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On recognizing proportionality: Does the ability to solve missing value proportional problems presuppose the conception of proportional reasoning?☆

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ABSTRACT

This paper investigates the relationship between the ability of middle school students to solve missing value proportional problems and their facility in differentiating proportional relationships from non-proportional relationships. Students in low- and high-proficiency groups in mathematics took a ratio-and-proportion test involving two typical missing value proportional (MVP) and two recognizing proportionality (RP) problems. The findings revealed that while the students generally performed better on MVP problems than on RP problems, the two groups differed in their performance on MVP problems, but not on RP problems. Moreover, of those students from both the groups who successfully solved the two MVP problems, a significantly greater proportion of students in the high-proficiency group were unsuccessful in solving either of the two RP problems than those in the low-proficiency group. An analysis of performance differences between items within the same student group showed that the effect of differences in the structural components of RP problems to some extent contradicted the previous findings on the effect of differences in the structural components of MVP problems. It is hoped that these findings can shed light on what might be missing in the teaching and learning of proportional reasoning.

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1. Introduction

Lesh, Post, and Behr (1988) considered proportional reasoning to be the "capstone of elementary arithmetic, number, and measurement concepts" and the "cornerstone of algebra and other higher level areas of mathematics" (p. 97). According to the National Council of Teachers of Mathematics (NCTM), a significant amount of middle school mathematics problem-solving needs to be contextualized around students' development of number sense and computational fluency in proportional reasoning. The concept of proportional reasoning represents "an important integrative thread that connects many of the mathematics topics studied in grades 6–8" (NCTM, 2000, p. 217); thus, it "merits whatever time and effort must be expended to assure its careful development" (NCTM, 1989, p. 82).

More specific to its relation with the development of number sense, proportional reasoning pertains to the core of introducing students to making sense of ratios. This experience becomes an even more determinative learning moment

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during their academic years in middle school when exposure to missing value proportional (MVP) problems becomes the main emphasis. MVP problems, the most common proportional reasoning problems (Tjoe & de la Torre, 2012), are designed to assess students' ability to evaluate one missing value, given three values in such a way that four of them are related proportionally (Cramer & Post, 1993; Kaput & West, 1994).

In terms of formal mathematical symbols, a proportional relationship between two rational relationships between two values each can be expressed as $a/b = c/d$. Among many structural components of MVP problems, Harel and Behr (1989) described the locations of the missing values: top-left (a), bottom-left (b), top-right (c), and bottom-right (d). They particularly maintained that students found MVP problems involving the bottom-right missing values easier than those involving the top-right missing values. In addition to the location of the missing value, other factors affecting student performance on MVP problems included the magnitude and direction of ratio scale (Freudenthal, 1983; Karplus, Pulos, & Stage, 1983). While the magnitude of ratio scale (presence of integer ratios) can be integer (e.g., the ratio of 3 to 6) or non-integer (e.g., the ratio of 3 to 5), the direction of ratio scale can be internal (within similar measure spaces, e.g., the ratio of a to c, or the ratio of 3 apples to 6 apples) or external (between different measure spaces, e.g., the ratio of a to b, or the ratio of 3 apples to 6 students).

Apart from solving MVP problems, students need to understand proportionality in its complete abstraction. It requires the ability to identify two values and to construct one single ratio that relates the two, as well as building up any subsequent ratios of two new values, in a joint manner that preserves the constant multiplicative relationship within the first single ratio and between that and subsequent ratios. To perform these tasks, students should demonstrate mental arithmetic readiness in proportionality, which can only progress from a proficiency that goes beyond comparing two values additively (Lamon, 2007).

Thus, students who lack the mastery to think proportionally tend to fall back on additive reasoning when posed with MVP problems. Niaz (1988) observed this phenomenon more frequently on MVP problems that involved certain ratio pairs suggestive of the use of additive reasoning (e.g., 3 is to 5 as 4 is to x) than on those that involved other ratio pairs (e.g., 4 is to 6 as 6 is to x). This was due to the fact that in the former case, there were non-integer ratios, and the additive difference between the given numbers was moreover relatively small. Kaput and West (1994) found that when ratios with non-integer multiples, rather than those with integer multiples, were involved in MVP problems, students' incorrect use of additive comparisons appeared to be more evident.

On the other hand, students who perform well on MVP problems often inappropriately apply a proportional reasoning strategy to solve problems involving certain numerical structures that commonly characterized MVP problems (i.e., non-proportional reasoning problems). For instance, a more recent initiative by Van Dooren, De Bock, Evers, and Verschaffel (2009) and Van Dooren, De Bock, Vleugels, and Verschaffel (2010) analyzed the effect of integer and non-integer ratios on students' overuse of proportionality in solving non-proportional reasoning problems. Their findings recognized that the presence of ratios with integer multiples in non-proportional reasoning problems prompted a higher chance that students would apply proportional reasoning inappropriately.

Using both non-standard MVP and non-proportional reasoning problems, Lim (2009) observed a similar pattern in pre-service teachers as Van Dooren et al. (2009) did in middle school students. Lim (2009) consequently recommended that students be exposed to solving problems involving not only proportional but also non-proportional situations. This recommendation echoed the need for reasoning beyond the mechanization of solving MVP problems. Lamon (2007) specifically called the students' attention to learning to "recognize the difference" between proportional relationships and non-proportional relationships (p. 647).

To the extent that ratios with integer multiple are involved, previous studies have hinted not only at students' high performance on MVP problems (Kaput & West, 1994; Karplus et al., 1983), but also at their low performance on non-proportional reasoning problems due to misapplications of proportional reasoning strategies (De Bock, Van Dooren, Janssens, & Verschaffel, 2002; Van Dooren, De Bock, Hessels, Janssens, & Verschaffel, 2005; Van Dooren et al., 2009, 2010). Such studies are valuable for assessing students' ability to solve MVP problems as well as non-proportional reasoning problems. Yet, little can be inferred about their conceptual ability to reason proportionally.

In particular, these studies do not systematically answer the fundamental question of what can be more eminent—perhaps exigent—in the teaching and learning of proportional reasoning in everyday classroom settings (Ebersbach, Van Dooren, Goudriaan, & Verschaffel, 2010; Esteley, Villarreal, & Alagia, 2010; Lamon, 2007; Lim, 2009; Lobato, Ellis, & Zbiek, 2010; Modestou & Gagatsis, 2010; Van Dooren & Greer, 2010; Vlahović-Štetića, Pavlin-Bernardića, & Rajtera, 2010). Specifically, is teaching to solve MVP problems synonymous with teaching to reason proportionally? Does the mechanism of solving MVP problems presuppose the conception of proportional reasoning? Another way to view this issue is to examine whether students who perform well on MVP problems necessarily understand the difference between situations that involve proportional relationships and those that involve non-proportional relationships.

In this study, we explored the possibility of evaluating students' ability to recognize proportionality. In doing so, we made an explicit effort to link the two different contexts of proportionality and non-proportionality in one single problem. Given student performance on such problems, we basically sought to find an immediate route that distinguishes between students who can differentiate a proportional situation from a non-proportional one and those who cannot. We later made the connection between students' mastery of MVP problems and their proficiency in recognizing proportional relationships. This paper concludes with a pedagogical recommendation based on the findings of the study.

2. Methodology

Given the prominence and importance of proportional reasoning in middle school mathematics, eighth grade students can be assumed to have successfully learned proportional reasoning toward the end of their school year. Eighth grade students from two different middle schools in two different suburban school districts in northeastern U.S. participated in this study. The first group of participants consisted of 242 eighth graders from a middle school with 46% of the eighth graders considered proficient in mathematics based on the 2010 state assessment. The second group of participants consisted of 151 eighth graders from a middle school with 89% of the eighth graders considered proficient in mathematics based on the 2010 state assessment. The state average for the eighth grade mathematics assessment in 2010 was 69%. In this study, the first and second groups of participants were referred to as the low- and high-proficiency groups, respectively. Near the end of their school year, students from both groups took a test consisting of 50 ratio-and-proportion-related problems, four of which are the main focus of the current study. Table 1 presents the four problems.

Like the other 46 problems, the four problems were in a multiple-choice format. The first two problems (MVP1 and MVP2) were typical MVP problems that could be found in standard mathematics textbooks or on mathematics tests. The second two problems (RP1 and RP2) were recognizing proportionality (RP) problems which required students to differentiate a proportional from a non-proportional situation. In these RP problems, students were asked to determine whether a particular proportion could be used to solve two given problems (one involving a proportional relationship and the other involving a non-proportional relationship).

The non-proportional relationship involved in RP1 was of an additive problem, which might be more appropriately solved using an additive comparison via a constant difference between two quantities. In contrast, the non-proportional relationship involved in RP2 was of an indirect variation problem, which might be more appropriately solved using a multiplicative reciprocal approach via an inverse proportionality between two values. It should be emphasized again that the decision to administer the test at the end of the school year was not arbitrary. In particular, considering that the standard curriculum on ratios and proportions at the middle school level incorporates constructing ratios and proportions, we assumed that by the end of the school year, most eighth graders would already be familiar with using formal mathematical symbols. Thus, we did not consider the presentation of the proportions as part of the two RP problems an issue for our participants.

Because of particular structural similarities between MVP1 and RP1, as well as between MVP2 and RP2, the four problems were presented to the students in the following order: MVP1, RP1, MVP2, and RP2; the other 46 problems appeared randomly in between, before, or after the four problems. MVP1 and RP1 had the similar structural location of a missing value on the

Table 1
Missing value proportional and recognizing proportionality problems.

Problem	Description
MPV1	3 apples cost \$4. 6 apples cost how much? A. \$5 B. \$6 C. \$7 D. \$8
MVP2	Ross makes 7 trips for a total of 14 km. How many trips will Ross make for a total of 20 km? A. 9 trips B. 10 trips C. 13 trips D. 27 trips
RP1	The proportion $1/2 = 10/x$ can be used to solve which of the following situations? Situation I: For every 1 boy, there are 2 girls in a classroom. If there are 10 boys in the classroom, how many girls are there? Situation II: Bob is 1 year old and Mary is 2 years old. When Bob is 10 years old, how old will Mary be? A. Situation I only B. Situation II only C. Both Situations I and II D. Neither Situation I nor II
RP2	The proportion $5/15 = x/9$ can be used to solve which of the following situations? Situation I: 5 pounds of tomatoes can make 15 cups of Brilliantly Yummy tomato juice. How many pounds of tomatoes does Tommy need if he wants to make 9 cups of Brilliantly Yummy tomato juice? Situation II: In a classroom, there are 5 rows of chairs and there are 15 chairs in each row. How many chairs are there in a row if the same chairs are rearranged into 9 rows of chairs? A. Situation I only B. Situation II only C. Both Situations I and II D. Neither Situation I nor II

Table 2
Item features.

Problem	MVP1	RP1	MVP2	RP2
Type of problem	MVP	RP	MVP	RP
Location of missing value	Bottom-right	Bottom-right	Top-right	Top-right
Magnitude of ratio scale	Integer	Integer	Integer	Integer
Direction of ratio scale	Internal	Internal	External	External
Type of non-proportional situation	N/A	Additive	N/A	Indirect variation

bottom-right of the proportion (e.g., $a/b = c/x$), whereas for MVP2 and RP2, the location was on the top-right of the proportion (e.g., $a/b = x/d$). Moreover, MVP1 and RP1 involved internal ratios, whereas MVP2 and RP2 involved external ratios. All ratios used in the four problems were integer multiples in order to preserve the simplicity of the assessment, and more importantly, to avoid confounding the focus of the study which was to compare student performance on MVP and RP problems. Table 2 displays the features of the four items.

The analysis of the findings started with a brief summary of students' overall performance on each of the four problems and continued with an item-by-item comparison between the two student groups, leading to a group-by-group comparison between similar items. In the analysis of student performance, responses to the four problems were scored for correctness. In addition to each individual MVP and RP problem, the sums of correct responses for both MVP and RP problems were calculated and coded as MVPs and RPs, respectively. (This means that the number of correct responses for MVP1, MVP2, RP1, and RP2 ranged from 0 to 1, whereas for MVPs and RPs, the range was 0–2.) In the analysis of performance difference across MVP and RP problems between the low- and high-proficiency groups, the two-proportion z-test was utilized; in the analysis of performance difference between items within the same student group, the chi-square test for within-subjects (i.e., the McNemar's test) was utilized; and in the analysis of association between ordinal scores of MVP and RP problems, the asymmetric measure of association (i.e., the Somers' d statistic) was utilized. All tests were carried out at the 0.05 level of significance.

3. Findings

Table 3 presents a brief summary of the performance of students in the low and proficiency groups on MVP1, RP1, MVP2, and RP2. For example, 94.0% of students in the high-proficiency group answered MVP1 correctly.

3.1. On missing value proportional problems

Regarding student performance on MVP problems, a significant difference between the two groups was evident: students in the high-proficiency group performed significantly better than students in the low-proficiency group. Students in the high-proficiency group scored 94.0% and 93.4% on MVP1 and MVP2, while students in the low-proficiency group scored 86.0% and 73.1% on the same problems. This phenomenon was consistent with the state classification on mathematics proficiency for the two groups of participants in this study. To a greater extent, the students' performance on MVP problems was a clear reflection of how the state assessed proficiency in proportional reasoning: by design, MVP1 and MVP2 were typical MVP problems commonly found in mathematics textbooks or on mathematics tests. In addition, the majority of students in the low- and high-proficiency groups, who did not answer MVP problems correctly, chose options C and D for MVP1 and MVP2, respectively, both of which could be obtained through an inappropriate use of additive reasoning. This result substantiated the findings of earlier studies that additive reasoning was often used by students who had not mastered proportional reasoning to solve MVP problems (Karplus et al., 1983; Lesh et al., 1988).

Although intended to be simple and posing no material challenge, students from both groups performed differently on each MVP problem. Students in the low-proficiency group did 12.9% worse on MVP2 than on MVP1, while students in the high-proficiency group did merely 0.6% worse on MVP2 than on MVP1. The McNemar's test further showed that the difference in the performances on the MVP problems was significant for the low-proficiency group, but not for the high-proficiency group (p-value = 0.0004 vs. p-value = 1.0000). In other words, compared with students in the high-proficiency group, students in the low-proficiency group were notably more affected by the structural differences in MVP1 and MVP2. This result confirmed the findings of earlier studies that MVP problems involving bottom-right missing values and internal ratios were found to be easier than those involving top-right missing values and external ratios (Harel & Behr, 1989; Kaput & West, 1994; Karplus et al., 1983). However, this study also showed that the structural difference might not be an issue when administered to students in the high-proficiency group.

Table 3
Brief summary of performance for the low- and high-proficiency groups.

Proficiency group	MVP1	RP1	MVP2	RP2
Low	86.0%	34.7%	73.1%	40.1%
High	94.0%	31.8%	93.4%	43.7%

Despite the differences in the nature of structural components involved in the two MVP problems, testing more than one MVP problem to assess students' mastery in the mechanism of solving MVP problems might be overdone for certain groups of students. While MVP1 evoked a reasonably significant difference of 8.0% in performance between the two groups, MVP2 attracted a significant difference of 20.3% (p-value = 0.0125 vs. p-value = 0.0000). This result demonstrated that although MVP2 could be regarded as an additional tool with which to measure student groups' ability to solve MVP problems, such a simple MVP problem as MVP1 might even do an adequate job of distinguishing students in the high-proficiency group from students in the low-proficiency group.

Specifically, MVP2 was a more effective supplementary diagnostic tool for students in the low-proficiency group, but not so much for students in the high-proficiency group. In fact, 75.5% of students in the low-proficiency group who correctly answered MVP1 were able to answer MVP2 correctly, while 88.7% of students in the low-proficiency group who correctly answered MVP2 were able to answer MVP1 correctly. In comparison, 93.7% of students in the high-proficiency group who correctly answered MVP1 were able to answer MVP2 correctly, while 94.3% of students in the high-proficiency group who correctly answered MVP2 were able to answer MVP1 correctly. In other words, mastery of MVP1 translated more strongly to mastery of MVP2, and vice versa, for students in the high-proficiency group than for students in the low-proficiency group. In this sense, a test consisting of one MVP problem might be sufficient for assessing students in the high-proficiency group, but not students in the low-proficiency group.

Even with its redundancy in separating students in the high-proficiency group based on their ability to solve different MVP problems, one might argue for the value of MVP2 in facilitating the developmental learning process for varying routine MVP problems while also reinforcing previously-learned cognitive processes, such as the need for more advanced mental arithmetic fluency in solving MVP2 rather than MVP1. This was particularly important because of the structural differences involved in the two MVP problems. Yet, neither MVP1 nor MVP2 assessed anything beyond arithmetic, given the rationale of including them as examples of the most common proportional reasoning problems. Indeed, critical thinking in proportionality needs to be considered. However, the students' relatively high performance on MVP problems did not help them think proportionally: all students performed considerably lower on RP problems regardless of their group membership.

3.2. On recognizing proportionality problems

There was no significant difference between the two groups with regard to student performance on RP problems. Students in the low-proficiency group scored 34.7% and 40.1% on RP1 and RP2, respectively, while students in the high-proficiency group scored 31.8% and 43.7% on the same problems. While students in the low-proficiency group performed 2.9% insignificantly better on RP1 than those in the high-proficiency group, the former performed 3.6% insignificantly better on RP2 than the latter (p-value = 0.5507 vs. p-value = 0.4779).

On their own, there was no performance difference on RP1 and RP2 for both groups of students. Still, the insignificant difference between the two groups was not in line with the phenomenon seen earlier on MVP problems, in which students in the high-proficiency group tended to perform significantly better than students in the low-proficiency group. In contrast, RP problems were much more demanding than MVP problems for both groups. Student performance was 85.0% on MVP problems, compared with 37.5% on RP problems. Although the state classification on mathematics proficiency served as an indicator to distinguish students from the high-proficiency group from those from the low-proficiency group in their performance on MVP problems, it failed to produce a parallel effect in measuring their facility to reason proportionally. More importantly, the present findings demonstrated that high performance on MVP problems did not consistently translate into an acceptably passing mark on RP problems (e.g., a cut-score of 50%).

Although performance difference was significant between the low- and high-proficiency groups, a more careful examination of student performance conditional on their performance on MVPs presented a different point of view. In this analysis, student performance on RPs was examined based on their score on MVPs (i.e., 0–2). First, the number of students who received a score of 0 on MVPs was too small and unreliable to conduct a chi-squared test on three different responses of RPs (i.e., 0–2). Second, a similar chi-squared test on the number of students in the low- and high-proficiency groups who correctly answered one of the two MVP problems produced no significant difference (p-value = 0.3155). However, a significant difference was found between the number of students in the low- and high-proficiency groups who obtained correct responses on both MVP problems, based on the same chi-squared test (p-value = 0.0146). Hence, we conditioned our subjects to those students with a score of 2 on MVPs.

Among these students, 29.3% in the low-proficiency group and 42.9% in the high-proficiency group could not answer any of the RP problems correctly. A two-proportion z-test substantiated that students in the high-proficiency group performed significantly worse than students in the low-proficiency group on the two RP problems (p-value = 0.0162). To a certain extent, "more experienced" problem-solvers might appear to have mastered the topic of proportional reasoning by focusing superficially on its procedural understanding, thus overlooking its conceptual understanding. Such mastery might give rise to their fixation to call mechanistically upon certain algorithms (e.g., a false attraction to the proportion given in the problem prior to applying proportional reasoning) and be rather inflexible in reasoning intuitively whether those algorithms might be appropriate. By choosing to accept that the given proportion could be used to solve both situations in RP problems, students revealed that they "blindly" applied proportional reasoning strategies to any problems involving certain numerical structures that commonly characterized MVP problems, as Van Dooren et al. (2009, 2010) found.

A further analysis using the McNemar's test demonstrated a significant difference between performances on the RP problems within the high-proficiency group, but not within the low-proficiency group (p-value = 0.0207 vs. p-value = 0.2631). Students in the low-proficiency group performed insignificantly different on RP1 than on RP2, but students in the high-proficiency group performed significantly worse on RP1 than RP2. This result might be related to the structural differences involved in RP1 and RP2, as well as the type of non-proportional situation.

From the structural component point of view, the fact that students in the high-proficiency group performed worse on RP1 than on RP2 deviated from previous literature on the effect of differences in the structural components of MVP problems. To some extent, neither this effect nor its deviation was apparent in students in the low-proficiency group. On the other hand, students in the high-proficiency group might find an indirect variation situation to be easier than an additive situation to contrast with a direct proportional situation. Indeed, 36.8% of students in the high-proficiency group with a score of 2 on MVPs chose option C for RP2, while 62.4% chose option C for RP1. In other words, the "best" students in the high-proficiency group confused an additive situation, about twice more than an indirect variation situation, with a direct proportional situation.

Additionally, the Somers' d statistic showed that the scores on MVP problems correlated significantly with those on RP problems for all students in the low-proficiency group, but not for all students in the high-proficiency group (p-value = 0.0000 vs. p-value = 0.3774). This result—that performance on MVP problems was a better association of performance on RP problems for students in the low-proficiency group—led to a possible conjecture that the few students who perhaps did not receive adequate formal classroom instruction and preparation had to learn "the hard way" to think proportionally in solving MVP problems and thus were more competent in recognizing proportionality. On the other hand, it might also be possible that the mathematics pedagogy received by students in the high-proficiency group over-emphasized the routine problem-solving of proportional situations, but exposed them to only few variations of non-proportional situations as suggested by Lim (2009).

4. Pedagogical and research implications

The current study revealed a variety of evidence that the ability to solve MVP problems to a certain degree had no association with or direct implication for solving RP problems. It particularly showed that teaching to solve MVP problems was not synonymous with the teaching to reason proportionally. At the same time, the mechanism of solving MVP problems did not appear to presuppose the conception of proportional reasoning. It became obvious that students' mathematics experience on proportional reasoning problems had evolved into mechanistic problem-solving practice leaving behind any appreciation for critical thinking. These findings demonstrated ongoing concerns described earlier by various mathematics educators on the overuse of proportional reasoning strategies (Lamon, 2007; Van Dooren et al., 2009, 2010; Van Dooren & Greer, 2010). Clearly, an assessment of students' ability to think proportionally that mainly emphasize MVP problems is not sufficient. It is important to improve classroom instruction and preparation by continuously presenting students with proportional relationships and previously learned mathematical relationships. Classroom discourse on comparing and contrasting different ways of solving proportional and non-proportional situations might be beneficial. Encouraging students to reason in abstraction in an early stage of their mathematics learning experience can facilitate higher-level mathematical thinking. To this end, such an assessment of the conceptual understanding of proportional reasoning skills was shown feasible by the creation of an RP problem. For instance, it might be used, in an alignment with the Common Core State Standards for Mathematics (Common Core State Standards Initiative, 2010), to assess middle school students' ability to appropriately model both proportional and non-proportional real-life problems with mathematics through the application of proportional reasoning at the sixth, seventh, and eighth grade levels.

The current study also demonstrated different ways in which student performance on MVP and RP problems might be affected by structural characteristics, such as the location of missing values and the direction of ratio scale. The current study was limited in several ways. It did not exhaust all possibilities of different types of problems, locations of missing value, magnitude and direction of ratio scale, and types of non-proportional situations. Different measures of classifying low- and high-proficiency group, other than the state eighth grade mathematics assessment, might be considered. It might be of interest to conduct a more comprehensive study that compares the effects of different combinations of structural characteristics and types of non-proportional situations on student performance on RP problems. In addition, future studies might consider the effect of presenting to students MVP and RP problems in a test, or proportional relationship and non-proportional relationship situation in RP problems, in a different order than in the order in which they were assessed in the current study. Student from different demographics might also be examined to validate the findings in this study.

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