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The identification and validation process of proportional reasoning attributes: an application of a cognitive diagnosis modeling framework

Hartono Tjoe · Jimmy de la Torre

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Abstract In this paper, we discuss the process of identifying and validating students' abilities to think proportionally. More specifically, we describe the methodology we used to identify these proportional reasoning attributes, beginning with the selection and review of relevant literature on proportional reasoning. We then continue with the deliberation and resolution of differing views by mathematics researchers, mathematics educators, and middle school mathematics teachers of what should be learned theoretically and what can be taught practically in everyday classroom settings. We also present the initial development of proportional reasoning items as part of the two-phase validation process of the previously identified attributes. In particular, we detail in the first phase of the validation process our collaboration with middle school mathematics teachers in the creation of prototype items and the verification of each item-attribute specification in consideration of the most common ways (among many different ways) in which middle school students would have solved these prototype items themselves. In the second phase of the validation process, we elaborate our think-aloud interview procedure in the search for evidence of whether students generally solved the prototype items in the way they were expected to.

Keywords Ratios · Proportions · Proportional reasoning · Assessment

Introduction

Proportional reasoning is widely viewed as one of the most reliable predictors of students' mathematical readiness for advancing from elementary school mathematics

H. Tjoe (✉)
The Pennsylvania State University, Berks Campus, 238 Gaige Building, Tulpehocken Road,
P.O. Box 7009, Reading, PA 19610, USA
e-mail: hht1@psu.edu

J. de la Torre
Rutgers University, 10 Seminary Place, New Brunswick, NJ 08901, USA
e-mail: j.delatorre@rutgers.edu

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to high school mathematics (Lesh et al. 1988). Lesh and colleagues particularly note that students' facility to reason proportionally develops progressively through a series of local competences, rather than emerges distinctly as the materialisation of a global ability. From the perspective of a cognitive diagnosis framework, local competences can be construed as students' mastery or non-mastery of fine-grained skills, cognitive processes, or problem-solving steps, commonly referred to as *attributes* (de la Torre 2009; de la Torre and Lee 2010). On this point, the stake in large-scale assessments of proportional reasoning can benefit from cognitive diagnosis models (CDMs), a recent development in the field of psychometrics.

In contrast to more traditional psychometric methods that employ unidimensional item response theory models (IRT; e.g., Misailidou and Williams 2003), CDMs have the advantage of not imposing constraints on the development of these competences (or lack thereof). Instead of positing a strict ordering, CDMs allow for partial ordering in local competences. By imposing a less restrictive structure, CDMs acknowledge that competence in one area does not necessarily imply competence in another area. This view of development is more consistent with the nature of proportional reasoning described by Lesh et al. (1988).

The current paper is divided into two main parts. In the first part, the attribute identification process, we examine an opportunity to generate an alternative theoretical groundwork of proportional reasoning as a basis to develop a proportional reasoning assessment relevant to the eighth grade level using a cognitive diagnosis modeling framework. More specifically, we describe the methodology we used to identify these proportional reasoning attributes, beginning with the selection and review of relevant literature on proportional reasoning. We then continue with the deliberation and resolution of differing views by mathematics researchers, mathematics educators, and middle school mathematics teachers of what should be learned theoretically and what can be taught practically in everyday classroom settings.

In the second part, the attribute validation process, we present the initial development of proportional reasoning items as part of the two-phase validation process of the previously identified attributes. In particular, we detail in the first phase of the validation process our collaboration with middle school mathematics teachers in the creation of prototype items and the verification of each item-attribute specification in consideration of the most common ways (among many different ways) in which middle school students would have solved these prototype items themselves. In addition, we elaborate in the second phase of the validation process our think-aloud interview procedure in the search for evidence of whether students generally solved the prototype items in the way they were expected to.

Purpose

The purpose of this paper is to formulate a list of psychometric properties (attributes) that account for the theoretical and operational understanding of proportional reasoning relevant to the eighth grade level. These attributes are hoped to serve as a foundation for designing effective tools to assess students' understanding of proportional reasoning, that is, an application of a cognitive diagnosis modeling framework. Not only do these attributes need to be identified, but they also need to be validated.

In the identification process, we explored the following research question: What are the skills necessary for an eighth grade student (age 13–14) to master to be deemed proficient in proportional reasoning? This identification process included two phases. In the first phase of the identification process, we aimed to produce a summary of literature on proportional reasoning. In the second phase of the identification process, we aimed to materialise a list of proportional reasoning attributes through dialogues and consultations with a group of experts in the field of proportional reasoning.

In the validation process, we aimed to examine the accuracy of the proportional reasoning attributes determined by the expert group in the identification process as supported and documented by the literature. This validation process included two phases. In the first phase of the validation process, we aimed to develop a number of prototype proportional reasoning items which would include the attributes identified in the second phase of the identification process. We sought to verify such item-attribute specifications by analysing the way in which expert problem solvers would solve the prototype items, as well as their understanding of the most common ways (among many different ways) a typical eighth grade student would solve them. In the second phase of the validation process, we aimed to evaluate the extent to which the previously identified attributes were used by eighth grade students to solve proportional reasoning problems. We sought to ascertain that eighth grade students would actually solve the prototype items in the manner that the item creators expected them to. Additional efforts would be taken to look for evidence of other attributes not previously recognised in the identification process.

Methodology

In the identification process, we sought to identify pedagogically meaningful and psychometrically measureable attributes. In particular, desired attributes are those that: 1) cut across various types of proportional reasoning problems; 2) are descriptive, diagnostic, and prescriptive; and 3) are at appropriate grain-size to be informative, yet amenable to practical implementation. Understanding these attributes is of paramount significance not only to explain students' success and failure in the process of acquiring proportional reasoning skills, but also to build a foundation for designing effective tools to assess students' understanding of proportional reasoning.

In the first phase of the identification process, we explored to better understand proportional reasoning and its underlying concepts. We searched several databases containing mathematics education literature and checked the citations of each article we found. Recognising that not all of these sources would be relevant for our purposes, we took the following steps to choose references from this list.

Reviews in the two handbooks of research published by the National Council of Teachers of Mathematics (Behr et al. 1992; Lamon 2007) did not focus exclusively on proportional reasoning; thus, the comprehensive review by Tourniaire and Pulos (1985) has not been updated for more than two decades. As a result, we primarily reviewed studies conducted since Tourniaire and Pulos's review, that is, from 1985 to 2010. Still, a few studies included in Tourniaire and Pulos's review are cited in our paper because of their pervasiveness throughout the literature. Moreover, we acknowledge that proportional reasoning is inherently tied to students' ideas about rational numbers.

Because a large body of research is devoted to these topics (e.g., Carpenter et al. 1993), we did not describe that research in our review. Rather, we focused exclusively on research on proportional reasoning, specifically relevant to the eighth grade level.

In the second phase of the identification process, we investigated the opinions and views of experts in the field of proportional reasoning to substantiate and synthesise the findings from the literature review from the first phase of the identification process. To this end, we held three meetings in collaboration with several mathematics researchers, mathematics educators, and middle school mathematics teachers. The selection of mathematics researchers and mathematics educators was based on the recommendations by their peers in the mathematics education community. Each member in this particular group of experts has conducted at least one nationally recognized project on proportional reasoning. The selection of the middle school mathematics teachers was based on the recommendations by their school district superintendents. Members in this particular group of experts have received a variety of awards for excellence in teaching and have been mathematics teachers at the middle school level for at least 15 years. As members of the current research project, a number of graduate students in psychometrics and mathematics education also participated in the meetings.

The first meeting was attended by a diverse group of four mathematics researchers, three mathematics educators, five middle school mathematics teachers, and five graduate students in psychometrics and mathematics education. The goal of this meeting was to initiate the identification of proportional reasoning attributes that are of assessable value in measurement settings and of practical value in classroom settings. Prior to this meeting, the attendees would be supplied with a summary of the findings from the first phase of the identification process, a previously prepared literature review on proportional reasoning; this was an anticipatory source of empirical evidence and theoretical support for any attributes that might be proposed during the first meeting.

In this first meeting, participants were randomly assigned into groups of three to four members. They were asked to work in group specifically to indicate, interpret, elucidate, and establish a list of skills for proficiency in proportional reasoning relevant to the eighth grade level. A voluntary spokesperson from each group would then present the list of attributes that his or her group proposed, while members from other groups would raise any questions, comments, or disagreements in the form of a dialogue. Through a series of exchanges and reasoned argumentations, we anticipated a thorough refinement of converging ideas of what counts as necessary proportional reasoning skills.

The second meeting was attended by a (not completely) different group of participants consisting of one mathematics educator, five middle school mathematics teachers, and four graduate students in psychometrics and mathematics education. In this meeting, we aimed to finalise a list of six measurable attributes initiated at the first meeting. We collected evidence from literature and associated relevant references to each of the initiated attributes as a way to verify, ascertain, support, and affirm the views of the expert group from the first meeting.

The third meeting was attended by four middle school mathematics teachers and four graduate students in psychometrics and mathematics education. In this meeting, we aimed to discuss and refine the intended meaning and measurability of the attributes. The refining step was carried out one attribute at a time, with each attendee asked to explain his or her interpretations of the particular attribute to the rest of the attendees and to convince each other until they reached an agreement.

In the first phase of the validation process, we developed a number of proportional reasoning items that confirmed particular assignments of the six attributes identified earlier, as part of the goal to understand the problem-solving behaviour of experienced problem solvers as they solved these proportional reasoning items. Two researchers and two middle school mathematics teachers participated in this process. They were to create proportional reasoning items that measured one or more attributes from the provided list of attributes. These items could be of a multiple-choice or open-ended format. For multiple-choice items, the two researchers and two middle school mathematics teachers were to create possible answer choices that corresponded to their assessment tasks. To help the two middle school mathematics teachers write such items, the researchers created sample assessment problems which were annotated with the attributes used, the reasons for the choice of attributes, and annotated answer choices. In supplying the solutions to the proposed items, the two middle school mathematics teachers were reminded to consider the most common ways in which eighth grade students would have solved these items. These participants were encouraged to write assessment items in the format of the sample items provided. In addition to the most common ways, we also encouraged these participants to solve their items in as many different ways as possible (using different attributes perhaps not included in the list of attributes identified earlier). The new proportional reasoning items that had been created by the four participants in the first phase of the validation process were referred to as the prototype items.

The first phase of the validation process involved four rounds of exchanges, with the aim to ensure that the proportional reasoning items were measuring the attributes that the item writers intended. In the first round, we compiled and randomly arranged into one document the prototype items (for which the intended attributes were not disclosed). The two researchers and two middle school mathematics teachers served as raters, individually working on the prototype items, providing detailed solution steps, and specifying the attributes used in each solution step. Upon completion, we summarised and tabulated the attribute specifications from each rater. The summary of attribute specifications was kept anonymous and, in the second round, was sent to the same four raters, who were asked to re-examine the items by considering their peers' anonymous attribute specifications. After reviewing the items and attributes, the four raters submitted their revised list of attributes for each item. A third round was completed following the same procedure of the second round. During the third round, the raters were also asked to explain their reasoning for choosing particular attributes whenever those attributes differed from their peers' attributes. Finally, the four raters met in person for the fourth round to discuss their opinions and differences in detail. At the end of the fourth round, a list of prototype items was expected to be finalised.

In the second phase of the validation process, we aimed to examine how students solved a number of the prototype items developed in the first phase of the validation process. Nineteen subjects were categorised into two different groups. The first group consisted of 8 seventh grade students from a suburban school district in New Jersey, and the second group consisted of 11 science and engineering first-year students from an urban university in the north-eastern area. Participation in this second phase of the validation process was voluntary. All 19 students were interviewed individually using the think-aloud protocol (Ericsson and Simon 1984; Van Someren et al. 1994). During the interviews, students were asked to explain clearly how they solved the items and

arrived at their solutions. We constantly and continuously probed students' explanations and justifications one item at a time until we were able to successfully ascertain the fine-grained cognitive processes that the students demonstrated as they solved the items.

The middle school student group was asked to solve 10 of the 13 items, whereas the college student group was asked to solve all 13 items. The first 10 items tested out to the college student group were the same 10 items tested out to the middle school student group. The three additional items for the college student group incorporated more advanced science-related items. The three items were deemed more appropriate for college students in the proportional reasoning context at a higher level of mathematics with applications in chemistry and physics.

These interviews were audio-taped and transcribed. Along with students' written solutions, the transcripts were coded to detect any presence of students' use of any of the six attributes. The following coding scheme was used. First, the coding sought to locate the source of students' use of attributes. The code (T) was used when the transcripts were considered in looking for evidence, and the code (S) was used when the written solutions were considered in looking for evidence. Second, the coding sought to examine the clarity of students' use of attributes. The code (0) was used when there was no evidence of students using a particular attribute in solving an item; the code (1) was used when there was some (complete or incomplete) evidence of students using a particular attribute in solving an item. In addition, students' correctness (C/I) in answering the item was coded. The code (0) was used when an item was incorrect, and the code (1) was used when an item was correct.

A total of 223 cases were coded from the 10 items solved by the 8 middle school students and the 13 items solved by the 11 college students. These 223 cases were randomised before the coding process began to avoid any bias from knowing a particular student's specific pattern of solving an item. Four researchers individually coded these 223 cases item per item to keep coding consistency within a particular item and throughout the 13 items in general. The four researchers met for three rounds to discuss differences in their coding before coming to a consensus on coding for each case. It should be made clear that the coding scheme used was not intended to score students' responses or mastery of any of the six attributes; rather, it aimed to examine which attributes the students used to solve the 13 items within the context of proportional reasoning.

Findings

First phase of the identification process

In the first phase of the identification process, a list of over 400 articles, books, and conference papers that described research on proportional reasoning was assembled. After following the procedure of narrowing down the literature, we arrived at a list of approximately 70 references that we used to summarise our review. A 40-page summary of findings from the literature review on proportional reasoning was produced. Our review points to two areas of research that we believe help identify proportional reasoning attributes: 1) construction of proportional reasoning tasks, including their structural and contextual components, and 2) students'

understanding of proportional reasoning. Readers interested in obtaining this review can contact the authors.

Second phase of the identification process

Table 1 summarises a list of attributes as a conclusion of the identification process consisting of three meetings in collaboration with several mathematics researchers, mathematics educators, and middle school mathematics teachers. It featured six attributes, each of which measured distinct cognitive mastery in proportional reasoning.

First meeting

The findings from the second phase of the identification process were based on three meetings. In the first meeting, although randomly assigned into groups of three or four members, the attendees basically fell into two main groups, each of which had its own theoretical points of view, technical approaches, and strategic decisions for identifying attributes of proportional reasoning. One group assumed a stance from the perspective of mathematics education research. It maintained that students' proportional reasoning skills should be assessed in terms of their conceptual understanding, in addition to their procedural understanding. The second group offered a perspective from everyday classroom practice. It favoured a pragmatic teaching and learning approach. What members of this group thought about students' necessary skills for proportional reasoning was based purely on how they believed proportional reasoning has been taught customarily, practically, and successfully in the classroom environment.

Ideally, an assessment formulated on a diagnostic framework should measure conceptual understanding based on recent research findings. Furthermore, it should provide specific information that can be used in classroom settings. In other words, a diagnostic assessment should have conceptual and theoretical underpinnings and, at the same time, should inform everyday pedagogical practice. Such an assessment can accommodate the two divergent views above, and we intended for the final form of the assessment from this research project to do so.

Table 1 List of six proportional reasoning attributes

Attribute	Description
A1	Prerequisite skills and concepts required in proportional reasoning
A2b	Comparing fractions
A2a	Ordering fractions
A3a	Constructing ratios
A3b	Constructing proportions
A4	Identifying a multiplicative relationship between sets of values
A5	Differentiating a proportional relationship from a non-proportional relationship
A6	Applying algorithms in solving proportional reasoning problems

However, because of limited time and resources, it was not immediately feasible to determine how to achieve both goals during the first meeting.

Second meeting

In the second meeting, a list of six measurable attributes initiated at the first meeting was finalised, despite a few remaining disagreements among the participants. The first attribute is the prerequisite skills and concepts required in proportional reasoning (A1). These relevant skills and concepts include mathematics knowledge taught before the eighth grade level. They can be classified as basic skills and concepts (e.g., addition, subtraction, multiplication, and division); intermediate skills and concepts (e.g., finding the least common multiples and the greatest common factors); and advanced skills and concepts (e.g., reducing fractions, understanding various representations of rational numbers, and interpreting linear graphs on a coordinate plane).

The second attribute consists of comparing and ordering fractions (A2). Following missing-value problems, comparison problems are considered the second most widely researched and most commonly instructed proportional reasoning problem (Clark et al. 2003; Karplus et al. 1983; Noelting 1980a, b; Watson et al. 2008; Watson and Shaughnessy 2004). Hence, it is crucial to integrate A2 into the list of attributes. A2 evaluates students' ability to compare two fractions (e.g., fractional forms of ratios) and to determine whether one of the two is less than, equal to, or greater than the other. Moreover, A2 assesses students' ability to put three or more fractions (e.g., fractional forms of ratios) in ascending or descending order. As such, comparing fractions is a prerequisite for ordering fractions: Students who perform well in fraction ordering are deemed to be sufficiently proficient in fraction comparison.

The third attribute is comprised of constructing ratios and proportions (A3). A3 assesses not only students' ability to construct a single ratio from a given situation, but also their ability to construct an appropriate proportion from a relevant proportional relationship situation. Research in proportional reasoning emphasises students' ability to construct formal ratios, or their variants such as ratio tables, as a foundation to facilitate more advanced conceptualisation of proportions (Cramer et al. 1993; Dole 2008; Middleton and van den Heuvel-Panhuizen 1995). Because proportions can be viewed as a collection of single ratios, constructing ratios is a prerequisite for constructing proportions: Students who can consistently construct appropriate proportions from various situations are assumed to be fairly conversant in constructing single ratios.

The fourth attribute measures students' ability to identify a multiplicative relationship between sets of values (A4). The need for this attribute surfaced in the first meeting, when three mathematics educators convincingly put forth their position that it is not enough for students to simply apply a certain procedure (as later described in A6) to solve a common missing-value proportion problem. They maintained that mastery in proportional reasoning requires students to be able to identify and think more conceptually about the presence of multiplicative relationship. In particular, when integer multiples are involved in a given proportional situation (e.g., $\frac{a}{b} = \frac{c}{d}$), students need to recognise the multiplicative relationships within single ratios (e.g., a and b, or c and d) and between two or more ratios (e.g., a and c, or b and d).

The importance of multiplicative reasoning as an essential foundation to think proportionally has long been affirmed by the literature (Mulligan 2002; Steffe 1994). Even so, students' ability to identify a multiplicative relationship given sets of values is seldom emphasised explicitly by the mathematics education community. Not until recently did Lamon (2007) incorporate into her definition of proportional reasoning this cognitive skill, reintroducing it as the knowledge of "covariance of quantities and invariance of ratios or products" (p. 638). The mathematics educators in the first and second meetings discussed the need for this attribute from a conceptual understanding point of view: Whereas covariance refers to the dynamic process of changing two values in a joint manner that preserves the constant relationship between them, invariance pertains to the static measure of maintaining this relationship in a multiplicative manner that balances the stretching or shrinking from one set of two values to another set of two values.

Arguably the more primitive solution method in reasoning proportionally, the skill of identifying a multiplicative relationship is more often observed in the proportional reasoning problem solving of elementary school students than in that of middle school students. Indeed, to some extent, A4 can be thought of as a convenient solution method used by middle school students to solve an easier problem. More specifically, because missing-value problems involving integer values are often considered easier than those involving non-integer values (Kaput and West 1994), missing-value problems with integer values can assess A4 (as opposed to A6, as discussed later) for our purpose. In the end, the participants in the first and second meetings came to an agreement that A4 needed to be included in the list of attributes for its predictive value in students' readiness and success in transitioning from multiplicative reasoning to proportional reasoning.

The fifth attribute assesses students' ability to determine whether or not two pairs of values form a proportional relationship (A5). In addition to recognising the presence of proportional relationships, students deemed competent in A5 are expected to demonstrate the ability to differentiate between situations where proportional reasoning is applicable and those where it is not (e.g., direct proportions vs. indirect proportions, constant ratio vs. constant difference or additive situations, and linear proportions vs. non-linear proportions).

Many studies note that elementary and middle school students struggle to decide between applying additive approaches to proportional reasoning problems and applying proportional reasoning approaches to non-proportional reasoning problems that appear to have similar structural components, as commonly found in missing-value proportional problems (Haja and Clarke 2011; Jacob and Willis 2001; Lamon 2007; MacGregor and Stacey 1995; Peled 2010; Singh 2000; Van Dooren et al. 2005, 2009). These studies address, among many difficulties and misconceptions, students' ability to discriminate between proportional situations and non-proportional situations, thereby highlighting A5 as one of the most important proportional reasoning skills. Indeed, the attendees of both the first and second meetings considered A5, compared with other attributes, to be cognitively more demanding because it functionally captures the essence of proportional reasoning in terms of its abstraction and conceptualisation. They also agreed that this attribute was of critical importance to the extent

that the absence of A5 may jeopardise students' development of problem-solving skills in areas of study that require higher-order mathematical understanding, for example, physics or chemistry.

The sixth attribute measures students' ability to apply algorithms in solving proportional reasoning problems (A6). The literature on mathematical thinking and the different ways that students solve proportional reasoning problems suggests that such algorithms include the building-up/down strategy, the abbreviated building-up/down strategy, the formal division strategy, the early-adjustment strategy, the late-adjustment strategy, the norming strategy, the unit-factor strategy, and the factor-of-change strategy (e.g., Battista and Borrow 1995; Christou and Philippou 2001; Cramer and Post 1993; Kaput and West 1994; Lamon 2007). Another solution method that is widely advocated in the mathematics curriculum and more commonly taught in everyday classroom practice is the cross-multiplication algorithm (Cramer and Post 1993; Heller et al. 1990; Lamon 2007).

Just as one can solve a mathematics problem from multiple perspectives, we recognise that the indicated list of solution methods may not be exhaustive; thus, other solution methods not listed here may also be considered part of A6. Nonetheless, it should be clear that while other attributes emphasise constructing proportions, understanding multiplicative relationships, or recognising proportional relationships, A6 specifically focuses on solving for missing values in proportional problems. As such, A6 pertains more significantly to a procedural algorithm than to an arithmetic computation.

Third meeting

In the third meeting, some of the attributes on the list were expanded. A2 (comparing and ordering fractions) was divided into two subcategories: A2a (comparing fractions) and A2b (ordering fractions). Similarly, A3 (constructing ratios and proportions) was split into A3a (constructing ratios) and A3b (constructing proportions). From the CDM framework point of view, two of the six attributes (i.e., A2 and A3) can be viewed as being structured into two subcategories or hierarchical skill levels (de la Torre et al. 2010) or as polytomous attributes (Chen and de la Torre 2012). Some important notes that the attendees agreed upon are included below.

First, it was noted that the fractions involved in the second attribute (A2a and A2b) are mostly common fractions with absolute values strictly less than one, generally known as proper fractions. These proper fractions can be readily analysed by comparing or ordering the values of the numerators or denominators of the fractions, given the fixed values of the others, for example, $\frac{5}{9} < \frac{7}{9}$ or $\frac{5}{9} < \frac{5}{7}$. The importance of instructing middle school students about these proper fractions has been substantiated by previous studies on children's developmental understanding of fractional equivalence (e.g., Carpenter et al. 1993; Pitkethly and Hunting 1996). Ball (1993) specifically documented the case of third grade students misattributing larger denominators as the cause of quantitatively greater fractions, describing their difficulties as possibly stemming from inabilities to convert and interpret graphical representations of fractions.

This second attribute also considers situations in which fractions are qualitatively compared in a more abstract sense. According to two of the mathematics educators from the first meeting, students need to comprehend the effect on the value of the entire fraction caused by an increase or a decrease in the values of the numerators or denominators of that fraction. It should be noted that most fractions in real-life applications as well as those commonly found in mathematics textbook problems are either proper fractions with unequal values of numerators or denominator, or improper fractions.

Other proportional reasoning items that involve such fractions will most likely elicit students' automatic use of other cognitive skills that are not accounted for exclusively by A2a or A2b. Indeed, all of the middle school mathematics teachers discussed and confirmed that converting fractions into decimal notations and resorting to drawing graphical representations of fractions are two of the most common practices in their mathematics classroom instruction of comparing or ordering fractions. In such a case, additional attributes other than A2a or A2b will be considered. We ascertained that proportional reasoning items that are intended to measure only A2a or A2b will involve only proper fractions with either fixed numerators or fixed denominators, while A1 will be adjoined when other fractions are involved.

Second, it was noted that A4 is not entirely synonymous with the multiplication skills included in A1. Mastery of A4 can be viewed as a more proactive analysis of recognising evidence of a multiplicative relationship between two or more multiple values. By contrast, the ability to multiply two or more values as measured in A1 presumes that two or more values are given or already identified.

Third, it was noted that the construction of any proportional reasoning items requiring A6 is generally inseparable from several aspects of A1, for example, addition, subtraction, multiplication, and division. For a proportional reasoning item to measure only A6, it is necessary to exhaust all different possible ways that students can solve such a problem without including any arithmetic computations. That is, from a psychometric point of view, a student's incorrect solution of an item measuring only A6 cannot necessarily be interpreted as his or her non-mastery of A6 if the student possesses an algorithm not assessed in that particular item, or if that particular item does not provide a choice involving an algorithm familiar to the student.

First phase of the validation process

After four rounds of exchanges in the first phase of the validation process, 16 prototype items were developed. Appendix A presents an example of the prototype item along with its item-attribute specification and explanations of possible incorrect answers. The attribute specifications of these prototype items were also verified to the extent that the two researchers and two middle school mathematics teachers solved them in consideration of the most common solutions by a typical eighth grade student.

On the basis of the findings from the first phase of the validation process—the 16 prototype items—some issues pertaining to A1 needed further clarification. When

identifying a list of attributes of proportional reasoning, mathematics researchers, mathematics educators, and middle school mathematics teachers agreed that a better understanding of mathematics at a more basic level was needed to solve many proportional reasoning problems. As mentioned earlier, we considered prerequisite skills and concepts (namely, A1) to be a necessary attribute. Two issues with A1 became evident in the validation phase of our study in that A1 was deemed to be pervasive and extensive.

The first issue was that, on the basis of the findings from the fourth round of the validation process, A1 clearly appeared in all 16 items, except Item 5. Although Item 5 demonstrated an early potential to create a proportional reasoning item requiring no prerequisite skills and concepts, it is important to consider carefully the pervasiveness of A1 throughout our assessment items. We could attempt to create more items that discount A1, but we needed to weigh the issue of whether these problems would be so sterile that they might not represent the proportional reasoning problems that students would actually solve beyond the assessment. While we could leave many of the items as they are, we also needed to acknowledge that this assessment would not provide much (if any) information for students who lacked basic skills and concepts, except for the simple information that they lacked basic skills and concepts. Pinpointing the specific basic skills and concepts that the students lack requires additional items that measure A1 purely.

The second issue was that, because A1 was a very coarse attribute and because identifying the specific basic skills and concepts that students lacked could not be done at the level anticipated in this assessment, the haphazard inclusion of prerequisite skills and concepts might lead to misleading inferences. For example, when a student correctly answers an item on finding an equivalent fraction of a given fraction (i.e., an item measuring A1), we do not necessarily know whether the student has mastered A1 in its entirety. Rather, the correct answer suggests that the student has mastered the notion of equivalent fractions. If this student incorrectly answered items in the assessment that measure other attributes in addition to A1, he or she may not understand the more advanced attributes of proportional reasoning (e.g., A5 or A6). Alternatively, the incorrect response may be traced to the student's lack of understanding the basic skills and concepts embedded in that particular item. That is, the basic skills and concepts used elsewhere in the assessment may be different from those intended to be measured in the particular fraction equivalent item. Indeed, this may likely be the case given the multitude of skills embedded in A1 and the limited space and time we have for the assessment. The argument for considering the practical comprehensiveness of A1 is indispensable. Thus, it was decided that because proportional reasoning is the main theme of this study's assessment, A1 should operate only to the extent that it supports, rather than leads, the positions of the other more pedagogically meaningful attributes of proportional reasoning.

Generalising further to the other attributes, we anticipated two challenges that were relatively similar to the issues examined for A1. The first challenge concerned the experience shared by the four raters during the validation process. Some items identified as missing-value problems tended to be easier to rate than other items from the list of 16 prototype items. Such easy-to-rate items apparently included items commonly

found in standard mathematics textbooks. We then determined that a good assessment should include a variety of novel items in addition to missing-value problems or straightforward tasks.

The second challenge concerned the question of the extent to which proportional reasoning items of a certain combination of attributes can or should be designed to form a well-rounded assessment that considers pedagogical circumstances and psychometric constraints. On the one hand, such items need to be deemed valuable because they foster the capacity to provide timely formative feedback from teachers to students. Although a new item can call for a total of 63 possible combinations from six attributes, not all of them can be constructed meaningfully. That is, certain combinations of attributes may not make any pedagogical sense. For example, an item that aims to measure both A2b (ordering fractions) and A4 (multiplicative relationship) may not be as revealing as two items that aim to measure A2b and A4 separately. On the other hand, to take full advantage of the parsimonious characteristic of the CDMs, such assessment should include only a sufficiently small number of effective items. That is, it can be targeted to capture a good mix of attributes at the same time. The test length issue is of an even greater concern when considering other common test conditions such as students' anxiety and time limitation.

Second phase of the validation process

Appendix B presents an example of coding for one of the 223 cases which was mutually agreed upon by the four researchers. This example shows that a student answered the item incorrectly. In addition, the transcript shows some evidence that the student had used A3a, and his or her written solution shows some evidence of using A1, A3b, and A6.

Appendix C summarises the findings of the item-by-item analysis for the middle school, college, and middle school/college student groups. In particular, it tabulates the percentage attribute usage by students for each item as evidenced in the transcripts and written solutions. For example, the first table shows that we found some evidence from either of the two sources that 50 %, 13 %, 50 %, 63 %, 0 %, 13 %, 0 %, and 0 % of the middle school students used A1, A2a, A2b, A3a, A3b, A4, A5, and A6 in solving Item 1, and that 0 % of them answered Item 1 correctly. Note that the shaded-cell attribute use for each item indicates an attribute for the particular item to measure, as agreed upon by the two researchers and two middle school mathematics teachers during the first phase of the validation process. For example, Item 1 was aimed at measuring students' mastery of A1 and A2b.

The findings from the second phase of the validation process revealed relatively poor performance in proportional reasoning problem solving. On average, the middle school student group correctly answered the 10 items about 34 % of the time, whereas the college student group correctly answered the 13 items about 65 % of the time. Nonetheless, those who did answer the items correctly demonstrated, to some extent, an agreeable pattern in the use of attributes previously validated by the two researchers and the two middle school mathematics teachers from the first phase of the validation process. This evidence was more apparent in the college student group than in the middle school student group. All in all, 7 out of the 13 items—namely, Items 1, 2, 4, 7, 8, 10, and 12—can be considered "on-target" for having evidence that at least 50 % of the students from either group demonstrated use of all of the previously validated (coloured-

cell) attributes in solving these items. For example, 73 % and 91 % of students in the college student group utilised A1 and A2b, respectively, to solve "on-target" Item 1, while 0 % of them utilised A4 to solve Item 3. (In addition to Item 7, due to the unfavourable test result, we decided that the other six items required modifications before we could administer them in future studies. However, this paper partly focuses more on the process of attribute validation than that of item development.)

Even so, one might have a different interpretation of the more complete picture. Although the previously validated (coloured-cell) attributes appeared relatively significant for most of the items, they were often accompanied by the presence of other attributes that were not intended to be assessed for those particular items. One explanation for this discrepancy may stem from the fact that students tended to perform more unnecessary (arithmetic) work than required, for example, the use of A1 as previously mentioned in the previous section. (We acknowledge the possibility that one might not be completely assured that some of the developed items with the required attributes were indeed "on-target" in general because of the noise from the attributes that were not intended to be measured by the items.)

More valuable than the success of developing the prototype items was the substantiation of attributes attached to them. Not only did the four researchers observe students' use of all six attributes previously established by the group of experts in the identification process, but they also gathered no material impact of incorporating additional attributes. There was no significant indication of cognitive processes exhibited by the students in the think-aloud interviews that pointed to the use of attributes not included in the list of six attributes. In other words, we believe that the six attributes constitute a comprehensive range of conceptual and procedural knowledge of proportional reasoning that students need to master to be deemed proficient in this topic relevant to the eighth grade level.

Future directions

In this paper, we presented the opportunities for and challenges to identifying a list of six proportional reasoning attributes based on a literature review and on consultations with mathematics researchers, mathematics educators, and middle school mathematics teachers. We also validated these six proportional reasoning attributes to examine whether or not the attribute(s) used by both experienced problem solvers and novice problem solvers to solve an item matched the attribute(s) specified or intended by the item writer.

A future study may benefit from analysing the development process of a more comprehensive list of items with desired psychometric properties as suggested in the list of six proportional reasoning attributes. Such an analysis will facilitate the administration of these test items into full use by middle school mathematics teachers to assess eighth grade students' mastery of proportional reasoning. A professional development workshop will also be necessary to help teachers interpret the psychometric results and make them more comprehensible so that they can improve their teaching and their students' learning of proportional reasoning. Another future direction may also include

conducting a longitudinal study to examine the relative mathematics performance of the same group of eighth graders once they have entered college.

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Appendixes

Appendix A. Example of prototype proportional reasoning item

Danny's 4-in. by 6-in. photo of his skateboard is enlarged into a 12-in. by 18-in. photo. If the length of his skateboard in the original photo is 5 in., how long is his skateboard in the enlarged photo?

A. 11 in.
B. 13 in.
C. 15 in.
D. 17 in.

Key: Option C.
Attributes intended and solution steps mapped to attributes intended:

Solution steps	Attribute intended
1. Students need to recognise that the dimensions of the original and enlarged photos form a proportional relationship.	1. A5: Determining whether two relations form a proportion.
2. Students need to construct an appropriate proportion to describe the proportional relationship between the dimensions of the original photo and those of the enlarge photo. That is, $\frac{4 \text{ inches}}{6 \text{ inches}} = \frac{12 \text{ inches}}{18 \text{ inches}}$.	2. A3b: Constructing proportions from a given situation.
3. Students need to figure out that the dimensions of the enlarged photo are 3 times as long as the dimensions of the original photo. That is, $12 \text{ inches} = 3 \times 4 \text{ inches}$ and $18 \text{ inches} = 3 \times 6 \text{ inches}$.	3. A4: Identifying a multiplicative relationship between sets of values.
4. Students need to make a connection between the knowledge of proportionality in Step 1, the multiplicative relationship in Step 3, and their multiplication skills to compute the length of the skateboard in the enlarged photo. That is, $3 \times 5 \text{ inches} = 15 \text{ inches}$.	4. A1: Prerequisite skills and concepts required in proportional reasoning (multiplication skills).

Explanations of the possible incorrect choices:

Option A: Students may think of an additive relationship: Since $6 \text{ inches} + 6 \text{ inches} = 12 \text{ inches}$, therefore $5 \text{ inches} + 6 \text{ inches} = 11 \text{ inches}$.
Option B: Students may think of an additive relationship: Since $4 \text{ inches} + 8 \text{ inches} = 12 \text{ inches}$, therefore $5 \text{ inches} + 8 \text{ inches} = 13 \text{ inches}$.
Option D: Students may think of an additive relationship: Since $6 \text{ inches} + 12 \text{ inches} = 18 \text{ inches}$, therefore $5 \text{ inches} + 12 \text{ inches} = 17 \text{ inches}$.

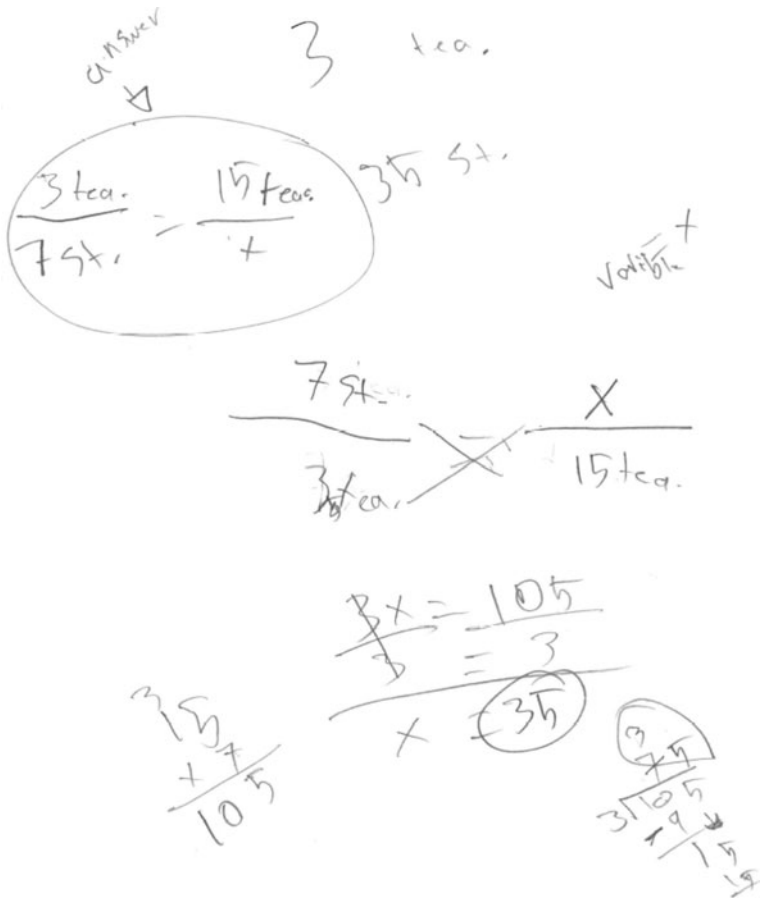
Appendix B. Example of coding for student's written solution

C/I	Attribute															
	A1		A2a		A2b		A3a		A3b		A4		A5		A6	
	T	S	T	S	T	S	T	S	T	S	T	S	T	S	T	S
0	0	1	0	0	0	0	1	0	0	1	0	0	0	0	0	1

Item 2:

A preschool principal made a rule that required 3 teaching assistants for a class of 7 students. Mary figured out that since there are 35 students in the preschool, 15 teaching assistants are needed. Write a proportion that could be used to show this proportional relationship.

Written Solution (S):



Appendix C. Summary of think-aloud findings: item-by-item analysis

Result from the Middle School Student Group									
		Percentage Attribute Usage							
Item	C/I	A1	A2a	A2b	A3a	A3b	A4	A5	A6
1	0%	50%	13%	50%	63%	0%	13%	0%	0%
2	38%	75%	0%	0%	25%	75%	25%	0%	38%
3	75%	100%	0%	0%	0%	0%	38%	0%	75%
4	13%	75%	0%	0%	0%	25%	0%	38%	13%
5	50%	75%	0%	0%	50%	13%	0%	13%	38%
6	25%	100%	0%	0%	13%	13%	38%	13%	13%
7	0%	75%	0%	0%	25%	13%	0%	88%	13%
8	50%	100%	0%	0%	0%	0%	0%	0%	88%
9	13%	38%	0%	0%	0%	0%	0%	25%	0%
10	75%	50%	0%	0%	25%	0%	13%	75%	0%

Result from the College Student Group									
		Percentage Attribute Usage							
Item	C/I	A1	A2a	A2b	A3a	A3b	A4	A5	A6
1	73%	73%	55%	91%	91%	0%	9%	0%	0%
2	36%	100%	0%	0%	55%	100%	45%	0%	55%
3	91%	100%	0%	0%	0%	0%	0%	27%	91%
4	73%	91%	0%	0%	0%	64%	0%	100%	45%
5	55%	82%	9%	0%	64%	18%	9%	18%	36%
6	73%	100%	0%	0%	18%	55%	73%	55%	27%
7	0%	82%	0%	0%	64%	0%	0%	82%	0%
8	100%	100%	0%	0%	0%	0%	0%	0%	100%
9	91%	100%	0%	0%	9%	64%	0%	91%	55%
10	100%	100%	0%	0%	0%	0%	0%	91%	0%
11	36%	100%	0%	0%	0%	45%	9%	55%	91%
12	73%	91%	0%	0%	36%	55%	0%	73%	73%
13	45%	91%	0%	0%	45%	0%	0%	9%	0%

Combined Result from the Middle School and College Student Groups									
		Percentage Attribute Usage							
Item	C/I	A1	A2a	A2b	A3a	A3b	A4	A5	A6
1	42%	63%	37%	74%	79%	0%	11%	0%	0%
2	37%	89%	0%	0%	42%	89%	37%	0%	47%
3	84%	100%	0%	0%	0%	0%	16%	16%	84%
4	47%	84%	0%	0%	0%	47%	0%	74%	32%
5	53%	79%	5%	0%	58%	16%	5%	16%	37%
6	53%	100%	0%	0%	16%	37%	58%	37%	21%
7	0%	79%	0%	0%	47%	5%	0%	84%	5%
8	79%	100%	0%	0%	0%	0%	0%	0%	95%
9	58%	74%	0%	0%	5%	37%	0%	63%	32%
10	89%	79%	0%	0%	11%	0%	5%	84%	0%

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