Notations**

\[ Y_{j:r_i} : \text{jth order statistic out of } r_i \text{ units of } Y \]

\[ Y_{j:r_i}^B : \text{BLUP of } Y_{j:r_i} \]

\[ Y_{j:r_i}^{OSM} : \text{OSMLP of } Y_{j:r_i} \]

\[ Y_{j:r_i}^{TSM} : \text{TSMLP of } Y_{j:r_i} \]
Notations – Continued

\[ Y_{j:r_i}^{OSMM} : \text{OSMMLP of } Y_{j:r_i} \]

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\[ Y_{j:r_i}^{TSAM} : \text{TSAMLP of } Y_{j:r_i} \]

\[ Y_{j:r_i}^{C} : \text{CMP of } Y_{j:r_i} \]
The Normal distribution

- \( f(x; \mu, \sigma) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{1}{2\sigma^2} (x-\mu)^2}, -\infty < \mu < \infty, \sigma > 0. \)

- \( Y = (Y_1, \cdots, Y_m). \)

- For the sake of simplicity, assume \( \sigma = 1 \) and will cover only the case of unknown scale parameter \( \mu. \)

- The case when both parameters \( \mu \) and \( \sigma \) are unknown is handled with little additional difficulty.
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The BLUP $Y_{j:r_i}^B$ of $Y_{j:r_i}$ is given by

$$Y_{j:r_i}^B = \mu^* + \alpha_1 + w'\Sigma^{-1}(Y - \mu^*1 - \alpha).$$

where the BLUE $\mu^*$ of $\mu$:

$$\mu^* = \frac{1'\Sigma^{-1}(Y - \alpha)}{1'\Sigma^{-1}1}.$$

$\alpha$ and $\Sigma$ are the vector of means and dispersion matrix, respectively, of the $m$ progressively Type-II right censored order statistics from the standard normal distribution.
The BLUP $Y_{j:r_i}^B$ of $Y_{j:r_i}$ is given by

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- $\alpha$ and $\Sigma$ are the vector of means and dispersion matrix, respectively, of the $m$ progressively Type-II right censored order statistics from the standard normal distribution.
**BLUP**

- \( \alpha_1 \) is the expected value of the \( j \)-th order statistic out of \( r_i \) units from the standard normal conditional distribution, with the condition that it is greater than \( Y_i \).

\[
\alpha_1 = c_{i-1} \sum_{k=1}^{i} a_{k,i} \sum_{l=0}^{\gamma_k-r_i-1} (-1)^l \binom{\gamma_k-r_i-1}{l} \times \frac{r_i! \ l!}{(r_i + l + 1)!} E(X_{j+l+1:r_i+l+1}).
\]
Progressive Censoring

Prediction Problem

Different Predictors

BLUP

- $\alpha_1$ is the expected value of the $j$-th order statistic out of $r_i$ units from the standard normal conditional distribution, with the condition that it is greater than $Y_i$.

\[
\alpha_1 = c_{i-1} \sum_{k=1}^{i} a_{k,i} \sum_{l=0}^{\gamma_k - r_i - 1} (-1)^l \binom{\gamma_k - r_i - 1}{l} \times \frac{r_i!}{(r_i + l + 1)!} E(X_{j+l+1:r_i+l+1}).
\]
BLUP

\[ \mathbf{w}' = (0, 0, \cdots, \sigma_{ij}, \cdots, 0) \text{ with } \sigma_{ij} = \text{Cov}(Y_i, Y_{j:r_i}). \]

\[
\text{Cov}(Y_i, Y_{j:r_i}) = E(Y_i Y_{j:r_i}) - E(Y_i)E(Y_{j:r_i}) = E(Y_i Y_{j:r_i}) - \alpha_1 E(Y_i),
\]
BLUP

- \( w' = (0, 0, \cdots, \sigma_{ij}, \cdots, 0) \) with \( \sigma_{ij} = \text{Cov}(Y_i, Y_{j:r_i}) \).

\[
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\]
BLUP

\[
E(Y_i Y_{j:r_i}) = c_{i-1} \sum_{k=1}^{i} a_{k,i} \sum_{l=0}^{\gamma_k-r_i-1} (-1)^l \binom{\gamma_k-r_i-1}{l} \\
\times \frac{r_i! \cdot l!}{(r_i + l + 1)!} \times E(X_{l+1:r_i+l+1} \cdot X_{j+l+1:r_i+l+1}),
\]

\[
E(Y_i) = c_{i-1} \sum_{k=1}^{i} a_{k,i} \frac{1}{\gamma_k} E[X_{1:\gamma_k}],
\]
BLUP

where

\[
\begin{align*}
\gamma_1 &= n \\
\gamma_k &= n - r_1 - \cdots - r_{k-1} - k + 1; \quad k = 2, \ldots, m \\
c_{l-1} &= c_{l-1}(\gamma_1, \cdots, \gamma_l) = \prod_{k=1}^{l} \gamma_k; \quad l = 1, \cdots, m \\
a_{k,l} &= a_{k,l}(\gamma_1, \cdots, \gamma_l) = \prod_{u=1, u \neq k}^{l} \frac{1}{\gamma_u - \gamma_k}. \quad 1 \leq k \leq l \leq m
\end{align*}
\]
Outline

1. Progressive Censoring
2. Prediction Problem
3. Different Predictors
   - BLUP
   - MLP
     - OSMLP
     - TSMLP
   - MMLP
     - OSMMLP
     - TSMMLP
   - AMLP
     - OSAMLPI
     - TSAMLPI
   - CMP
Let $y = (y_1, y_2, \cdots, y_m)$ with $y_1 \leq y_2 \cdots \leq y_m$ and $y = y_{j:r_i}$ denote the observed value of $Y$ and the unobserved value of $Y_{j:r_i}$, respectively.

The predictive likelihood function (PLF) of $Y_{j:r_i}$ and $\theta$ is given by

$$
L = L(y, \theta; y)
= cf(y) [F(y) - F(y_i)]^{j-1} [1 - F(y)]^{r_i-j} \\
\times \prod_{l=1}^{m} f(y_l) \prod_{l=1, l \neq i}^{m} [1 - F(y_l)]^{r_l}, \ y \geq y_i,
$$

where $c$ denotes a constant factor.
Let $y = (y_1, y_2, \cdots, y_m)$ with $y_1 \leq y_2 \cdots \leq y_m$ and $y = y_{j:r_i}$ denote the observed value of $Y$ and the unobserved value of $Y_{j:r_i}$, respectively.

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where $c$ denotes a constant factor.
Progressive Censoring

Prediction Problem

MLP

Let \( y = (y_1, y_2, \cdots, y_m) \) with \( y_1 \leq y_2 \cdots \leq y_m \) and \( y = y_{j:r_i} \) denote the observed value of \( Y \) and the unobserved value of \( Y_{j:r_i} \), respectively.

The predictive likelihood function (PLF) of \( Y_{j:r_i} \) and \( \theta \) is given by

\[
L = L(y, \theta; y) \\
= cf(y) [F(y) - F(y_i)]^{j-1} [1 - F(y)]^{r_i-j} \\
\times \prod_{l=1}^{m} f(y_l) \prod_{l=1, l \neq i}^{m} [1 - F(y_l)]^{r_l}, \quad y \geq y_i,
\]

where \( c \) denotes a constant factor.
MLP

If \( Y^L_{j:r_i} = t(Y) \) and \( \theta^{**} = u(Y) \) are statistics for which

\[
L(t(y), u(y); y) = \sup_{y, \theta} L(y, \theta; y),
\]

then \( t(Y) \) is said to be the MLP of \( Y_{j:r_i} \), and \( u(Y) \) the predictive maximum likelihood estimator (PMLE) of \( \theta \).
If $Y_{j:r_i}^L = t(Y)$ and $\theta^{**} = u(Y)$ are statistics for which

$$L(t(y), u(y); y) = \sup_{y, \theta} L(y, \theta; y),$$

then $t(Y)$ is said to be the MLP of $Y_{j:r_i}$, and $u(Y)$ the predictive maximum likelihood estimator (PMLE) of $\theta$. 
MLP

The PLF of $Y_{j:r_i} = Y$ and $\mu$ is given by

$$L(Z, \mu; Z) = c_1 c_2 f(Z) \prod_{l=1}^{m} f(Z_l) \prod_{l=1, l \neq i}^{m} [1 - F(Z_l)]^{r_l}$$

$$\times [F(Z) - F(Z_i)]^{j-1} [1 - F(Z)]^{r_i-j}, \ Z \geq Z_i,$$

where

$Z = Y - \mu, Z_l = Y_l - \mu \ (l = 1, \cdots, m), Z = (Z_1, Z_2, \cdots, Z_m),$

c_1 = c_1(j, r_i) = \frac{r_i!}{(j-1)! (r_i-j)!}$ and

c_2 = c_2(n, m, r_1, \cdots, r_m) =

$$n(n - r_1 - 1)(n - r_1 - r_2 - 2) \cdots (n - r_1 - \cdots - r_{m-1} - m + 1).$$
The PLF of \(Y_{j:r_i} = Y\) and \(\mu\) is given by

\[
L(Z, \mu; Z) = c_1 c_2 f(Z) \prod_{l=1}^{m} f(Z_l) \prod_{l=1, l \neq i}^{m} [1 - F(Z_l)]^{r_i} \\
\times [F(Z) - F(Z_i)]^{i-1} [1 - F(Z)]^{r_i-j}, \quad Z \geq Z_i,
\]

where

\[
Z = Y - \mu, \quad Z_l = Y_l - \mu \ (l = 1, \cdots, m), \quad Z = (Z_1, Z_2, \cdots, Z_m),
\]

\[
c_1 = c_1(j, r_i) = \frac{r_i!}{(j-1)!(r_i-j)!} \quad \text{and}
\]

\[
c_2 = c_2(n, m, r_1, \cdots, r_m) = \frac{n(n-r_1-1)(n-r_1-r_2-2) \cdots (n-r_1-\cdots-r_{m-1}-m+1)}{n!}.
\]
MLP

This PLF can be written as a product of two likelihood functions

\[ L_1(\mu; Z) = c_2 \prod_{l=1}^{m} \{ f(Z_l) [1 - F(Z_l)]^{r_l} \} \quad \text{and} \]

\[ L_2(Z; \mu, Z) = c_1 f(Z) \frac{[F(Z) - F(Z_i)]^{j-1}}{[1 - F(Z_i)]^{r_i}} [1 - F(Z)]^{r_i-j}, \quad Z \geq Z_i. \]
This PLF can be written as a product of two likelihood functions

\[
L_1(\mu; Z) = c_2 \prod_{l=1}^{m} \{f(Z_l) [1 - F(Z_l)]^{r_l}\} \quad \text{and}
\]

\[
L_2(Z; \mu, Z) = c_1 f(Z) \frac{[F(Z) - F(Z_i)]^{j-1}}{[1 - F(Z_i)]^{r_i}} \left[1 - F(Z)\right]^{r_i-j}, \quad Z \geq Z_i.
\]
MLP

We need:

\[
h(Z) = \frac{\phi(Z)}{1 - \Phi(Z)},
\]

\[
h(Z_l) = \frac{\phi(Z_l)}{1 - \Phi(Z_l)}; \quad l = 1, \ldots, m,
\]

\[
h_1(Z_i, Z) = \frac{\phi(Z_i)}{\Phi(Z) - \Phi(Z_i)}; \quad Z > Z_i,
\]

\[
h_2(Z_i, Z) = \frac{\phi(Z)}{\Phi(Z) - \Phi(Z_i)}; \quad Z > Z_i.
\]
OSMLP

- OSMLP $Y_{j:r_i}^{OSM}$ is given by

$$\frac{\partial \log L}{\partial \mu} = Z + \sum_{l=1}^{m} Z_l + \sum_{l=1, l \neq i}^{m} r_l h(Z_l) + (j - 1) [h_1(Z_i, Z) - h_2(Z_i, Z)] + (r_i - j) h(Z) = 0,$$

$$\frac{\partial \log L}{\partial Y} = -Z + (j - 1) h_2(Z_i, Z) - (r_i - j) h(Z) = 0,$$
OSMLP

OSMLP $Y_{j:r_i}^{OSM}$ is given by

$$\frac{\partial \log L}{\partial \mu} = Z + \sum_{l=1}^{m} Z_l + \sum_{l=1, l \neq i}^{m} r_l h(Z_l) + (j - 1) [h_1(Z_i, Z) - h_2(Z_i, Z)] + (r_i - j) h(Z) = 0,$$

$$\frac{\partial \log L}{\partial Y} = -Z + (j - 1) h_2(Z_i, Z) - (r_i - j) h(Z) = 0,$$
TSMLP

TSMLP $Y_{j:r_i}^{TSM}$ is given by

$$\frac{\partial \log L_1}{\partial \mu} = \sum_{l=1}^{m} Z_l + \sum_{l=1}^{m} r_l h(Z_l) = 0,$$

$$\frac{\partial \log L_2}{\partial Y} = -Z + (j - 1) h_2(Z_i, Z) - (r_i - j) h(Z) = 0.$$
TSMLP

- TSMLP $Y_{j:r_i}^{TSM}$ is given by

$$\frac{\partial \log L_1}{\partial \mu} = \sum_{l=1}^{m} Z_l + \sum_{l=1}^{m} r_l h(Z_l) = 0,$$

$$\frac{\partial \log L_2}{\partial Y} = -Z + (j - 1)h_2(Z_i, Z) - (r_i - j)h(Z) = 0.$$
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     - TSMMLP
   - AMLP
     - OSAMLPI
     - TSAMLPI
   - CMP
MMLP

Replacing $h(Z), h(Z_i), h_1(Z_i, Z), h_2(Z_i, Z)$ by their expected values. For MMLP, we need:

- Lemma 1

\[
E[h(Z_l)] = B_l = c_{l-1} \sum_{k=1}^{l} \frac{a_{k,l}}{(\gamma_k - 1)\gamma_k} E(Z_1:\gamma_k); l = 1, 2, \ldots, m.
\]
Replacing $h(Z)$, $h(Z_i)$, $h_1(Z_i, Z)$, $h_2(Z_i, Z)$ by their expected values. For MMLP, we need:

Lemma 1

$$E[h(Z_l)] = B_l = c_{l-1} \sum_{k=1}^{l} \frac{a_{k,l}}{(\gamma_k - 1)\gamma_k} E(Z_{1:\gamma_k}); \ l = 1, 2, \cdots, m.$$
**MMLP**

- **Lemma 2**
  \[ E[h(Z)] = \frac{1}{r_i-j} E_i(j + 2) \]
  where,

  \[
  E_i(a) = c_{i-1} \sum_{k=1}^{i} a_{k,i} \sum_{l=0}^{\gamma_k-r_i-1} (-1)^l \binom{\gamma_k-r_i-1}{l} \times \frac{r_i!!}{(r_i+l+1)!} \sum_{m=l+a} E(Z_{m:r_i+l+1}).
  \]

- **Lemma 3**
  \[ E[h_1(Z_i, Z)] = \frac{1}{j-1} E_i(2). \]

- **Lemma 4**
  \[ E[h_2(Z_i, Z)] = \frac{1}{j-1} E_i(j + 1). \]
**MMLP**

- **Lemma 2**
  \[ E[h(Z)] = \frac{1}{r_i-j}E_i(j + 2) \]

  \[
  E_i(a) = c_{i-1} \sum_{k=1}^{i} \sum_{l=0}^{\gamma_k-r_i-1} (-1)^l \binom{\gamma_k-r_i-1}{l} \frac{r_i!!}{(r_i+l+1)!} \sum_{m=l+a} E(Z_m:r_i+l+1).
  \]

- **Lemma 3**
  \[ E[h_1(Z_i, Z)] = \frac{1}{j-1}E_i(2). \]

- **Lemma 4**
  \[ E[h_2(Z_i, Z)] = \frac{1}{j-1}E_i(j + 1). \]
MMLP

Lemma 2

\[ E[h(Z)] = \frac{1}{r_i-j} E_i(j + 2) \]

where,

\[ E_i(a) = c_{i-1} \sum_{k=1}^{i} a_{k,i} \sum_{l=0}^{\gamma_k-r_i-1} (-1)^l \binom{\gamma_k-r_i-1}{l} \times \frac{r_i!!}{(r_i+l+1)!!} \sum_{m=l+a} E(Z_m:r_i+l+1). \]

Lemma 3

\[ E[h_1(Z_i, Z)] = \frac{1}{j-1} E_i(2). \]

Lemma 4

\[ E[h_2(Z_i, Z)] = \frac{1}{j-1} E_i(j + 1). \]
Lemma 2

\[ E[h(Z)] = \frac{1}{r_i - j} E_i(j + 2) \]

where,

\[ E_i(a) = c_{i-1} \sum_{k=1}^{i} a_{k,i} \sum_{l=0}^{\gamma_k - r_i - 1} (-1)^l \binom{\gamma_k - r_i - 1}{l} \times \frac{r_i!! l!!}{(r_i + l + 1)!} \sum_{m=l+a} E(Z_m: r_i + l + 1). \]

Lemma 3

\[ E[h_1(Z_i, Z)] = \frac{1}{j-1} E_i(2). \]

Lemma 4

\[ E[h_2(Z_i, Z)] = \frac{1}{j-1} E_i(j + 1). \]
OSMMLP

OSMMLP $Y_{j:r_i}^{OSMM}$ is given by

$$Y_{j:r_i}^{OSMM} = \begin{cases} 
\hat{\mu}_{OSMM} + D_i(j + 1) & \text{if } 2 \leq j \leq r_i - 1 \\
\hat{\mu}_{OSMM} - E_i(3) & \text{if } j = 1 \\
\hat{\mu}_{OSMM} + E_i(r_i + 1) & \text{if } j = r_i 
\end{cases}$$

where

$$\hat{\mu}_{OSMM} = \frac{1}{m} \left[ \sum_{l=1}^{m} Y_l + \sum_{l=1, l \neq i}^{m} r_l B_l \right]$$

and

$$D_i(c) = c_{i-1} \sum_{k=1}^{i} a_{k,i} \sum_{l=0}^{\gamma_k-r_i-1} (-1)^l \binom{\gamma_k-r_i-1}{l} \times \frac{1}{(r_i+l+1)!} \frac{1}{r_i-j} E(Z_l + c : r_i + l + 1).$$
OSMMLP

OSMMLP $Y_{j:r_i}^{OSMM}$ is given by

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\hat{\mu}_{OSMM} + D_i(j + 1) & \text{if } 2 \leq j \leq r_i - 1 \\
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OSMMLP $Y_{j:r_i}^{OSMM}$ is given by

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\hat{\mu}_{OSMM} + E_i(r_i + 1) & \text{if } j = r_i 
\end{cases}$$

where

$$\hat{\mu}_{OSMM} = \frac{1}{m} \left[ \sum_{l=1}^{m} Y_l + \sum_{l=1, l \neq i}^{m} r_l B_l \right]$$

and

$$D_i(c) = c_{i-1} \sum_{k=1}^{i} a_{k,i} \sum_{l=0}^{\gamma_k - r_i - 1} (-1)^l \binom{\gamma_k - r_i - 1}{l} \times \frac{1}{(r_i + l + 1)!} \frac{1}{r_i - j} E(Z_{l+c:r_i+l+1}).$$
TSMMLP

TSMMLP $Y^{TSMM}_{j:r_i}$ is given by

$$
Y^{TSMM}_{j:r_i} = \begin{cases} 
\hat{\mu}^{TSMM} + D_i(l + j + 1) & \text{if } 2 \leq j \leq r_i - 1 \\
\hat{\mu}^{TSMM} - E_i(3) & \text{if } j = 1 \\
\hat{\mu}^{TSMM} + E_i(r_i + 1) & \text{if } j = r_i 
\end{cases}
$$

where

$$
\hat{\mu}^{TSMM} = \frac{1}{m} \sum_{l=1}^{m} [Y_l + r_lB_l] .
$$
Outline

1. Progressive Censoring
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     - OSAMLNP
     - TSAMLNP
   - CMP
Replacing $h(Z_l); l = 1, \cdots, m, h(Z), h_1(Z_l, Z), h_2(Z_l, Z)$ by their Taylor Series expansions around the point $F^{-1}(p_l); l = 1, \cdots, m$, $F^{-1}(\pi_{i,j}), (F^{-1}(p_i), F^{-1}(\pi_{i,j}))$ and $(F^{-1}(p_i), F^{-1}(\pi_{i,j}))$ respectively, where $\pi_{i,j} = \frac{j}{r_i+1}$ and $p_i$'s can be obtained by

\[
\begin{align*}
p_1 &= p_1^*, \\
p_2 &= p_1 + p_2^* \cdot \frac{1-p_1}{1-q_1-p_1^*}, \\
p_3 &= p_2 + p_3^* \cdot \frac{1-p_2}{1-q_1-q_2-p_1^*-p_2^*}, \\
& \quad \vdots \\
p_m &= p_{m-1} + p_m^* \cdot \frac{1-p_{m-1}}{1-\sum_{i=1}^{m-1} q_i - \sum_{i=1}^{m-1} p_i^*}.
\end{align*}
\]
AMLPP

Replacing \( h(Z_l); l = 1, \cdots, m, h(Z), h_1(Z_l, Z), h_2(Z_l, Z) \) by their Taylor Series expansions around the point \( F^{-1}(p_l); l = 1, \cdots, m, F^{-1}(\pi_{i,j}), (F^{-1}(p_i), F^{-1}(\pi_{i,j})) \) and \( (F^{-1}(p_i), F^{-1}(\pi_{i,j})) \) respectively, where \( \pi_{i,j} = \frac{j}{r_i + 1} \) and \( p_i \)’s can be obtained by

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    p_3 &= p_2 + p_3^* \cdot \frac{1-p_2}{1-q_1-q_2-p_1^*-p_2^*}, \\
    & \quad \cdots \\
    p_m &= p_{m-1} + p_m^* \cdot \frac{1-p_{m-1}}{1-\sum_{i=1}^{m-1} q_i - \sum_{i=1}^{m-1} p_i^*}.
\end{align*}
\]
Replacing $h(Z_l); l = 1, \cdots, m$, $h(Z), h_1(Z, Z), h_2(Z, Z)$ by their Taylor Series expansions around the point $F^{-1}(p_l); l = 1, \cdots, m$, $F^{-1}(\pi_{i,j}), (F^{-1}(p_i), F^{-1}(\pi_{i,j}))$ and $(F^{-1}(p_i), F^{-1}(\pi_{i,j}))$ respectively, where $\pi_{i,j} = \frac{j}{r_i+1}$ and $p_i$’s can be obtained by

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p_3 &= p_2 + p_3^* \cdot \frac{1-p_2}{1-q_1-q_2-p_1^*-p_2^*}, \\
& \quad \vdots \\
p_m &= p_{m-1} + p_m^* \cdot \frac{1-p_{m-1}}{1-\sum_{i=1}^{m-1} q_i - \sum_{i=1}^{m-1} p_i^*}. 
\end{align*}
\]
\( q_i \) and \( p_i^*(i = 1, 2, \cdots, m) \) are defined as follows:

- Suppose in a progressive censoring, \( r_i \) items are randomly withdrawn after \( s_i \) (additional) failures for \( i = 1, 2, \cdots, m \).
- Now assume \( \frac{s_i}{n} \to p_i^* \) and \( \frac{r_i}{n} \to q_i \).

The Taylor series approximations are given by

\[
\begin{align*}
    h(Z_i) &\approx \alpha_l + \beta_l Z_i; l = 1, \cdots, m, \\
    h(Z) &\approx \alpha^* + \beta^* Z, \\
    h_1(Z_i, Z) &\approx \gamma^* - \rho^* Z_i - \nu^* Z, \\
    h_2(Z_i, Z) &\approx \gamma + \rho Z_i - \nu Z.
\end{align*}
\]
$q_i$ and $p_i^*(i = 1, 2, \cdots, m)$ are defined as follows:

- Suppose in a progressive censoring, $r_i$ items are randomly withdrawn after $s_i$ (additional) failures for $i = 1, 2, \cdots, m$.
- Now assume $\frac{s_i}{n} \to p_i^*$ and $\frac{r_i}{n} \to q_i$.

The Taylor series approximations are given by

\[ h(Z_l) \approx \alpha_l + \beta_l Z_l; l = 1, \cdots, m, \]
\[ h(Z) \approx \alpha^* + \beta^* Z, \]
\[ h_1(Z_i, Z) \approx \gamma^* - \rho^* Z_i - \nu^* Z, \]
\[ h_2(Z_i, Z) \approx \gamma + \rho Z_i - \nu Z. \]
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The Taylor series approximations are given by

\[
\begin{align*}
    h(Z_i) & \approx \alpha_l + \beta_i Z_i; l = 1, \cdots, m, \\
    h(Z) & \approx \alpha^* + \beta^* Z, \\
    h_1(Z_i, Z) & \approx \gamma^* - \rho^* Z_i - \nu^* Z, \\
    h_2(Z_i, Z) & \approx \gamma + \rho Z_i - \nu Z.
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The Taylor series approximations are given by

\[
\begin{align*}
h(Z_l) & \approx \alpha_l + \beta_l Z_l; l = 1, \cdots, m, \\
h(\bar{Z}) & \approx \alpha^* + \beta^* \bar{Z}, \\
h_1(Z_i, \bar{Z}) & \approx \gamma^* - \rho^* Z_i - \nu^* \bar{Z}, \\
h_2(Z_i, \bar{Z}) & \approx \gamma + \rho Z_i - \nu \bar{Z}.
\end{align*}
\]
The constants $\alpha_l, \beta_l, \alpha^*, \beta^*, \gamma^*, \rho^*, \nu^*, \gamma, \rho$ and $\nu$ are given by

\[
\alpha_l = f(\zeta_l) \left[ (1 + \zeta_l^2)(1 - p_l) - \zeta_l f(\zeta_l) \right] / (1 - p_l)^2,
\]

\[
\beta_l = f(\zeta_l) \left[ f(\zeta_l) - (1 - p_l)\zeta_l \right] / (1 - p_l)^2,
\]

\[
\alpha^* \equiv \alpha_{j:r_i}^* = f(\eta_j) \left[ (1 + \eta_j^2)(1 - \pi_{i,j}) - \eta_j f(\eta_j) \right] / (1 - \pi_{i,j})^2,
\]

\[
\beta^* \equiv \beta_{j:r_i}^* = f(\eta_j) \left[ f(\eta_j) - (1 - \pi_{i,j})\eta_j \right] / (1 - \pi_{i,j})^2,
\]

\[
\gamma^* \equiv \gamma_{j:r_i}^* = f(\zeta_i) \left[ (1 + \zeta_i^2)(\pi_{i,j} - p_i) + \eta_j f(\eta_j) - \zeta_l f(\zeta_l) \right] / (\pi_{i,j} - p_i)^2,
\]
\[ \rho^* \equiv \rho_{j:ri} = f(\zeta_i) \left[ -f(\zeta_i) + \zeta_i(\pi_{i,j} - p_i) \right] / (\pi_{i,j} - p_i)^2, \]

\[ \nu^* \equiv \nu_{j:ri} = f(\zeta_i)f(\eta_j) / (\pi_{i,j} - p_i)^2, \]

\[ \gamma \equiv \gamma_{j:ri} = f(\eta_j) \left[ (1 + \eta^2_j)(\pi_{i,j} - p_i) + \eta_j f(\eta_j) - \zeta_j f(\zeta_i) \right] / (\pi_{i,j} - p_i)^2, \]

\[ \rho \equiv \rho_{j:ri} = f(\zeta_i)f(\eta_j) / (\pi_{i,j} - p_i)^2, \]

\[ \nu \equiv \nu_{j:ri} = f(\eta_j) \left[ \eta_j(\pi_{i,j} - p_i) + f(\eta_j) \right] / (\pi_{i,j} - p_i)^2, \]

where \( \zeta_i = F^{-1}(p_i) \), \( \zeta_l = F^{-1}(p_l) \), \( \eta_j = F^{-1}(\pi_{i,j}) \).
OSAMLP  $Y_{j:r_i}^{OSAM}$ is given by

$$Y_{j:r_i}^{OSAM} = \begin{cases} 
\hat{Y}_{j:r_i} & \text{if } \hat{Y}_{j:r_i} > Y_i \\
Y_i & \text{if } \hat{Y}_{j:r_i} \leq Y_i 
\end{cases}$$

where $\hat{Y}_{j:r_i}$ is given by

$$\hat{\mu}^{OSAM} [(j - 1)(\nu - \rho) + (r_i - j)\beta^* + 1] + (j - 1) [\gamma + \rho Y_i] - (r_i - j)\alpha^*$$

$$(j - 1)\nu + (r_i - j)\beta^* + 1$$

and

$$\hat{\mu}^{OSAM} = \frac{m\bar{Y} + \sum_{l=1, l \neq i}^{m} r_l(\alpha_l + \beta_l Y_l) + (j - 1) [\gamma^* - \rho^* Y_i - \nu^* \hat{Y}_{j:r_i}]}{m + \sum_{l=1}^{m} r_l \beta_l - (j - 1)(\rho^* + \nu^*)}.$$
OSAMLPPROGRESSIVE CENSORING

Different Predictors

OSAMLPPREDICTION PROBLEM

**OSAMLP**

OSAMLP \( Y_{j:r_i}^{OSAM} \) is given by

\[
Y_{j:r_i}^{OSAM} = \begin{cases} 
\hat{Y}_{j:r_i} & \text{if } \hat{Y}_{j:r_i} > Y_i \\
Y_i & \text{if } \hat{Y}_{j:r_i} \leq Y_i
\end{cases}
\]

where \( \hat{Y}_{j:r_i} \) is given by

\[
\hat{\mu}_{OSAM} \left[ (j - 1)(\nu - \rho) + (r_i - j)\beta^* + 1 \right] + (j - 1) [\gamma + \rho Y_i] - (r_i - j)\alpha^* \frac{(j - 1)\nu + (r_i - j)\beta^* + 1}{(j - 1)\nu + (r_i - j)\beta^* + 1},
\]

and

\[
\hat{\mu}_{OSAM} = \frac{m\bar{Y} + \sum_{l=1, l \neq i}^m r_l(\alpha_l + \beta_l Y_l) + (j - 1) [\gamma^* - \rho^* Y_i - \nu^* \hat{Y}_{j:r_i}]}{m + \sum_{l=1, l \neq i}^m r_l\beta_l - (j - 1)(\rho^* + \nu^*)}.
\]
OSAMLp $Y_{j:r_i}^{OSAM}$ is given by

$$Y_{j:r_i}^{OSAM} = \begin{cases} \hat{Y}_{j:r_i} & \text{if } \hat{Y}_{j:r_i} > Y_i \\ Y_i & \text{if } \hat{Y}_{j:r_i} \leq Y_i \end{cases}$$

where $\hat{Y}_{j:r_i}$ is given by

$$\hat{\mu}_{OSAM} [(j-1)(\nu - \rho) + (r_i - j)\beta^* + 1] + (j-1) [\gamma + \rho Y_i] - (r_i - j)\alpha^*$$

$$= \frac{(j-1)\nu + (r_i - j)\beta^* + 1}{(j-1)\nu + (r_i - j)\beta^* + 1},$$

and

$$\hat{\mu}_{OSAM} = m\bar{Y} + \sum_{l=1, l \neq i}^m r_l(\alpha_l + \beta_l Y_l) + (j-1) [\gamma^* - \rho^* Y_i - \nu^* \hat{Y}_{j:r_i}]$$

$$= \frac{m + \sum_{l=1}^m r_l\beta_l - (j-1)(\rho^* + \nu^*)}{m + \sum_{l=1}^m r_l\beta_l - (j-1)(\rho^* + \nu^*)}.$$
**TSAM LP**

**TSAMLP** $Y^{TSAM}_{j:r_i}$ is given by

$$Y^{TSAM}_{j:r_i} = \begin{cases} \hat{Y}^{TSA}_{j:r_i} & \text{if } \hat{Y}^*_{j:r_i} > Y_i \\ Y_i & \text{if } \hat{Y}^*_{j:r_i} \leq Y_i. \end{cases}$$

where $\hat{Y}^*_{j:r_i}$ is given by

$$\hat{\mu}^{TSA}_{TSA} = \frac{(j - 1)(\nu - \rho) + (r_i - j)\beta^* + 1}{{(j - 1)\nu + (r_i - j)\beta^* + 1}$$

and

$$\hat{\mu}^{TSA} = \frac{m\bar{Y} + \sum_{l=1}^{m} r_l(\alpha_l + \beta_l Y_l)}{m + \sum_{l=1}^{m} r_l \beta_l}.$$
TSAMLP $Y_{j:r_i}^{TSAM}$ is given by

$$Y_{j:r_i}^{TSAM} = \begin{cases} \hat{Y}_{j:r_i} & \text{if } \hat{Y}_{j:r_i} > Y_i \\ Y_i & \text{if } \hat{Y}_{j:r_i} \leq Y_i. \end{cases}$$

where $\hat{Y}_{j:r_i}^*$ is given by

$$\hat{\mu}^{TSA} [(j - 1)(\nu - \rho) + (r_i - j)\beta^* + 1] + (j - 1)[\gamma + \rho Y_i] - (r_i - j)\alpha^*,$$

$$(j - 1)\nu + (r_i - j)\beta^* + 1$$

and

$$\hat{\mu}^{TSA} = \frac{m\bar{Y} + \sum_{l=1}^{m} r_l(\alpha_l + \beta_l Y_l)}{m + \sum_{l=1}^{m} r_l \beta_l}.$$
TSAMLMP $Y^{TSAM}_{j:r_i}$ is given by

$$Y^{TSAM}_{j:r_i} = \begin{cases} 
\hat{Y}^{TSA}_{j:r_i} & \text{if } \hat{Y}^*_{j:r_i} > Y_i \\
Y_i & \text{if } \hat{Y}^*_{j:r_i} \leq Y_i.
\end{cases}$$

where $\hat{Y}^*_{j:r_i}$ is given by

$$\hat{\mu}^{TSA} \left[ (j - 1)(\nu - \rho) + (r_i - j)\beta^* + 1 \right] + (j - 1) \left[ \gamma + \rho Y_i \right] - (r_i - j)\alpha^*$$

$$\frac{(j - 1)\nu + (r_i - j)\beta^* + 1}{(j - 1)\nu + (r_i - j)\beta^* + 1},$$

and

$$\hat{\mu}^{TSA} = \frac{m\bar{Y} + \sum_{l=1}^{m} r_l(\alpha_l + \beta_l Y_l)}{m + \sum_{l=1}^{m} r_l\beta_l}.$$
Outline

1. Progressive Censoring
2. Prediction Problem
3. Different Predictors
   - BLUP
   - MLP
     - OSMLP
     - TSMLP
   - MMLP
     - OSMMLP
     - TSMMLP
   - AMLP
     - OSAMLMLP
     - TSMMLP
   - CMP
A statistic $T$ is called the CMP of $Y_{j:r_i}$ if it is the median of the conditional distribution of $Y_{j:r_i}$ given $Y_i$.

The CMP $Y_{j:r_i}^C$ of $Y_{j:r_i}$ is such that

$$\int_{y_i}^{Y_{j:r_i}^C} f(y|y_i) dy = \frac{1}{2}$$

in which $f(y|y_i)$ is given by

$$\frac{r_i!}{(j-1)!(r_i-j)!} \left[ \frac{\int_{y_i}^{y} e^{-\frac{(x-\mu^*)^2}{2}} dx}{\int_{y_i}^{\infty} e^{-\frac{(x-\mu^*)^2}{2}} dx} \right]^{j-1} e^{-\frac{(y-\mu^*)^2}{2}} \left[ \frac{\int_{y_i}^{\infty} e^{-\frac{(x-\mu^*)^2}{2}} dx}{\int_{y_i}^{y} e^{-\frac{(x-\mu^*)^2}{2}} dx} \right]^{r_i-j}.$$  

where $\mu^*$ is the BLUE of $\mu$.  

A statistic \( T \) is called the CMP of \( Y_{j:r_i} \) if it is the median of the conditional distribution of \( Y_{j:r_i} \) given \( Y_i \).

The CMP \( Y_{j:r_i}^C \) of \( Y_{j:r_i} \) is such that

\[
\int_{y_i}^{Y_{j:r_i}^C} f(y|y_i) \, dy = \frac{1}{2}
\]

in which \( f(y|y_i) \) is given by

\[
\frac{r_i!}{(j - 1)! (r_i - j)!} \left[ \int_{y_i}^{y} e^{-\frac{(x-\mu^*)^2}{2}} \, dx \right]^{j-1} e^{-\frac{(y-\mu^*)^2}{2}} \left[ \int_{y}^{\infty} e^{-\frac{(x-\mu^*)^2}{2}} \, dx \right]^{r_i-j}
\]

\[
\frac{\left[ \int_{y_i}^{\infty} e^{-\frac{(x-\mu^*)^2}{2}} \, dx \right]^{r_i}}{\left[ \int_{y_i}^{\infty} e^{-\frac{(x-\mu^*)^2}{2}} \, dx \right]^{r_i}}
\]

where \( \mu^* \) is the BLUE of \( \mu \).
A statistic $T$ is called the CMP of $Y_{j:r_i}$ if it is the median of the conditional distribution of $Y_{j:r_i}$ given $Y_i$.

The CMP $Y_{j:r_i}^C$ of $Y_{j:r_i}$ is such that

$$\int_{y_i}^{Y_{j:r_i}} f(y|y_i) dy = \frac{1}{2}$$

in which $f(y|y_i)$ is given by

$$\frac{r_i!}{(j-1)!(r_i-j)!} \left[ \int_{y_i}^{y} e^{-\frac{(x-\mu^*)^2}{2}} dx \right]^{j-1} e^{-\frac{(y-\mu^*)^2}{2}} \left[ \int_{y_i}^{\infty} e^{-\frac{(x-\mu^*)^2}{2}} dx \right]^{r_i-j} \left[ \int_{y_i}^{\infty} e^{-\frac{(x-\mu^*)^2}{2}} dx \right]^{r_i}.$$ 

where $\mu^*$ is the BLUE of $\mu$. 

**CMP**
A statistic $T$ is called the CMP of $Y_{j:r_i}$ if it is the median of the conditional distribution of $Y_{j:r_i}$ given $Y_i$.

The CMP $Y^C_{j:r_i}$ of $Y_{j:r_i}$ is such that

$$
\int_{y_i}^{Y^C_{j:r_i}} f(y|y_i) dy = \frac{1}{2}
$$

in which $f(y|y_i)$ is given by

$$
\frac{r_i!}{(j-1)!(r_i-j)!} \left[ \int_{y_i}^{y} e^{-\frac{(x-\mu^*)^2}{2}} dx \right]^{j-1} e^{-\frac{(y-\mu^*)^2}{2}} \left[ \int_{y}^{\infty} e^{-\frac{(x-\mu^*)^2}{2}} dx \right]^{r_i-j} \left[ \int_{y_i}^{\infty} e^{-\frac{(x-\mu^*)^2}{2}} dx \right]^{r_i-j}.
$$

where $\mu^*$ is the BLUE of $\mu$. 
THANK YOU