A PECULIAR COIN-TOSSING MODEL

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1. Coin tossing according to de Finetti

A coin is drawn at random from a finite set of coins.

Each coin generates an i.i.d. sequence of outcomes (heads or tails).

The following facts are the gist of de Finetti's analysis of coin tossing in terms of an exchangeable stochastic process.

(A) a Bayesian agent's prior belief regarding the infinite sequence of outcomes that he will observe can be represented by a nonatomic probability measure on $\{\mathsf{H}, \mathsf{T}\}^{\mathbb{N}}$ ($\mathbb{N}$ denotes $\{1, 2, \dots\}$);

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(B) learning continues forever, and the agent's posterior beliefs about which coin is being tossed converge almost surely;

(C) that limit is a tail-measurable random variable on the probability space described in (A); and

(D) the agent's posterior beliefs about the coin converge to certainty (that is, for any particular coin, the posterior belief that it is the one being tossed converges to either 0 or 1) almost surely.

This example will be generalized here by assuming that successive tosses of each particular coin are independently, but not necessarily identically, distributed. Property (B) continues to be satisfied, but none of the other properties is necessarily satisfied.

2. How things fall apart—a quick survey

Consider two idealized trays of coins, T_1 and T_2.

Each tray T_i holds, in a fixed order, an infinite sequence $\langle c_{in} \rangle_{n \in \mathbb{N}}$ of coins.

A tray will be selected at random.

The agent has prior probability $q_0 \in (0, 1)$ for the event that T_1 will be chosen.

The coins in a tray are not identical to one another.

A Bayesian agent has belief p_{in} that coin c_{in} will land heads when it is tossed.

He believes that each coin c_n^1 is more heavily biased toward tails than is the corresponding c_n^2. That is, for each n,

$$0 < p_{1n} < p_{2n} < 1 \tag{1}$$

He believes that, for large n, both p_{1n} and p_{2n} are very close to 0.

$$\sum_{n=1}^{\infty} p_{2n} < \infty \qquad \left(\text{so} \quad \sum_{n=1}^{\infty} (q_0 p_{1n}) + ((1 - q_0) p_{2n}) < \infty \right) \tag{2}$$

[$(q_0 p_{1n}) + ((1 - q_0) p_{2n})$ is the prior probability that the outcome of the nth coin toss will be heads.]

The coins in the selected tray are tossed in their proper sequence.

The agent believes that, conditionally on the tray that has been chosen, the tosses are independent.

After having observed the outcomes of the first n tosses of an infinite sequence $\sigma = \langle \sigma_n \rangle_{n\in\mathbb{N}}$ of coin-toss outcomes (that is, each σ_n is either H or T), the agent has posterior probability $q_n(\sigma) \in (0,1)$ that T_1 has been chosen.

Here is what will happen.

Learning continues forever: Since $0 < p_{1n} < p_{2n} < 1$ and $q_n \in (0,1)$, $\forall t\ q_{n+1} \neq q_n$ a.s.

Define $q_\infty \colon \{\mathsf{H},\mathsf{T}\}^{\mathbb{N}} \to [0,1]$ by $q_\infty(\sigma) = \lim_{n\to\infty} q_n(\sigma)$.

Posterior probability is a bounded martingale, so $q_\infty(\sigma)$ is defined almost surely and $\mathsf{E}[q_\infty] = q_0$.

Since $\mathsf{E}[q_1|\mathsf{T}] > \mathsf{E}[q_1|\mathsf{H}]$ and q is a martingale, q_∞ is not almost surely constant.

By the Borel-Cantelli Lemma, (2) implies that, almost surely, only finitely many coin-toss outcomes will be heads.

The set of coin-toss-outcome sequences with only finitely many heads outcomes is a countable set. Thus, P must be purely atomic. (A fails.)

Since the outcome sequence is almost surely asymptotically T, the set of which sequences is an atom of the tail σ-algebra, and since q_∞ is not almost surely constant, q_∞ is not a tail-measurable random variable. (C fails.)

It will be shown below that the agent's posterior belief about the tray almost surely fails to converge to certainty. (D fails.)

3. Formalizing an example

Let $(\Omega, \mathcal{B})$ be a measurable space.

$\mathcal{B}$ is partitioned into events T_1 and T_2, with $\mathcal{T} = \{T_1, T_2\}$.

$T(\omega)$ denotes the element of $\mathcal{T}$ to which ω belongs.

For each $n \in \mathbb{N}$, let $\mathcal{C}_n = \{C_{n\mathsf{H}}, C_{n\mathsf{T}}\} \subseteq \mathcal{B}$ be a partition of Ω. $\mathcal{C}_n$ represents the toss of the nth coin.

Call (F, f) a *finite toss-event sequence* if F is a non-empty, finite subset of $\mathbb{N}$, $f \colon F \to \mathcal{B}$, and $\forall n \in F\ f(n) \in \mathcal{C}_n$.

If P is a probability measure on $(\Omega, \mathcal{B})$, then $\langle \mathcal{C}_n \rangle_{n\in\mathbb{N}}$ is *conditionally independent* iff, for both $T \in \mathcal{T}$ and for every finite toss-event sequence (F, f),

$$\mathrm{P}[\bigcap_{n\in F} f(n))|T] = \prod_{n\in F} \mathrm{P}[f(n)|T] \tag{3}$$

A peculiar coin-tossing model (PCTM) satisfies the following conditions.

$$\mathrm{P}(T_1) > 0 \text{ and } \mathrm{P}(T_2) > 0 \tag{4}$$

$$\forall n \in \mathbb{N}\ \ 0 < \mathrm{P}[C_{n\mathsf{H}}|T_1] < \mathrm{P}[C_{n\mathsf{H}}|T_2] < 1 \tag{5}$$

$$\sum_{n\in\mathbb{N}} \mathrm{P}[C_{n\mathsf{H}}|T_2] < \infty \tag{6}$$

$$\langle \mathcal{C}_n \rangle_{n\in\mathbb{N}} \text{ is conditionally independent} \tag{7}$$

$$\prod_{n\in\mathbb{N}} \frac{\mathrm{P}[C_{n\mathsf{T}}|T_2]}{\mathrm{P}[C_{n\mathsf{T}}|T_1]} > 0 \tag{8}$$

A coin-tossing model satisfies conditions (4), (5), and (7) (but not necessarily conditions (6) and (8)).

4. Existence of a PCTM

A PCTM is proved to exist via two lemmas. Lemma 1 guarantees the existence of a coin-tossing model. The proof is a standard measure-theoretic construction. Lemma 2 guarantees that conditions (6) and (8) can be satisfied simultaneously.

Lemma 1. *If $0 < q_0 < 1$ and, for each $n \in \mathbb{N}$, $0 < p_{1n} < p_{2n} < 1$, then there is a coin-tossing model that satisfies*

$$\mathrm{P}(T_1) = q_0; \text{ and} \tag{9}$$

$$\forall n \in \mathbb{N} \quad \mathrm{P}[C_n\mathsf{H}|T_1] = p_{1n} \text{ and } \mathrm{P}[C_n\mathsf{H}|T_2] = p_{2n} \tag{10}$$

Lemma 2. *If $\forall n \in \mathbb{N}$ $0 < p_{1n} < p_{2n} < 1$ and $\sum_{n\in\mathbb{N}} p_{2n} < \infty$, then $\prod_{n\in\mathbb{N}}[(1-p_{2n})/(1-p_{1n})] > 0$.*

Proof. First, note that $[(1-p_{2n})/(1-p_{1n})] > 1 - p_{2n}$.

Thus, because $\sum_{n\in\mathbb{N}} p_{2n} < \infty$, it is sufficient to prove that

$$\sum_{n\in\mathbb{N}} p_{2n} < \infty \implies \prod_{n\in\mathbb{N}} (1-p_{2n}) > 0$$

If $0 < x < 1/2$, then

$$\ln(1-x) = -\int_{1-x}^{1} u^{-1}\,du > -\int_{1-x}^{1} 2\,du = -2x$$

That $\sum_{n\in\mathbb{N}} p_{2n} < \infty$ implies that, for some $N \in \mathbb{N}$,

$$\forall n > N \; p_{2n} < 1/2$$

That is, $\forall n > N \; \ln(1-p_{2n}) > -2p_{2n}$.

$$\begin{aligned}
\prod_{n\in\mathbb{N}}(1-p_{2n}) &= \left[\prod_{n=1}^{N}(1-p_{2n})\right]\left[\prod_{n=N+1}^{\infty}(1-p_{2n})\right] \\
&= \left[\prod_{n=1}^{N}(1-p_{2n})\right]\left[\exp\Big(\sum_{n=N+1}^{\infty}\ln(1-p_{2n})\Big)\right] \qquad (11)\\
&> \left[\prod_{n=1}^{N}(1-p_{2n})\right]\left[\exp\Big(-2\sum_{n=N+1}^{\infty}p_{2n}\Big)\right]
\end{aligned}$$

$\sum_{n=N+1}^{\infty} p_{2n} < \infty$, so $\exp(-2\sum_{n=N+1}^{\infty} p_{2n}) > 0$.
Therefore, by (11), $\prod_{n\in\mathbb{N}}(1-p_{2n}) > 0$ □

By the two lemmas,

Proposition 1. *If $0 < q_0 < 1$ and*

$$\forall n \in \mathbb{N} \;\; 0 < p_{1n} < p_{2n} < 1 \text{ and}$$
$$\sum_{n\in\mathbb{N}} p_{2n} < \infty,$$

then the coin-tossing model specified by conditions (9) and (10) is a PCTM.

5. The probability measure of a PCTM

An event B in a σ-algebra $\mathcal{C}$ is an *atom* of $\mathcal{C}$ if there does not exist a partition $\mathcal{P} \subseteq \mathcal{C}$ of B.

Measure P is *purely atomic* if there exists a countable set $\mathcal{A}$ of atoms such that $\mathrm{P}(\Omega) = \sum_{B \in \mathcal{A}} \mathrm{P}(B)$.

In a PCTM, the events pertaining to coin-toss outcomes are $C_{n\mathsf{H}}$ and $C_{n\mathsf{T}}$.

Define $\mathcal{C}$ to be the σ-algebra generated by $\bigcup_{n \in \mathbb{N}} \mathcal{C}_n$.

Proposition 2. *The measure space* $(\Omega, \mathcal{C}, \mathrm{P} \restriction \mathcal{C})$ *derived from a PCTM is purely atomic. Specifically, defining*

$$Z_\tau = \{\omega | \exists N \, \forall n \geq N \; \omega \in C_{n\mathsf{T}}\}$$

Z_τ *is a countable union of atoms and* $\mathrm{P}(Z_\tau) = 1$.

Proof. An event is an atom of $\mathcal{C}$ if and only if it is

$$C_\sigma = \bigcap_{n \in \mathbb{N}} C_{n\sigma_n} \tag{12}$$

for some $\sigma \in \{\mathsf{H}, \mathsf{T}\}^{\mathbb{N}}$. Define $G \subseteq \{\mathsf{H}, \mathsf{T}\}^{\mathbb{N}}$ by

$$G = \{\sigma | \{n | \sigma_n = \mathsf{H}\} \text{ is finite}\} \tag{13}$$

G is a countable set.

Note that $Z_\tau = \bigcup_{\sigma \in G} \bigcap_{n \in \mathbb{N}} C_{n\sigma_n}$, so Z_τ is a countable union of atoms of $\mathcal{C}$.

It will now be shown that $\mathrm{P}(Z_\tau) = 1$.

$$\begin{aligned} \mathrm{P}(C_{n\mathsf{H}}) &= \mathrm{P}((C_{n\mathsf{H}} \cap T_2) \cup (C_{n\mathsf{H}} \cap T_1)) \\ &= \mathrm{P}[C_{n\mathsf{H}} | T_2]\mathrm{P}(T_2) + \mathrm{P}[C_{n\mathsf{H}} | T_1]\mathrm{P}(T_1) \\ &< \mathrm{P}[C_{n\mathsf{H}} | T_2] \end{aligned} \tag{14}$$

$\sum_{n \in \mathbb{N}} \mathrm{P}(C_{n\mathsf{H}}) < \sum_{n \in \mathbb{N}} \mathrm{P}[C_{n\mathsf{H}} | T_2] < \infty$, so $\mathrm{P}(Z_\tau) = 1$ by the Borel-Cantelli Lemma. □

The *tail σ-algebra* is the sub-σ-algebra of $\mathcal{C}$ consisting of events that can be characterized in terms of asymptotic properties of an infinite sequence of coin tosses.

Formally, let $\mathcal{Z}_N$ denote the σ-algebra generated by $\bigcup_{n=N}^{\infty} \mathcal{C}_n$ and define the tail σ-algebra by $\mathcal{Z} = \bigcap_{N \in \mathbb{N}} \mathcal{Z}_N$.

An atom of $\mathcal{Z}$ is an event $Z_\sigma = \bigcup_{N \in \mathbb{N}} \bigcap_{n \geq N} C_{n\sigma_n}$, where $\sigma \in \{\mathsf{H}, \mathsf{T}\}^{\mathbb{N}}$.

In particular, letting $\tau = \langle \mathsf{T}, \mathsf{T}, \mathsf{T}, \ldots \rangle$, Z_τ is an atom of $\mathcal{Z}$.

By proposition 2, $\mathrm{P}(Z_\tau) = 1$. This proves:

Corollary 1. *The probability measure of a PCTM assigns probability 1 to the atom Z_τ of the tail σ-algebra.*

6. Conditional probabilities of events in $\mathcal{T}$

Let $\sigma \in \{\mathsf{H}, \mathsf{T}\}^{\mathbb{N}}$ and $\omega \in C_\sigma$ and $N \in \mathbb{N}$.

Define the conditional probability of tray 1 after having observed N outcomes to be

$$\begin{aligned} Q_N(\omega) &= \mathrm{P}\left[T_1 | \bigcap_{n \leq N} C_{n\sigma_n}\right] \\ &= \frac{\mathrm{P}(T_1) \prod_{n \leq N} \mathrm{P}[C_{n\sigma_n} | T_1]}{\mathrm{P}(T_1) \prod_{n \leq N} \mathrm{P}[C_{n\sigma_n} | T_1] + \mathrm{P}(T_2) \prod_{n \leq N} \mathrm{P}[C_{n\sigma_n} | T_2]} \end{aligned} \tag{15}$$

Define $\mathcal{C}^N$ to be the algebra generated by $\bigcup_{n \leq N} \mathcal{C}_n$.

$\langle Q_N \rangle_{N \in \mathbb{N}}$ is a bounded martingale adapted to the filtration $\langle \mathcal{C}^N \rangle_{N \in \mathbb{N}}$.

Lemma 3. *The martingale $\langle Q_N \rangle_{N\in\mathbb{N}}$ converges almost surely and in L^1 to a $\mathcal{C}$-measurable random variable, Q_∞, with mean $\mathrm{P}(T_1)$.*

$$Q_{N+1}(\omega) = \frac{\mathrm{P}(T_1)[\prod_{n\le N}\mathrm{P}[C_{n\sigma_n}|T_1]]\mathrm{P}[C_{(N+1)\sigma_{N+1}}|T_1]}{\sum_{T\in\mathcal{T}}\mathrm{P}(T)[\prod_{n\le N}\mathrm{P}[C_{n\sigma_n}|T]]\mathrm{P}[C_{(N+1)\sigma_{N+1}}|T]} \tag{16}$$

Equation (16) implies the following result.

Lemma 4. *Suppose that $\forall n \le N$ $\sigma_n = \sigma'_n$ and $\sigma_{N+1} = \mathsf{H}$ and $\sigma'_{N+1} = \mathsf{T}$, and that $\omega \in C_\sigma$ and $\omega' \in C_{\sigma'}$. Then $Q_{N+1}(\omega) < Q_N(\omega) = Q_N(\omega') < Q_{N+1}(\omega')$.*

Setting $N = 0$ in lemma 4 yields the result that that $\mathsf{E}[Q_1|C_{1\mathsf{H}}] < \mathsf{E}[Q_1|C_{1\mathsf{T}}]$.

By the martingale property and lemma 3, it follows that $\mathsf{E}[Q_\infty|C_{1\mathsf{H}}] < \mathsf{E}[Q_\infty|C_{1\mathsf{T}}]$.

Both $C_{1\mathsf{H}}$ and $C_{1\mathsf{T}}$ are positive-probability events.

Therefore, Q_∞ is not constant on an event of probability 1. But, by corollary 1, it would be so, if it were tail measurable. This proves that

Proposition 3. *Q_∞ is not tail measurable.*

The final result to be proved here, is that $Q_\infty(\omega) \in (0,1)$ almost surely.

By (15), if $\sigma \in \{\mathsf{H},\mathsf{T}\}^{\mathbb{N}}$ and $\omega \in C_\sigma$ and $N \in \mathbb{N}$, then

$$Q_N^{-1}(\omega) = 1 + \frac{\mathrm{P}(T_2)\prod_{n\le N}\mathrm{P}[C_{n\sigma_n}|T_2]}{\mathrm{P}(T_1)\prod_{n\le N}\mathrm{P}[C_{n\sigma_n}|T_1]} \tag{17}$$

If $\omega \in Z_\tau$, then $\sigma \in G$ (defined in (13)) and, for some $K \in \{0\}\cup\mathbb{N}$, $\forall n > K$ $\sigma_n = \mathsf{T}$.

In this case, (17) implies that, for $K < N$,

$$\begin{aligned} Q_N^{-1}(\omega) &= 1 + \frac{\mathrm{P}(T_2)\left[\prod_{n\le K}\mathrm{P}[C_{n\sigma_n}|T_2]\right]\left[\prod_{K<n\le N}\mathrm{P}[C_{n\sigma_n}|T_2]\right]}{\mathrm{P}(T_1)\left[\prod_{n\le K}\mathrm{P}[C_{n\sigma_n}|T_1]\right]\left[\prod_{K<n\le N}\mathrm{P}[C_{n\sigma_n}|T_1]\right]} \\ &= 1 + \frac{\mathrm{P}(T_2)\prod_{n\le K}\mathrm{P}[C_{n\sigma_n}|T_2]}{\mathrm{P}(T_1)\prod_{n\le K}\mathrm{P}[C_{n\sigma_n}|T_1]}\prod_{K<n\le N}\frac{\mathrm{P}[C_{n\mathsf{T}}|T_2]}{\mathrm{P}[C_{n\mathsf{T}}|T_1]} \end{aligned} \tag{18}$$

Therefore, by (8) $\omega \in \mathrm{P}(Z_\tau)$ implies

$$\begin{aligned} Q_\infty &= \left[\lim_{N\to\infty} Q_N^{-1}(\omega)\right]^{-1} \\ &= \left[1 + \frac{\mathrm{P}(T_2)\prod_{n\le K}\mathrm{P}[C_{n\sigma_n}|T_2]}{\mathrm{P}(T_1)\prod_{n\le K}\mathrm{P}[C_{n\sigma_n}|T_1]}\lim_{N\to\infty}\prod_{K<n\le N}\frac{\mathrm{P}[C_{n\mathsf{T}}|T_2]}{\mathrm{P}[C_{n\mathsf{T}}|T_1]}\right]^{-1} \\ &\in (0,1) \end{aligned} \tag{19}$$

Since $\mathrm{P}(Z_\tau) = 1$, that (19) holds for all $\omega \in Z_\tau$ implies that

Proposition 4. *$Q_\infty(\omega) \in (0,1)$ almost surely.*

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